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ON THE MATHEMATICAL THEORY OF TURBULENCE AND ITS RELATION TO CHAOS AND FRACTALS

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Abstract
The Navier-Stokes differential equations describe the motion of fluids which are incompressible. The three-dimensional Navier-Stokes equations misbehave very badly although they are relatively simple-looking. The solutions could wind up being extremely unstable even with nice, smooth, reasonably harmless initial conditions. A mathematical understanding of the outrageous behaviour of these equations would dramatically alter the field of fluid mechanics. The Orr-Sommerfeld equation is also described. In this paper the author adopts a reasoned, practical approach towards resolving the issue and proposes a practical, statistical kind of mathematical solution.

Keywords: unpredictable, probability, estimate

2000 Mathematics Subject Classification: 76Fxx Turbulence

1. Navier-Stokes Equation

The general equations of motion for a viscous fluid were obtained by Sir George Stokes in 1845. The following is the fundamental equation (in vectorial form) governing the flow of a viscous fluid:

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$$-\frac{\partial v}{\partial t} + (v.\nabla)v = -\frac{1}{p}\nabla Pe - \nabla\varphi + \frac{\eta}{p}\nabla 2v\ ,$$

where v is the velocity of the fluid (as a function of position), Pe the pressure, φ the gravitational potential, p the density and η the viscosity.

A fluid in motion could be characterised by its velocity field (velocity as a function of position) but because of the complex nature of the forces affecting fluids (in general, forces of both compression and viscosity) the result of applying basic principles such as Newton's second law is a set of nonlinear equations. Computational methods thus play a large part in fluid dynamics. (Newton's second law states that the rate of change of momentum p of a body equals the total force F acting upon it:

$$F = \partial p/\partial t\ .$$

If, as is normally the case, the mass of the body is constant, $F = \partial(mv)/\partial t$ reduces to $F = m\partial v/\partial t$ or $F = ma$, where a is the acceleration of the body. Note that the force and acceleration are vectors. The first law is the null case of the second law (if $F = 0$ then $a = 0$).)

The Navier-Stokes equation is a miracle of brevity, relating a fluid's velocity, pressure, density and viscosity; fluid flow governed by this partial differential equation is deterministic and predictable in two dimensions. But this equation fails when the fluid becomes turbulent as turbulence represents three-dimensional flow of the fluid, for which the Navier-Stokes equation fares very poorly. Whereas fluid flow under normal conditions tends to be laminar in turbulence it becomes irregular and develops eddies, ripples and whorls. But yet there is some sort of order found within this disorder or turbulence which could be described as self-similar or fractal. What mathematical technique could now be utilised to describe this state?

The Navier-Stokes equations are nonlinear and do not submit to any general method of solution. Each new problem has to be carefully formulated as to geometry and proper boundary conditions. Then some scheme of attack may be chosen with the hope of reaching a solution. In most cases all attempts to obtain an exact solution fail. Approximate solutions have to suffice. In a few cases exact solutions can be obtained. The possibility that perhaps the flow of the fluid is unidirectional, i.e., $v\ (x, y, t) = 0$, is not an assumption. It is rather an intuitive guess we pursue until we either find a solution or become convinced that it does not lead to a solution, in which case we mark it as an unsuccessful trial.

Substitution of viscosity in the Navier-Stokes equations with viscosity = 0 reduces them to a form called the Euler equations:

$$p\ \underline{Dq} = pg - \nabla p \quad \text{(in vectorial form)}$$

Dt

The Euler equations were formulated earlier than the Navier-Stokes equations and were considered an approximation. The Euler equations are of the first order and cannot in general satisfy the boundary conditions. We can therefore conclude that the Euler equations do not form a good approximation near a rigid boundary. Far from a boundary and where viscosity = 0 is a fair estimate, they have an important role as approximations and are generally easier to solve than the full Navier-Stokes equations.

The Navier-Stokes equations require for their solution initial conditions as well as boundary conditions. The proper boundary conditions for a velocity on a rigid boundary are:

$q_n = q_t = 0$,

where q_n is the normal component of the velocity relative to the solid boundary, and q_t is the tangential component. These conditions are also termed the no-penetration ($q_n = 0$) and no-slip ($q_t = 0$) viscous boundary conditions. When the region occupied by the fluid is not closed, i.e., the fluid is not completely confined, additional conditions are still required on some surfaces which completely enclose the domain of the solution. These may represent some real physical surfaces or they may be chosen quite arbitrarily, provided the velocity on them is known. The pressure, which is also a dependent variable, requires boundary conditions too. The Navier-Stokes equations are then satisfied and we now know the resulting pressure field. This flow can exist only if the obtained pressure is possible. An acceptable boundary condition may be: $p = p_\infty$ = const at $r \rightarrow \infty$, which then implies: $p = p_\infty - \frac{pQ^2}{8\Pi^2} \cdot \frac{1}{r^2}$. We also note that in the solution for the pressure there is no trace of the viscosity.

This pressure thus also satisfies the Euler equations. (As viscosity in a fluid enables it to smooth out or overcome the ripples, eddies and whorls of turbulence, a viscous fluid is in effect not so much affected by turbulence than a non-viscous fluid. Therefore, the Navier-Stokes equations, as they relate to viscous fluids, present a better solution for incompressible fluids which are viscous and subject to turbulence than the Euler equations for non-viscous fluids.) [Ref. 8, 9, 10, 11, 22 & 23]

2. Orr-Sommerfeld Equation

The Orr-Sommerfeld equation is another equation which could be utilised in fluid dynamics and is considered in some quarters as the more relevant equation for describing turbulence. It is an eigenvalue equation that describes the linear two-dimensional modes of disturbance found in a viscous parallel flow. However, like the Navier-Stokes equation, it would not be of much help for turbulent fluid flow. The equation is obtained by solving a linearized version of the Navier-Stoke equation for the perturbation velocity field:

$\mathbf{u} = (U(z) + u'(x, z, t), 0, w'(x, z, t))$, where $(U(z), 0, 0)$ is the unperturbed or basic flow. For the perturbation velocity, the following is the wave-like solution:

$$\mathbf{u}' \propto \exp(i\alpha(x - ct)) .$$

The following dimensional form of the Orr-Sommerfeld equation is obtained by using this knowledge and the streamfunction representation for the flow:-

$$\frac{\mu}{i\alpha p}\left(\frac{d^2}{dz^2} - \alpha^2\right)^2 \varphi = (U - c)\left(\frac{d^2}{dz^2} - \alpha^2\right)\varphi - U''\varphi ,$$

where μ is the fluid's dynamic viscosity, p is its density and φ is the streamfunction. By measuring velocities according to a scale set by some chartacteristic velocity U_0 and by measuring lengths according to channel depth h, the equation could be written in non-dimensional form. The equation then takes the following form:-

$$\frac{1}{i\alpha Re}\left(\frac{d^2}{dz^2} - \alpha^2\right)^2 \varphi = (U - c)\left(\frac{d^2}{dz^2} - \alpha^2\right)\varphi - U''\varphi ,$$

where $Re = \frac{pU_0h}{\mu}$ is the Reynolds number of the basic flow. The relevant no-slip boundary conditions at the channel top and bottom $z = z_1$ and $z = z_2$ are as follows:

$$\alpha\varphi = \frac{d\varphi}{dz} = 0 \text{ at } z = z_1 \text{ and } z = z_2 .$$

c is the eigenvalue parameter of the problem and φ is the eigenvector. The base flow would be unstable and the small perturbation introduced to the system would be amplified in time if the imaginary part of the wave velocity c is positive. [Ref. 6 & 19]

3. Newton's Law Of Viscosity

The equation below is known as Newton's law of viscosity:-

$$T_{yx} = \mu \frac{du}{dy}$$

A fluid obeying this law is called a Newtonian fluid. This equation states that in unidirectional flow the shear stress in a Newtonian fluid is directly proportional to the transverse velocity gradient, du/dy, which is also known as the rate of shear strain or the rate of shear deformation.

