Journal of mathematics and computer science 8 (2014), 245-250

Contents list available at JMCS
Journal of Mathematics and Computer Science
Journal Homepage: www.tjmcs.com

J-TYPE MAPPINGS AND FIXED POINT THEOREMS IN MENGER SPACES

E. Feizi and Z. Hosseini

Faculty of Science, Mathematics Department, Bu-Ali Sina University, Hamedan, Iran
E-mail: efeizi@basu.ac.ir
Faculty of Science, Mathematics Department, Bu-Ali Sina University, Hamedan, Iran
E-mail: z.hoseini@basu.ac.ir

Article history:
Received July 2013
Accepted August 2013
Available online August 2013

Abstract

J. Garcia-Falset, E. Llorens-Fuster and S. Prus in [2] studied the existence of fixed point of J-type mappings in Banach spaces. In this paper, we extend these mappings in Menger spaces and prove the fixed point theorems of these mappings in complete Menger spaces. In this paper, we also prove theorems for the new class of mappings which is called altering J-type.

Keywords: Fixed point, Menger space, J-type mapping, Altering J-type mapping.

1. Introduction

There are some of generalization of metric space, one of them is Menger space commenced by Karl Menger [7] which the distance $d(x, y)$ between x and y replaced by a distribution function $F(x, y)$. In fact, $F_{x,y}(t)$ can be interpreted as the probability that the distance between x and y is less than t. The study of probabilistic metric space (for short, PM-space) was initiated by Schweizer and Sklar [12]. Fixed point theorems are very important in the PM-spaces and have several interesting applications. For example, these theorems are the main tools to study the problem of the existence of a solution for a system of differential equations in these spaces [11]. The study of fixed point theorems in probabilistic Menger space is a topic of recent interest and forms an active direction of research, is investigated by many authors [1, 3, 4, 6, 8, 9, 10, 13].
Let us recall some concept of the theory of PM-spaces and then Menger spaces. In the present work, $\mathbb{R}$ is the set of all real numbers and $\mathbb{R}^+$ is the set of all nonnegative real numbers.

Definition 1.1. A mapping $F: \mathbb{R} \rightarrow \mathbb{R}+$ is called a distribution function if it be a non decreasing and left continuous function satisfying the following conditions

$$\inf_{t \in \mathbb{R}} F(t) = 0, \qquad \sup_{t \in \mathbb{R}} F(t) = 1$$

Definition 1.2. A probabilistic metric space (PM-space) is an ordered set (S, F) Where S is a nonempty set and F is a function defined on S×S to the set of distribution functions which satisfy the following conditions

(i) $F_{x,y}(0) = 0$

(ii) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$

(iii) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in \mathbb{R}$

(iv) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$, then $F_{x,z}(t_1+t_2) = 1$.

Definition 1.3. A triangular norm (t-norm) is a function T: [0,1] × [0,1]→[0,1] which satisfies the following

(i) T(1,a) = a

(ii) T(a, b) = T(b, a)

(iii) T(c, d) ≥ T(a, b) whenever c ≥ a and d ≥ b

(iv) T(T(a, b), c) = T(a, T(b, c)).

Definition 1.4. A Menger space is a triplet (S, F, T) where Sis a nonempty set, F is a function defined on S×S to the set of distribution functions and T is a t-norm such that the followings are satisfied

(i) $F_{x,y}(0) = 0$ for all $x, y \in S$

(ii) $F_{x,y}(s) = 1$ for all $s > 0$ if and only if $x = y$

(iii) $F_{x,y}(s) = F_{x,y}(s)$ for all $x, y \in S$ and $s > 0$

(iv) $F_{x,y}(u+v) \geq T(F_{x,z}(u), F_{z,y}(v))$ for all $x, y, z \in S$ and $u, v > 0$.