There is no obvious reason why real fluids should obey this law. As a matter of fact, there are more fluids that do not obey this equation than those that do. Fluids that do not obey this law are called non-Newtonian. It is fortunate that the three most abundant fluids, air, water and petroleum, obey Newton's law of viscosity quite closely. The typical non-Newtonian fluids are paints, polymer solutions and melts, blood and many liquid food products, such as jellies, soups, etc.. [Ref. 1, 8, 22 & 23]

4. Computation Of Reynolds Numbers

The following is the formula for the Reynolds number:-

$$Re = v\ L\ p\ /\ n\ ,$$

where p (kg/m^3) is the fluid's density, v (m/s) is a typical fluid velocity, L (m) is some characteristic length, e.g., diameter of a pipe through which the fluid flows, and, n (kgm/s^2) is the coefficient of viscosity (Gooey substances have a higher n value than runny ones like petrol. Viscous force = ability to smooth out turbulent whorls, eddies and ripples.).

The Reynolds number is the ratio between two forces, the initial force and the viscous force (frictional force).

When the Reynolds number is below a few hundred the flow of the fluid is smooth.

When the Reynolds number exceeds about 2000 to 3000 the flow is completely turbulent.

Between these values the flow is sometimes smooth, sometimes turbulent.

The greater the viscosity of the fluid is the greater is its capability in overcoming or smoothing out the whorls, eddies and ripples of turbulence. [Ref. 19 & 22]

5. Fluid Behaviours

According to the Law of Continuity, water flowing from a wide pipe to a narrower pipe speeds up, while water flowing from a narrow pipe to a wider one slows down, and, the slow-moving water in the wide pipe would always have a higher pressure than the fast-moving water in the narrower pipe. Turbulent fluid flow could cause a pipe to give way.

The following are some equations pertaining to fluid flow:-

Energy (E) = Mass (m) x Speed (v^2) (i)

Altitude (A) + Energy (E) = Constant (ii)

Energy (E) = Density of Fluid (p) x Speed (v^2) (iii)

Pressure (P) + Energy (E) = Constant (iv)

Pressure (P) + Density of Fluid (p) x Speed (v^2) = Constant (v) [Ref. 1, 8, 22 & 23]

6. Model For Chaos

The logistic equation has been a popular method used for the modelling of chaos and turbulence. Invented by the Belgian mathematician, Pierre Francois Verhulst, the formula is very simple but because the process has to be repeated over and over again it ends up being extremely complicated. The equation works this way:-

For instance, if X is the population now, then the population next year is given by:

Xnext = rX (1 - X) ,

where r is some constant which could be adjusted according to the population being modelled. It is simplest if values of X between 0 and 1 are taken, so that 1 is the maximum population and 0 represents extinction. We may, e.g., take an arbitrary value for r of 2.6, and hence begin.

Let X = 0.2. Then 1 - X = 0.8, and, X (1 - X) = 0.2 x 0.8 = 0.16.

Then multiply this result by 2.6 and we would get 0.416.

Repeat the process. Start with X = 0.416 and we would get 0.6317. The population increases.

Start with 0.6317 and we would get 0.6049. The population falls.

Start with 0.6049 and we would get 0.6214. The population goes up again.

Repeating or iterating this process over and over again we would obtain the following population figures:

0.6117, 0.6176, 0.6141, 0.6162, 0.6150, 0.6156, 0.6152, 0.6155, 0.6153, 0.6154, 0.6153, 0.6154, 0.6154, 0.6154.

The population rises and falls but converges on a fixed number.

Scientists have however tried to model chaos or turbulence using this equation. [Ref. 2, 3, 4 & 12]

7. Fourier Analysis Of Waves As A Way Of Analysing Motions Of Fluids

The surface water in a wave moves in a circular path at an angular velocity w = 2π/T where T is the period of rotation. Deeper water moves in ellipses of decreasing size and increasing eccentricity.

The superpositions of simple sinusoidal oscillations in fluids could produce more complicated patterns of oscillations. The inverse mathematical operation of Fourier analysis could reduce any complicated oscillation into a sum of its simple sinusoidal components, each with a different period and amplitude.

With waves, there is a phenomenon which oscillates both in time and space. It might seem that this would considerably complicate any mathematical attempt to describe the superposition of waves. We could practically analyse complicated wave shapes either by freezing them in time or by freezing them in space. In the time domain, this gives us the wave's frequency components, while in the space domain we get the corresponding spectrum of wavelengths. These two approaches could each stand on their own, one being transformable to the other, because the product fλ is a constant (the wave speed), the longer wavelengths corresponding to the lower frequencies (i.e., the longer periods). The wavelength spectrum could be computed directly from the frequency spectrum by noting that for each harmonic component λ = v/f = v/T - this holds for all of the wave's Fourier components (e.g., λ_o = 3 metres, f_0 = 20 cycles per second and wave speed v = 60 m/s). [Ref. 1, 12, 15, 20 & 21]

8. Fourier Series And Circles

A Fourier series could be interpreted geometrically as the projection of a system of superimposed circular motions.

A circle rides upon the nest of spinning circles beneath it, and, we project the motion of a point on its circumference onto a line.

The result is a periodic but distinctly nonsinusoidal motion which could be described mathematically as follows:-

$$y(t) = (1)\sin\left(\frac{2\pi t}{T}\right) + \left(\frac{1}{3}\right)\sin\left(\frac{3.2\pi t}{T}\right) + \left(\frac{1}{5}\right)\sin\left(\frac{5.2\pi t}{T}\right) + \left(\frac{1}{7}\right)\sin\left(\frac{7.2\pi t}{T}\right)$$

where T = fundamental period of oscillation = time for one rotation of the largest circle

This is the Fourier series that has been truncated after the fourth term for greater simplicity. If more terms are included (i.e., more circles turning upon circles), the resulting graph would more nearly approximate a series of alternating horizontal line segments. [Ref. 1, 12, 15, 20 & 21]

9. Probability Waves As A Model

We now proceed to examine several related important ideas in quantum theory. Schrodinger had found an equation which could be applied to any physical system in which the mathematical form of the energy is known, which is as follows:-

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0$$

where ∂^2 is the second derivative with respect to x, x is the position of the particle, ψ is the Schrodinger wave function, or, the probability amplitude for an electron in the state n to scatter into the direction m, E is energy and V is potential energy.

The Schrodinger equation is a deterministic time-symmetrical description of nature. In classical mechanics, when we say that a quantum system is in a particular "state", we mean that the state is a point in phase space. It is here described by a wave function whose evolution over time is expressed by the following equation:-

$$ih / 2\pi \, \partial\psi (t) / \partial t = H_{op} \, \psi (t)$$

This equation identifies the time derivative of the Schrodinger wave function ψ with the action of the Hamiltonian operator on ψ. It is not derived but assumed at the start, and could thus be validated only by experiment. In quantum theory, it is the fundamental law of nature. Here, ψ is the probability amplitude for an electron - it is only an abstraction (having no physical reality). ψ is also, in a sense, the electron's own intensity wave. When it is squared and the absolute value is taken, it turns out to be a physical probability of the associated particle's presence.

Later, Born stated that the probability of the existence of a state is given by the square of the normalized amplitude of the individual wave function (i.e.ψ^2). This was another new concept, i.e., the probability that a certain quantum state exists. Born had said there were no more exact answers in atomic theory, but just probabilities. The wave Ψ determines the likelihood that the electron would be in a particular position, and, unlike the electromagnetic field, has no physical reality.

According to Dirac, light could be treated as waves or particles. In fact, in quantum mechanics, particles are regarded as waves. The behaviour of these particles could be predicted, as it were, and, they are thus known as probability waves or Dirac wave particles. There is a wave/particle duality here. When the particle is not observed, it remains a wave (a probability wave), but upon being observed it becomes a particle.

The formal solution of the Schrodinger equation is:-

$$\psi (t) = U (t) \, \psi (0)$$

where $U (t) = e^{-iHt}$, U (t) is the evolution operator that links the value of the wave function at time t to that at the initial time t = 0. Both future and past play the same role, since $U (t_1) \, U (t_2) = U (t_1 + t_2)$, whatever the sign of t_1 and t_2. This property defines a dynamical group.

This description of how quantum particles behave could not be strictly applied to the macro world of fluid flow. Nevertheless, the above "probability" principles pertaining to how quantum particles behave could be somewhat broadly adapted and used as a guide for the interpretation of turbulent fluid flow.

In another important theory, the Uncertainty Principle, propounded by Heisenberg, it is postulated that the very act of observing a quantum particle affects its behaviour. According to this theory, the position and the momentum of an elementary particle could not be known simultaneously. The reason for this is that if an electron could be held still long enough for its position to be determined, then its momentum could no longer be determined. A special point is that the product of two uncertainties (or spreads of possible values) is always at least a certain minimum number. From the de Broglie/Einstein relation, $\Delta p \sim h/\lambda$, Heisenberg obtained the imprecision in the momentum. Multiplying the two inaccuracies together, he showed that the product, $\Delta x \, \Delta p$, would always be greater than or equal to ($\geq$) a certain amount, as follows:-

$$(\Delta x)(\Delta p) \geq (\lambda)(h/\lambda) \geq h, \text{ or, ...} \qquad \text{(i)}$$

$$\Delta x \, \Delta p \geq h \qquad \text{(ii)}$$

where Δp and h/λ represent the de Broglie relation, and, Δx and λ are from the diffraction limit.

The frustrated researcher seeking certainty must always make a compromise, knowledge gained about time, for instance, is paid for in uncertainty about frequency and vice versa. Though we do not notice Heisenberg Uncertainty Principle in our everyday experience with the gross macroscopic world, the wave/particle duality defeats the atomic experimentalist who seeks perfection.

Another important implication of uncertainty that is worthy of comment is its effect on causality - the relation of cause to effect. Cause produces an effect. In classical physics if we understand fully the nature of a particular cause, we could then predict the effect. Cause and effect and predictability were cornerstones of classical physics and now they were under question. If it is impossible to measure precisely both the position and velocity of an electron (or any other particle) at the same moment, then it is also impossible to predict exactly where that electron would be at any given time afterward. An experimenter could send off two electrons in the same direction, at the same speed, and they would not necessarily end up in the same place. In the language of physics, the same cause could produce different effects. There are serious philosophical consequences in this idea.

All this in effect represents “chaotic” behaviour in the quantum world. Remember, chaotic behaviour is beyond prediction; the above-mentioned describes the behaviour of quantum particles in a probabilistic, uncertain sort of way - it represents a branch of physics known as statistical mechanics. The corollary of this is the macro-world phenomenon of three-dimensional turbulent fluid flow which defies the solution of the Navier-Stokes equations. Perhaps, we should, like the case for the quantum particles, incorporate a probability function, Φ, in the Navier-Stokes equation. [Ref. 1, 5, 14 & 18]

10. Nature Of Chaos Or Turbulence

It has been found that the transition from steady state through several splittings or bifurcations to chaos is similar to phase transitions - transitions which take place when a substance changes from a gas to a liquid or a liquid to a solid - and all these transitions are also similar in that scaling is involved. It is believed that the presence of strange attractors, which could be defined as an endless path in phase space where the future depends sensitively on the initial conditions, are responsible for the presence of chaos or turbulence; a strange attractor has the following characteristics:-

i) It is generated by a simple set of differential equations.
ii) It attracts and thus all nearby trajectories in phase space converge toward it.
iii) It has a great or very sensitive dependence on initial conditions, i.e., tiny differences or errors in the initial conditions lead quickly to large differences in the trajectory.
iv) It is fractal, i.e., there is self-similarity or some familiar pattern within it.

The problem with fluid flow in conditions of turbulence is that the path taken by the fluid is continuous but nowhere differentiable. Turbulence results in whorls, eddies and ripples in the fluid. However, there is a self-similar structure or pattern within the fluid - whorls would be found within whorls - this is in accordance with the well-established self-similarity concept which has been developed by Mitchell Feigenbaum in the 1970s and which brought him fame - according to this concept there is a tendency of identical mathematical structures to recur on many levels, and, within a given structure there would be smaller copies of the same structure, their sizes being determined by the scaling factor, which is 4.669 and found to be a constant like pi (3.142). (This means there is some kind of order or pattern found in turbulence, chaos or disorder.) If we were to plot a curve to describe the fluid's movement under conditions of turbulence we could expect the curve to be rough and nonlinear (which means it is not possible to derive the differential equations for describing this curve, thus making predictions relating to the fluid's movement very difficult if not impossible). But, for viscous fluids, for which the Navier-Stokes equations are formulated, they are able to smooth out or overcome the ripples, eddies and whorls of turbulence, as mentioned above, the more viscous the fluids the more able they are in doing so. Thus, the more viscous the fluid the more successful the Navier-Stokes equations should be in describing the motion of the fluid.