Definition 1.5. A sequence $\{x_n\} \subseteq S$ is said to converge to $x \in S$ if for all $\varepsilon > 0$ and $\lambda > 0$, we can find a positive integer $N_{\varepsilon,\lambda}$ such that $F_{x_n, x}(\varepsilon) > 1-\lambda$ for all $n > N_{\varepsilon,\lambda}$. The sequence is said to be Cauchy if for all $\varepsilon > 0$ and $\lambda > 0$, there exists $N_{\varepsilon,\lambda}$ such that $F_{x_n,x_m}(\varepsilon) > 1-\lambda$ for all $n, m > N_{\varepsilon,\lambda}$. A Menger space (S, F, T) is said to be complete if every Cauchy sequence is convergent.

2. J-type and altering J-type mappings

In this section, we prove fixed point theorems for class of nonlinear mappings in menger spaces which were studied by J. Garcia-Falset et al. [2] in Banach spaces. Before that, a basic definition is presented.

Definition 2.1. Let B be a closed and bounded subset of S. $y_0 \in S$ is called a center for mapping $f: B \rightarrow S$ if there exists $k \in (0,1)$ such that for each $x \in B$ and $s > 0$, $F_{x,y_0}(s) > 1-s$ implies

$$F_{f(x),y_0}(ks) > 1 - ks$$

Definition 2.2. The mapping f: B → S is called J-type mapping whenever, it is continuous and it has some center $y_0 \in S$.

We denote by $Z_m(f)$ the set of all centers of map f. The following notation is remarkable.

Remark2.3. Let (S, F, T) be a menger space, then $Z_m(f)$ is a closed subset of S.

Proof. Let $z \in clos(Z_m(f))$, then there is a sequence $\{z_n\} \subseteq Z_m(f)$ converges to z. From definition we have $k \in (0,1)$ such that for each $x \in S$ and $s > 0$, $F_{x,z_n}(s) > 1 - s$ implies $F_{f(x),z_n}(ks) > 1 - ks$.

So

$$F_{f(x),z}(2ks) > T(F_{f(x),z_n}(ks), F_{z_n,z}(ks)) > 1 - ks > 1 - 2ks$$

for minimum t-norm T. It is sufficient to put $k_1 = 2ks$. □

Several immediate consequences of the above definitions are considered the follow.

Remark 2.4. (i) It is obvious that if the center $y_0 \in B$, then y_0 is a fixed point of f,

(ii) If a mapping f: B → S has the center $y_0 \in S$, then the restriction of f to every subset of B does too,

(iii) If a mapping f: B → B has the center $y_0 \in S$, then f^n: B → B has the same center y_0 for all $n \in N$. The following theorem is closely inspired Theorem (1) of [5].

Theorem 2.5. Let (S, F, T) be a complete Menger space and B is a closed subset of S and f: B → S be J-type mapping. Suppose there exists $x_0 \in B$ that for all $n \geq 0$, $f^n(x_0) \in B$. If $\liminf_{n\to\infty}\{F_{f^n(x0),\, Z_m(f)}(s)\} = 1$ ($s > 0$), Then f has a fixed point $x^\star$ and sequence $\{f_n(x)\}$ converges to $x^\star$.

Proof. Let $\varepsilon > 0$ has been given. Since $\liminf_{n\to\infty}\{F_{f^n(x0),\, Z_m(f)}(s)\} = 1$, there exists positive number N such that for every $n > N$

$$\text{Inf}\,\{F_{f^n(x0),\, z}(s) \mid z \in Z_m(f)\} > 1 - \varepsilon.$$

that is, there exists $z_0 \in Z_m(f)$ such that $F_{f^n(x0),\, z0}(s) > 1 - \varepsilon$. So

$$F_{f^n(x0),\, f^m(x0)}(2s) \geq T(F_{f^n(x0),\, z0}(s), F_{f^m(x0),\, z0}(s))$$
$$> 1 - \varepsilon.$$

Above inequality is concluded from being minimum t-norm T and belonging z_0 to $Z_m(f)$. Therefore, the sequence $\{f^n(x_0)\}$ is a Cauchy sequence. So there exists $x^\star \in B$ such that

$$F_{f^n(x0),\, x^\star}(s) > 1 - \varepsilon.$$

From hypothesis, we have

$$F_{x\star,z0}(2s) \geq T(F_{x\star, f^{n}(x0)}(s), F_{f^{n}(x0), z0}(s))$$

$$> 1 - \varepsilon.$$

Since $Z_m(f)$ is closed, then $x\star \in Z_m(f)$, beside, $x\star \in B$. Hence $x\star$ is a fixed point off and the proof is completed. □

In order to proceed, a basic definition is presented.