If chaos or turbulence could be predicted by a mathematical equation or equations then it is not really chaos or turbulence. We have to bear in mind that chaos, as the term implies, results in disorder, having no discernable pattern, confusion and puzzlement, which is the contrary of the state of being orderly, having an obvious pattern, being deterministic and being predictable. However, whatever method we adopt for describing chaos we still need to confirm its validity through experiments, just as the validity of the Navier-Stokes equations for two-dimensional fluid motions has been confirmed by physical experiments. Therefore, we should first proceed with the physical experiments to get a better grasp of chaos in incompressible fluids. [Ref. 7, 10, 11, 13, 15, 16 & 17]

11. Relation Between Viscosity Of Fluid And Turbulence

The more viscous (sticky or gooey) the incompressible fluid is the less able it is to flow or move freely, the less runny it is, and, as mentioned above, the more able it is to overcome or smooth out the whorls, eddies and ripples of turbulence. A fluid may be so viscous and its flow so restricted that it appears almost like a solid. Such fluids that come to mind are, e.g., paints and polymer solutions. What does all this imply?

It is evident that there is a threshold or cut-off point in the viscosity of a fluid at and above which turbulence does not affect it much, which means that the effects of turbulence are only marked if a fluid has a viscosity which is less than this threshold or cut-off point. At or beyond this threshold, this cut-off point, we could expect the Navier-Stokes equations not to fare too badly.

How would the introduction of a large initial force affect a viscous fluid whose viscosity is at or above this threshold? With the introduction of a large initial force this fluid could be expected to stay "lumpy", instead of moving more freely or becoming more runny, i.e., turbulence could not be expected to make an appearance.

The contrary is true below this threshold. Below this cut-off point or threshold, i.e., if the fluid is not too viscous, with the initial force large enough, the fluid's Reynolds number stands a chance of exceeding 2000 to 3000, with turbulence setting in, and whorls, eddies and ripples forming in the fluid. The Navier-Stokes equations would face difficulties. [Ref. 6, 19, 22 & 23]

12. Solutions For Two-Dimensional Flows

Solutions of the Navier-Stokes equations result in velocity vectors, q, and pressures, p, that satisfy both the momentum equations and the continuity equation. If one were given such a combination, [q, p], one could check whether it constitutes a solution by substitution into the equations. How to find such a solution is something else. Any general step leading to this goal is helpful. For two-dimensional flows, it is possible to get rid of the continuity equation from the system of equations by using only functions that satisfy the continuity equation. This elimination is a formal step towards a solution. The functions which affect this elimination are the stream functions.

A flow can be defined as two-dimensional when its description in Cartesian coordinates shows no z-component of the velocity and no dependence on the z-coordinate. A flow like this can be described in the z = 0 plane, with the velocity vector and the streamlines lying in this plane. Moreover, the z = C planes, which are parallel to the z = 0 plane, show a flow pattern which is identical to that in the z = 0 plane. The z = 0 plane is called the representative plane. [Ref. 8, 9, 22 & 23]

13. Practical Solutions For Three-Dimensional Flows

Historically, the dynamics of chaos or turbulence is the corollary of Poincare's "three-body" problem concerning planetary motions which Poincare found to be very complex and unsolvable.

Geometrically, turbulence in fluids cannot be described by the use of the "Poincare section", i.e., there is no periodic solution for it. (In the use of the "Poincare section", for there to be a periodic solution, a circular curve must return to the section at its exact starting point. In the condition of turbulence the curve will not return to its exact starting point and there is thus no periodic solution.)

Our solutions here are to be implemented as much as possible in the spirit of the Navier-Stokes equations which pertain to the "behaviour" of an incompressible viscous fluid in three dimensions (x, y, z). We aim to overcome the obstacle faced by the Navier-Stokes equations in the three-dimensional case.

There are at least two methods of experimentation that can be carried out to get a good picture of turbulence in incompressible fluids. This "picture" is however only an "approximation", with a "factor of accuracy" - which comprises an upper limit and a lower limit. The first method involves bouncing laser beams off a reflective strip which is immersed in the turbulent viscous fluid, making use of the Doppler effect, to determine the velocity of the strip, and, hence, the velocity of the fluid. But this method only provides data obtained from one point, axis or dimension, though we could extrapolate the data for the other two dimensions to get a three-dimensional picture. The second method involves the simulation of fluid motions in conditions of turbulence with powerful computers. The fact that chaos is unpredictable implies that the result of fluid motions in turbulent conditions is anybody's guess. The guesses could vary widely in scope but each of them has a "probability" of being correct. The author would strongly recommend a statistical method of analysing fluid motions in conditions of turbulence, which is in some way similar in principle to the above-mentioned statistical mechanics. Though statistical mechanics could be used as an example for describing the behaviours of incompressible fluids under conditions of turbulence (e.g., the way it is being used to predict the behaviours and positions of quantum particles - only in a probabilistic manner), applying its methods is bound to face practical difficulties. (How do we assign "probability ratings" to the behaviours of quantum particles? What are the bases for such "ratings"? Are the "ratings" based on experimental proofs or data - which they should?) The author would like to suggest a method of analysing fluid flows under conditions of turbulence based on statistical data obtained through actual, rigorous experiments, the utilisation of statistical and interpolation/extrapolation methods, and, sound common sense or logic.

The second method, which the author strongly recommends, involves carrying out an experiment by way of computer simulation. Computer simulation is very powerful and is commonly used today. Instead of carrying out an experiment involving real fluids and real turbulence, which would be cumbersome, we would simulate this experiment with powerful computers, which would be more practicable and would cut down costs and save time considerably. (In computer simulations electrons are actually being utilised to represent the object or objects being simulated and these simulations could be carried out in three dimensions - in this case viscous, incompressible fluid in turbulence with Reynolds numbers in excess of 3000 being simulated in three dimensions as described hereafter.) As the flow of the fluid is completely turbulent when the Reynolds number exceeds about 2000 to 3000, we could via our powerful computers simulate fluid flows under conditions of turbulence for Reynolds numbers of, e.g., 3050, 3300, 3550, 3800, 4050, 4300, 4550, 4800 and 5050 respectively and thereby obtain the velocity results of fluid motion (by the process of iteration) for each of these Reynolds numbers.

For these computer simulations, we would use three powerful computers or work stations which work at very high speeds. For each of the above-mentioned nine Reynolds numbers (3050, 3300, 3550, 3800, 4050, 4300, 4550, 4800 and 5050) the three computers or work stations would carry out the following. The first computer would perform the simulation of fluid movement under conditions of turbulence and the fluid's velocity at a point, P, in the fluid would be measured in the x direction or axis (front view or plane) by a probe at this point, P, and recorded. The second computer would perform the same simulation but at a different plane or view, say, the side view (y), and the fluid's velocity at point P in the fluid would be measured in the y direction or axis (side view or plane) by a probe at point P and recorded. The third computer would perform the same simulation at a yet different plane or view, now, the top view (z), and the fluid's velocity at point P in the fluid would be measured in the z direction or axis (top view or plane) by a probe at point P and recorded. Thus, we now have the fluid's velocities (v = m/s) at each of the three planes or axes, x, y, z, for the same Reynolds number, which are as follows:-

(i) The fluid's velocity in the x axis - v (x)
(ii) The fluid's velocity in the y axis - v (y)
(iii) The fluid's velocity in the z axis - v (z)

For each of the nine Reynolds numbers, the three computers or work stations would each carry out (iterate) the simulations, say, 1 million times (the more simulations the better) to produce respectively 1 million values for v (x), 1 million values for v (y) and 1 million values for v (z).

(The principles involved in measuring the velocities of the fluid in the x, y, z directions or axes are explained here. The probes mentioned above should be made of thin material and plate-like and should offer minimal resistance against the flow of the fluid, i.e., offer minimal interference with the flow of the fluid. For each axis or plane, the flat surface of the probe should face the direction of the axis. The velocity of the fluid at each axis, plane or dimension is a function of the fluid pressure on the flat surface of the probe, the higher the fluid pressure the higher the velocity of the fluid and vice versa. The probe in each axis or plane should be able to measure both positive (+) velocities and negative (-) velocities, i.e., velocities in the opposite directions.)

We would carry out this process for all the above-mentioned nine Reynolds numbers, giving a total of 27 million simulations, and, 9 million values for v (x), 9 million values for v (y) and 9 million values for v (z).

Next, we apply the statistical method of time-series analysis (which is a method of statistical analysis designed to eliminate seasonal variations in trends, used for forecasting and prediction, but implemented with some modification here). For each of the above-mentioned nine Reynolds numbers, we compute (with the aid of the computer) the moving averages (m) for the 1 million v (x)'s, 1 million v (y)'s and 1 million v (z)'s, regardless of whether v (x), v (y) and v (z) are positive (+) or negative (-), by only adding (no subtracting) and dividing, as follows:-

(i) $v_1(x) + v_2(x) \div 2 = m_1(x)$
(ii) $v_1(x) + v_2(x) + v_3(x) \div 3 = m_2(x)$
(iii) $v_1(x) + v_2(x) + v_3(x) + v_4(x) \div 4 = m_3(x)$
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.
.
(*q) $v_1(x) + v_2(x) + v_3(x) + v_4(x) + \ldots\ldots\ldots + v_{q+1}(x) \div q + 1 = m_q(x)$

(*: q = 999,999)

whereby $m_1(x)$, $m_2(x)$, $m_3(x)$ . . . . $m_q(x)$ are the 999,999 moving averages (m) for v (x).

(The same applies to the 1 million v (y)'s and the 1 million v (z)'s.)

Thus, we have 999,999 (q) moving averages (m) each for each of the 1 million v (x)'s, 1 million v (y)'s and 1 million v (z)'s for each of the above-mentioned nine Reynolds numbers (giving a total of 26,999,973 moving averages (m) for all the above-mentioned nine Reynolds numbers).

For each of the above-mentioned nine Reynolds numbers, substituting 999,999 with q, we get the qth. moving averages (m_q) for v (x), v (y) and v (z), which are actually each respectively the 999,999th. average velocity of the fluid in each of the three dimensions or axes, x, y, z, which are as follows:-

(i) $m_q(x)$
(ii) $m_q(y)$
(iii) $m_q(z)$

On the completion of all the computer simulations and computations of the moving averages (m), we would produce a statistical table with the fluid velocities, $m_q(x)$, $m_q(y)$ and $m_q(z)$, in the three dimensions or axes, x (front view), y (side view) and z (top view), for each of the nine Reynolds numbers: 3050, 3300, 3550, 3800, 4050, 4300, 4550, 4800 and 5050. For each of the three velocities, $m_q(x)$, $m_q(y)$ and $m_q(z)$, for each of these nine Reynolds numbers, we would include an upper limit of accuracy, u (x), u (y) and u (z) (each of which is respectively the largest moving average (m) that has been obtained through carrying out the 1 million simulations for each of the three dimensions or axes, x, y, z), and, a lower limit of accuracy, l (x), l (y) and l (z) (each of which is respectively the smallest moving average (m) which has been obtained through carrying out the 1 million simulations for each of the three dimensions or axes, x, y, z). E.g., for each of these nine Reynolds numbers we would have the following velocities with their respective upper and lower limits of accuracy:-

(i) x dimension or axis - m_q (x), u (x), l (x)
(ii) y dimension or axis - m_q (y), u (y), l (y)
(iii) z dimension or axis - m_q (z), u (z), l (z)

However, for the Reynolds numbers between these nine Reynolds numbers, i.e., the "intermediate" Reynolds numbers (such as 3200, 4400 and 4950, e.g.), we would obtain the velocity results with their respective upper and lower limits of accuracy, i.e., the m_q's and their respective u's and l's, through interpolation (i.e., estimate the velocity results and their respective upper and lower limits of accuracy - some methods of interpolation are the Lagrange interpolation and the Gregory-Newton interpolation).

With the above-mentioned statistical data for the nine Reynolds numbers we would apply the rules of vector calculus to obtain for each of these nine Reynolds numbers the "resultant" velocity of the fluid in three dimensions or axes (x, y, z) and its upper and lower limits of accuracy. E.g., we could obtain the "resultant" velocity and its upper and lower limits of accuracy for unidirectional fluid flow in the three dimensions or axes (x, y, z) as follows:-

m_q (x, y, z), u (x, y, z), l (x, y, z) = + m_q (x) + m_q (y) + m_q (z); + u (x) + u (y) + u (z); + l (x) + l (y) + l (z)

where m_q (x, y, z) is the "resultant" velocity of the three moving averages m_q (x), m_q (y) and m_q (z), whereby m_q (x) would be treated as positive (+) if more than 50% of the 1 million v (x)'s are positive (+) and treated as negative (-) if more than 50% of the 1 million v (x)'s are negative (-), m_q (y) would be treated as positive (+) if more than 50% of the 1 million v (y)'s are positive (+) and treated as negative (-) if more than 50% of the 1 million v (y)'s are negative (-), and, m_q (z) would be treated as positive (+) if more than 50% of the 1 million v (z)'s are positive (+) and treated as negative (-) if more than 50% of the 1 million v (z)'s are negative (-).

Thus, when more than 50% of the 1 million v (x)'s, v (y)'s or v (z)'s are negative (-), i.e., move in the opposite direction, and their moving average (m_q) is hence treated as negative (-), we have to subtract m_q (x), m_q (y) or m_q (z), e.g., if m_q (z) is negative (-), we have to compute the "resultant" velocity m_q (x, y, - z) as follows:-

m_q (x, y, - z) = + m_q (x) + m_q (y) - m_q (z), i.e., subtract m_q (z) from + m_q (x) + m_q (y),

with its upper and lower limits of accuracy computed as follows:

(i) u (x, y, - z) = + u (x) + u (y) - u (z), where u (x) is the largest moving average (m (x)) among the 999,999 (q) moving
averages (m) for the 1 million v (x)'s, u (y) is the largest moving average (m (y)) among the 999,999 (q) moving averages
(m) for the 1 million v (y)'s, and, u (z) is the largest moving average (m (z)) among the 999,999 (q) moving averages (m) for
the 1 million v (z)'s.

(ii) l (x, y, - z) = + l (x) + l (y) - l (z), where l (x) is the smallest moving average (m (x)) among the 999,999 (q) moving
averages (m) for the 1 million v (x)'s, l (y) is the smallest moving average (m (y)) among the 999,999 (q) moving averages
(m) for the 1 million v (y)'s, and, l (z) is the smallest moving average (m (z)) among the 999,999 (q) moving averages (m)
for the 1 million v (z)'s.