(∗) $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing and continuous function such that $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.6. Let (S, F, T) be a Menger space. A mapping $f: S \to S$ is called altering J-type mapping if there exists $k \in (0,1)$ such that for each $x, y \in S$ and $s > 0$ the following implication holds, $F_{x,y0}(\psi(s)) > 1 - s$ implies

$$F_{f(x), y0}(\psi(ks)) > 1 - ks$$

where ψ satisfies in condition (∗)

Theorem 2.7. Let (S, F, T) be a complete Menger space and B is a closed subset of S. Let f be an altering J-type self-mapping on B. Then with an extra condition on f, there is a point $z \in B$ such that $f(z) = z$.

Proof. Since f is an altering J-type mapping on B, there exist $y_0 \in S$, $k \in (0,1)$ such that $F_{x,y0}(\psi(s)) > 1 - s$ implies $F_{f(x),y0}(\psi(ks)) > 1 - ks$ where ψ satisfies in condition (∗) and $x \in B$. Considering $t > 1$, we have for any $r > 0$

$$F_{x, y0}(\psi(r)) > 1 - t \ (x \in B)$$

So by above definition $F_{f(x), y0}(\psi(kr)) > 1 - kt$, that is

$$F_{f^{-1} f f(x),y0}(\psi(kr)) > 1 - kt \text{ or } F_{f^{-1} f^{2}(x), y0}(\psi(kr)) > 1 - kt$$

By Definition (2.6) we have again

$$F_{f f^{-1} f f(x), y0}(\psi(kr)) > 1 - kt \text{ or } F_{f^{2}(x), y0}(\psi(kr)) > 1 - kt$$

Therefore, for all $n \in N$ this results

(2.1) $$\inf\{F_{f^{n}(x0), y0}(\psi(knr))\} > 1 - k^{n}t.$$

Let $0 < \lambda < 1$ be an arbitrary real number. Since $0 < k < 1$ then $k^n t \to 0$ as $n \to \infty$, then there exists $N_1(\lambda) \in \mathbb{R}+$ such that

(2.2) $$1 - k^{n}t > 1 - \lambda \ (n > N_1(\lambda)).$$

Also suppose that $\varepsilon > 0$ be arbitrary, then there exists $N_2(\varepsilon) \in \mathbb{R}^+$ such that

(2.3) $$\varepsilon_2 > \psi(k^{n}r) \ \ (n > N_2(\varepsilon)).$$

Now we have from (2.1), (2.2) and (2.3)

$$F_{f^n(x), y_0}(\varepsilon_2) > F_{f^n(x), y_0}(\psi(k^n r)) > 1 - k^n t > 1 - \lambda.$$

for all $n > N = \max\{N_1(\lambda), N_2(\varepsilon)\}$ and $x \in B$. Moreover, we consider a sequence $\{x_n\} \subseteq B$ and let $m > n > N$.

(2.4) $$F_{f^n(x_{m-n}), f^n(x_0)}(\varepsilon) \geq T(F_{f^n(x_{m-n}), y_0}(\frac{\varepsilon}{2}), F_{f^n(x_0), y_0}(\frac{\varepsilon}{2})).$$

We assume extra condition $F_{f^n(x_{m-n}), y_0}(\lambda) \leq F_{f^n(x_0), y_0}(\lambda)$ for all $0 < \lambda < 1$.