"Velocity" diagrams (or vector diagrams) could be produced for all the *possible* "resultant" velocities indicated below, showing their directions of flow, which vary:-

(i) For + m_q (x) + m_q (y) + m_q (z); + u (x) + u (y) + u (z); + l (x) + l (y) + l (z), we get the "resultant" velocity m_q (x, y, z),
u (x, y, z), l (x, y, z).

(ii) For - m_q (x) - m_q (y) - m_q (z); - u (x) - u (y) - u (z); - l (x) - l (y) - l (z), we get the "resultant" velocity m_q (- x, - y, - z),
u (- x, - y, - z), l (- x, - y, - z).

(iii) For - m_q (x) - m_q (y) + m_q (z); - u (x) - u (y) + u (z); - l (x) - l (y) + l (z), we get the "resultant" velocity m_q (- x,
- y, z), u (- x, - y, z), l (- x, - y, z) (where - m_q (x) - m_q (y) > + m_q (z)).

(iv) For + m_q (x) - m_q (y) - m_q (z); + u (x) - u (y) - u (z); + l (x) - l (y) - l (z), we get the "resultant" velocity m_q (x, - y,
- z), u (x, - y, - z), l (x, - y, - z) (where - m_q (y) - m_q (z) > + m_q (x)).

(v) For - m_q (x) + m_q (y) - m_q (z); - u (x) + u (y) - u (z); - l (x) + l (y) - l (z), we get the "resultant" velocity m_q (- x, y,
- z), u (- x, y, - z), l (- x, y, - z) (where - m_q (x) - m_q (z) > + m_q (y)).

(vi) For - m_q (x) + m_q (y) + m_q (z); - u (x) + u (y) + u (z); - l (x) + l (y) + l (z), we get the "resultant" velocity m_q (- x,
y, z), u (- x, y, z), l (- x, y, z) (where + m_q (y) + m_q (z) > - m_q (x)).

(vii) For + m_q (x) - m_q (y) + m_q (z); + u (x) - u (y) + u (z); + l (x) - l (y) + l (z), we get the "resultant" velocity m_q (x,
- y, z), u (x, - y, z), l (x, - y, z) (where + m_q (x) + m_q (z) > - m_q (y)).

(viii) For + m_q (x) + m_q (y) - m_q (z); + u (x) + u (y) - u (z); + l (x) + l (y) - l (z), we get the "resultant" velocity m_q (x,
y, - z), u (x, y, - z), l (x, y, - z) (where + m_q (x) + m_q (y) > - m_q (z)).

So far, in the above "velocity" scenarios, for those with either, one negative velocity (- m_q) and two positive velocities (+ m_q), or, one positive velocity (+ m_q) and two negative velocities (- m_q), we have assumed that for the former the two positive velocities (+ m_q) combined together are larger than the one negative velocity (- m_q), and, for the latter we have assumed that the two negative velocities (- m_q) combined together are larger than the one positive velocity (+ m_q). Now, what happens when for the former, the two positive velocities (+ m_q) combined together are equal to the one negative velocity (- m_q), and, for the latter, the two negative velocities (- m_q) combined together are equal to the one positive velocity (+ m_q)? In each of these cases, the "resultant" velocity (m_q (x, y, z), u (x, y, z), l (x, y, z)) would be equal to zero, i.e., there would be no "resultant" velocity.

Next, what happens when the negative velocity, - m_q (x), - m_q (y), or, - m_q (z), is larger than the other two positive velocities combined together, e.g., - m_q (x) > + m_q (y) + m_q (z), - m_q (y) > + m_q (x) + m_q (z), or, - m_q (z) > + m_q (x) + m_q (y)? For such cases we have the following "resultant" velocities, whose directions of flow differ from each other:-

(i) For - m_q (x) + m_q (y) + m_q (z); - u (x) + u (y) + u (z); - l (x) + l (y) + l (z), where - m_q (x) > + m_q (y) + m_q (z), we
get the "resultant" velocity m_q (- x, y, z), u (- x, y, z), l (- x, y, z).

(ii) For + m_q (x) - m_q (y) + m_q (z); + u (x) - u (y) + u (z); + l (x) - l (y) + l (z), where - m_q (y) > + m_q (x) + m_q (z), we
get the "resultant" velocity m_q (x, - y, z), u (x, - y, z), l (x, - y, z).

(iii) For + m_q (x) + m_q (y) - m_q (z); + u (x) + u (y) - u (z); + l (x) + l (y) - l (z), where - m_q (z) > + m_q (x) + m_q (y), we
get the "resultant" velocity m_q (x, y, - z), u (x, y, - z), l (x, y, - z).

What happens then when the positive velocity, + m_q (x), + m_q (y), or, + m_q (z) is larger than the other two negative velocities combined together, e.g., + m_q (x) > - m_q (y) - m_q (z), + m_q (y) > - m_q (x) - m_q (z), or, + m_q (z) > - m_q (x) - m_q (y)? For such cases we have the following "resultant" velocities, whose directions of flow differ from each other:-

(i) For + m_q (x) - m_q (y) - m_q (z); + u (x) - u (y) - u (z); + l (x) - l (y) - l (z), where + m_q (x) > - m_q (y) - m_q (z), we get the "resultant" velocity m_q (x, - y, - z), u (x, - y, - z), l (x, - y, - z).

(ii) For - m_q (x) + m_q (y) - m_q (z); - u (x) + u (y) - u (z); - l (x) + l (y) - l (z), where + m_q (y) > - m_q (x) - m_q (z), we
get the "resultant" velocity m_q (- x, y, - z), u (- x, y, - z), l (- x, y, - z).

(iii) For - m_q (x) - m_q (y) + m_q (z); - u (x) - u (y) + u (z); - l (x) - l (y) + l (z), where + m_q (z) > - m_q (x) - m_q (y), we
get the "resultant" velocity m_q (- x, - y, z), u (- x, - y, z), l (- x, - y, z).

For the three velocities in the x, y, z dimensions or axes, i.e., m_q (x), m_q (y) and m_q (z), there are 14 possible "resultant" velocities, excluding the null "resultant" velocities, of which there are six possible cases (which are not expected to be likely to occur in turbulence), as shown above. Each of these "resultant" velocities would have an upper limit of accuracy (u (x, y, z)) and a lower limit of accuracy (l (x, y, z)). For fluid velocities (m_q (x), m_q (y), or, m_q (z)) in the opposite direction their values are negative. We could expect turbulence to be characterized by any of these 14 "resultant" velocities.

However, there is a possibility that one or more of the velocities, m_q (x), m_q (y) and m_q (z), might be equal to zero, though the chances of this occurring in turbulence might be remote:-

(i) If, e.g., m_q (z) = 0, then m_q (x, y, z = 0), u (x, y, z = 0), l (x, y, z = 0) would be a "resultant" velocity in two
dimensions or axes only, with the fluid moving in the two dimensions or axes, x and y, only.

(ii) For m_q (x) = 0, we get the "resultant" velocity m_q (x = 0, y, z), u (x = 0, y, z), l (x = 0, y, z).

(iii) For m_q (y) = 0, we get the "resultant" velocity m_q (x, y = 0, z), u (x, y = 0, z), l (x, y = 0, z).

As in the three-dimensional case above, there is the possibility in the two-dimensional case that one or both of the two velocities are negative or positive velocities. If, in the case whereby one of the two velocities is negative while the other is positive, the negative velocity is equal to the positive velocity (of which there are six possible cases), then the "resultant" velocity is null (zero). The following are the possible "resultant" velocities:-

(i) For m_q (z) = 0, we get the "resultant" velocity m_q (x, - y, z = 0), u (x, - y, z = 0), l (x, - y, z = 0), when m_q (x) is
positive while m_q (y) is negative, whereby - m_q (y) > + m_q (x).

(ii) For m_q (z) = 0, we get the "resultant" velocity m_q (x, - y, z = 0), u (x, - y, z = 0), l (x, - y, z = 0), when m_q (x) is
positive while m_q (y) is negative, whereby + m_q (x) > - m_q (y).

(iii) For m_q (z) = 0, we get the "resultant" velocity m_q (- x, y, z = 0), u (- x, y, z = 0), l (- x, y, z = 0), when m_q (y) is positive while m_q (x) is negative, whereby - m_q (x) > + m_q (y).

(iv) For m_q (z) = 0, we get the "resultant" velocity m_q (- x, y, z = 0), u (- x, y, z = 0), l (- x, y, z = 0), when m_q (y) is positive while m_q (x) is negative, whereby + m_q (y) > - m_q (x).

(v) For m_q (z) = 0, we get the "resultant" velocity m_q (- x, - y, z = 0), u (- x, - y, z = 0), l (- x, - y, z = 0), when m_q (x) and m_q (y) are both negative.

(vi) For m_q (z) = 0, we get the "resultant" velocity m_q (x, y, z = 0), u (x, y, z = 0), l (x, y, z = 0), when m_q (x) and m_q (y) are both positive.

(vii) For m_q (x) = 0, we get the "resultant" velocity m_q (x = 0, - y, z), u (x = 0, - y, z), l (x = 0, - y, z), when m_q (z) is positive while m_q (y) is negative, whereby - m_q (y) > + m_q (z).

(viii) For m_q (x) = 0, we get the "resultant" velocity m_q (x = 0, - y, z), u (x = 0, - y, z), l (x = 0, - y, z), when m_q (z) is positive while m_q (y) is negative, whereby + m_q (z) > - m_q (y).

(ix) For m_q (x) = 0, we get the "resultant" velocity m_q (x = 0, y, - z), u (x = 0, y, - z), l (x = 0, y, - z), when m_q (y) is positive while m_q (z) is negative, whereby - m_q (z) > + m_q (y).

(x) For m_q (x) = 0, we get the "resultant" velocity m_q (x = 0, y, - z), u (x = 0, y, - z), l (x = 0, y, - z), when m_q (y) is positive while m_q (z) is negative, whereby + m_q (y) > - m_q (z).

(xi) For m_q (x) = 0, we get the "resultant" velocity m_q (x = 0, - y, - z), u (x = 0, - y, - z), l (x = 0, - y, - z), when m_q (y) and m_q (z) are both negative.

(xii) For m_q (x) = 0, we get the "resultant" velocity m_q (x = 0, y, z), u (x = 0, y, z), l (x = 0, y, z), when m_q (y) and m_q (z) are both positive.

(xiii) For m_q (y) = 0, we get the "resultant" velocity m_q (- x, y = 0, z), u (- x, y = 0, z), l (- x, y = 0, z), when m_q (z) is positive while m_q (x) is negative, whereby - m_q (x) > + m_q (z).

(xiv) For m_q (y) = 0, we get the "resultant" velocity m_q (- x, y = 0, z), u (- x, y = 0, z), l (- x, y = 0, z), when m_q (z) is
positive while m_q (x) is negative, whereby + m_q (z) > - m_q (x).

(xv) For m_q (y) = 0, we get the "resultant" velocity m_q (x, y = 0, - z), u (x, y = 0, - z), l (x, y = 0, - z), when m_q (x) is
positive while m_q (z) is negative, whereby - m_q (z) > + m_q (x).

(xvi) For m_q (y) = 0, we get the "resultant" velocity m_q (x, y = 0, - z), u (x, y = 0, - z), l (x, y = 0, - z), when m_q (x) is
positive while m_q (z) is negative, whereby + m_q (x) > - m_q (z).

(xvii) For m_q (y) = 0, we get the "resultant" velocity m_q (- x, - y = 0, z), u (- x, - y = 0, z), l (- x, - y = 0, z), when
m_q (x) and m_q (z) are both negative.

(xviii) For m_q (y) = 0, we get the "resultant" velocity m_q (x, y = 0, z), u (x, y = 0, z), l (x, y = 0, z), when m_q (x) and
m_q (z) are both positive.

(xix) For m_q (x) = 0 and m_q (y) = 0, we get the "resultant" velocity m_q (x = 0, y = 0, z), u (x = 0, y = 0, z), l (x = 0,
y = 0, z) in one dimension or axis, which is positive.

(xx) For m_q (x) = 0 and m_q (y) = 0, we get the "resultant" velocity m_q (x = 0, y = 0, - z), u (x = 0, y = 0, - z), l (x = 0,
y = 0, - z) in one dimension or axis, which is negative.

(xxi) For m_q (x) = 0 and m_q (z) = 0, we get the "resultant" velocity m_q (x = 0, y, z = 0), u (x = 0, y, z = 0), l (x = 0, y,
z = 0) in one dimension or axis, which is positive.

(xxii) For m_q (x) = 0 and m_q (z) = 0, we get the "resultant" velocity m_q (x = 0, - y, z = 0), u (x = 0, - y, z = 0), l (x =
0, - y, z = 0) in one dimension or axis, which is negative.

(xxiii) For m_q (y) = 0 and m_q (z) = 0, we get the "resultant" velocity m_q (x, y = 0, z = 0), u (x, y = 0, z = 0), l (x, y =
0, z = 0) in one dimension or axis, which is positive.

(xxiv) For m_q (y) = 0 and m_q (z) = 0, we get the "resultant" velocity m_q (- x, y = 0, z = 0), u (- x, y = 0, z = 0), l (- x, y = 0, z = 0) in one dimension or axis, which is negative.