Therefore from (2.4) it is resulted

$$F_{f^n(x_{m-n}), f^n(x_0)}(\varepsilon) \geq F_{f^n(x_{m-n}), y_0}(\frac{\varepsilon}{2}) > 1 - \lambda.$$

Hence $\{f^n(x_m)\}$ is a Cauchy sequence in (S, F, T). The completeness of this space and closeness of B in S imply that there exists $z \in B$ such that $f^n(x_m) \to f^n(z)$ as $m \to \infty$. We clime that z is fixed point for map f. Let $s > 0$

(2.5) $$F_{f^n(z), z}(s) \geq T(F_{f^n(z), f^n(x_m)}(s_1), F_{f^n(x_m), z}(s_2)).$$

Where $s_1, s_2 \geq 0$ and $s_1 + s_2 = s$. Using the properties of condition (∗), we can find two positive numbers $\varepsilon_1, \varepsilon_2$ such that

(2.6) $$s_1 > \psi(\varepsilon_1) \text{ and } s_2 > \psi(\varepsilon_2).$$

On the other hand, since $f^n(x_m) \to f^n(z)$ as $m \to \infty$ then for $0 < \lambda_1 < 1$ there exists N_3 such that $n > N_3$

(2.7) $$F_{f^n(z), f^n(x_m)}(\psi(\varepsilon_1)) \geq 1 - \lambda_1.$$

and

(2.8) $$F_{f^n(z), f^n(x_{m-1})}(\psi(\varepsilon_2 k)) \geq 1 - \lambda_1.$$

From (2.2) and (2.8) it can concluded

(2.9) $$F_{f^{n+1}(z), f^{n+1}(x_{m-1})}(\psi(\varepsilon_2)) \geq 1 - k\lambda_1.$$

From (2.5) with using (2.6), (2.7) and (2.9), we have

$$F_{f^n(z), z}(s) \geq T(1-\lambda 1, F_{f^n(x_{m-1}), f^n(z)}(\varepsilon_2)) \ (s > 0)$$

Then $f^n(z) = z$ for all $n \in N$, especially $n = 1$ that it implies $f(z) = z$. □

ACKNOWLEDGEMENTS

The author would like to express his pleasure to referee(s) for his/her careful reading and making some useful corrections on the first draft of this paper which improved the presentation and readability.

References

[1] Binayak S. Chaudhury and Krishnapada Das, A coincidence point result in Menger spaces using a control function, Solitons and fractals. 42(2009) 3058-3063.
[2] J. Garcia-Falset, E. Llorens-Fuster and S. Prus The fixed point property for mappings admitting a center, Nonlinear Analysis., 66 (2007), 1257-1274.
[3] Jin-Xuan Fang, Common fixed point theorems of compatible and weakly compatible maps in Menger spaces, Nonlinear Analysis. 71(2009) 1833-1843.
[4] Jin-Xuan Fang and Yang Gao, Common fixed point theorems under strict contractive conditions in menger spaces, Nonlinear Analysis, 70(2009) 184-193.
[5] K. Goebel and W. A. Kirk, Classical theory of non expansive mappings, Handbook of metric fixed point theoy, Kluwer. A. P., (2001).
[6] M. Imdad, Javad Ali and M. Tanveer, Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces, Solitons and fractals. 42(2009) 1321-1329.
[7] Menger. K. Statistical metrics, Proc. Nat. Acad. Sci. USA 28(1942), 535-537.
[8] José R. Morales, E. M. Rojas, Altering distance functions and fixed point theorems through rational expression, Mathematic and Computer Science. (2012),
[9] Z. Mustafa, Mixed g-monotone property and quadruple fixed point theorems in partially ordered G-metric spaces using $(\phi - \psi)$ contractions, Mathematic and Computer Science. (2012), 1-22.
[10] B. D. Pant and Sunny Chauhan, A fixed point theorem in Menger space, 7(2011), 9-17.
[11] A. Razani and K. Fuladgar, Extension of contractive maps in the Menger probabilistic metric space, Solitons and fractals, 34(2007) 1724-1731.
[12] Schweizer B and Sklar A, Probabilistic metric space, New York. Elsevier. (1983).
[13] B. Singh and S. Jain, A fixed point theorem in Menger space through weak compatibility, J. Math. Anal. Appel,301(2005) 439-448.