After all the simulations and computations of moving averages (m) for the 27 million fluid velocities (v) in the x, y, z dimensions or axes, we firstly obtain a table, which would act as a rough statistical guide for fluid flow under conditions of turbulence, with the following statistical data, for each of the nine Reynolds numbers (3050, 3300, 3550, 3800, 4050, 4300, 4550, 4800 and 5050):-

(1) (i) The fluid's average velocity in the x dimension or axis - m_q (x)
(ii) This fluid average velocity's upper limit of accuracy - u (x) (i.e., largest moving average (m) for v (x))
(iii) This fluid average velocity's lower limit of accuracy - l (x) (i.e., smallest moving average (m) for v (x))

(2) (i) The fluid's average velocity in the y dimension or axis - m_q (y)
(ii) This fluid average velocity's upper limit of accuracy - u (y) (i.e., largest moving average (m) for v (y))
(iii) This fluid average velocity's lower limit of accuracy - l (y) (i.e., smallest moving average (m) for v (y))

(3) (i) The fluid's average velocity in the z dimension or axis - m_q (z)
(ii) This fluid average velocity's upper limit of accuracy - u (z) (i.e., largest moving average (m) for v (z))
(iii) This fluid average velocity's lower limit of accuracy - l (z) (i.e., smallest moving average (m) for v (z))
(4) (i) The fluid's "resultant" velocity in the x, y, z dimensions or axes - m_q (x, y, z) (i.e., +/- m_q (x) +/- m_q (y) +/- m_q (z))
(ii) This fluid "resultant" velocity's upper limit of accuracy - u (x, y, z) (i.e., +/- m (x) +/- m (y) +/- m (z))
(iii) This fluid "resultant" velocity's lower limit of accuracy - l (x, y, z) (i.e., +/- m (x) +/- m (y) +/- m (z))

We should also include the "velocity" diagrams depicting the various "resultant" velocities and the directions of flow, a total of 27 diagrams, three "velocity" diagrams for each of the nine Reynolds numbers - one for m_q (x, y, z), one for u (x, y, z) and one for l (x, y, z). We may title this table "Statistical Data Of An Incompressible Fluid's Velocities For The Nine Reynolds Numbers: 3050, 3300, 3550, 3800, 4050, 4300, 4550, 4800 And 5050".

For the "intermediate" Reynolds numbers, e.g., 3200, 4400 and 4950, we would interpolate the fluid's respective velocities and their upper and lower limits of accuracy with the above-mentioned statistical table (Statistical Data Of An Incompressible Fluid's Velocities For The Nine Reynolds Numbers: 3050, 3300, 3550, 3800, 4050, 4300, 4550, 4800 And 5050) and the complementary table (Velocity Results Of The 27 Million Simulations In The X, Y, Z Dimensions) described below as guides, i.e., estimate them - some methods of interpolation that could be adopted are the Lagrange interpolation and the Gregory-Newton interpolation. For velocities for Reynolds numbers outside the range of the above nine Reynolds numbers (3050 to 5050), e.g., for Reynolds numbers 2500 and 5200, we would extrapolate them, including their upper and lower limits of accuracy, using the same tables as guides.

With the above techniques (and with proper simulations) we therefore have a statistical table of fluid flow velocity results (Statistical Data Of An Incompressible Fluid's Velocities For The Nine Reynolds Numbers: 3050, 3300, 3550, 3800, 4050, 4300, 4550, 4800 And 5050) under conditions of turbulence (with their respective ranges of possibilities or probabilities, i.e., their respective upper limits and lower limits of accuracy) for the various Reynolds numbers which are more or less rigorously based on actual experiments (simulations in this case), a table which is somewhat like, e.g., the statistical tables for t-distribution and chi-squared distribution; in the above-mentioned experiments (or simulations) we could expect more accurate results with more simulations being carried out (more data being collected), the more the simulations carried out the more accurate the results are likely to be. This represents a practical, logical way of roughly approximating the velocities of fluid motions under conditions of turbulence (in three dimensions) at the various Reynolds numbers. Given a fluid velocity (m_q (x, y, z), u (x, y, z), l (x, y, z)) from the above-mentioned statistical table we could in principle tell (or predict) for a particular Reynolds number the position (position after distance travelled, d_2, relative to the position at $d_1 = 0$) of an object, e.g., an aluminium strip, which could be taken to represent a section of the fluid, carried along by the fluid in motion, at a point of time, $t_2 > 0$, with the initial time being $t_1 = 0$, or, the distance ({$d_2 > 0$} - {$d_1 = 0$}) this object travelled at a point of time, $t_2 > 0$, with the initial time being $t_1 = 0$, by computation with the equation: $d_2 - d_1$ = (m_q (x, y, z), u (x, y, z), l (x, y, z)) x ($t_2 - t_1$). However, in this instance, wherein turbulence rules, this prediction of position or distance is not expected to be accurate (but should be regarded only as a rough approximation) and should be subject to the statistical rule of "probability" - in this case the upper limit ({u (x, y, z)} x {$t_2 - t_1$}) and the lower limit ({l (x, y, z)} x {$t_2 - t_1$}) of the accuracy of the result obtained, as indicated by the above-mentioned statistical table. The "velocity" diagrams in the statistical table would act as a guide with regards to the positioning of the object and the direction it would travel. Here the movement of the object, which is the aluminium strip, in effect represents the movement of the fluid which carries it along. However, the data reflected in this first statistical table (Statistical Data Of An Incompressible Fluid's Velocities For The Nine Reynolds Numbers: 3050, 3300, 3550, 3800, 4050, 4300, 4550, 4800 And 5050) should be regarded only as a rough approximation. With the further assistance of the complementary table (Velocity Results Of The 27 Million Simulations In The X, Y, Z Dimensions) described below this approximation could be refined.

With the data from the above-mentioned statistical table (Statistical Data Of An Incompressible Fluid's Velocities For The Nine Reynolds Numbers: 3050, 3300, 3550, 3800, 4050, 4300, 4550, 4800 And 5050) we could plot a curve for the fluid flow velocity results (m_q (x, y, z)) for the above-mentioned nine Reynolds numbers. If this curve is smooth and linear, which is unlikely, from the gradient of the slope of this curve we could derive the differential equations for this curve, whereby forecasts or predictions would be possible. However, if this curve is rough and nonlinear, which is likely to be the case, then these differential equations would not be obtainable, and, we would have to rely on the above-mentioned table of fluid flow velocity results (Statistical Data Of An Incompressible Fluid's Velocities For The Nine Reynolds Numbers: 3050, 3300, 3550, 3800, 4050, 4300, 4550, 4800 And 5050), and, the complementary table (Velocity Results Of The 27 Million Simulations In The X, Y, Z Dimensions) described below, as statistical guides for our approximation (including interpolation and extrapolation) of velocity results for fluid flows at the various Reynolds numbers. Both these statistical tables with their "velocity" diagrams should complement one another and should be a great help for this approximation process, the more copious the data available there the more effective the approximation should be. We should hence be able to approximate not only fluid velocities but directions of fluid motions as well (with the aid of the "velocity" diagrams). We could continue to plot graphs or curves with these statistical data and attempt to interpolate or extrapolate with them, looking out for trends or patterns, and arrive at some forecasts or predictions, which is expected to be a difficult task; by doing so we could at least get a "feel" or intuitive understanding of the whole situation, which should make our job of forecasting or predicting the outcome easier, an evidently challenging undertaking.

The above-mentioned statistical table, Statistical Data Of An Incompressible Fluid's Velocities For The Nine Reynolds Numbers: 3050, 3300, 3550, 3800, 4050, 4300, 4550, 4800 And 5050, presents the average velocities for the nine Reynolds numbers; these average velocities each represents the velocity "trend" for each of the nine Reynolds numbers. But carrying out the approximation for the respective velocities for the respective Reynolds numbers, wherein complete turbulence is involved, might not be that easy. We have to remember that the Navier-Stokes equations fare very badly when turbulence sets in, when a viscous, incompressible fluid behaves in an unpredictable, irregular, chaotic, and, "nonlinear" manner. To complement the first statistical table (Statistical Data Of An Incompressible Fluid's Velocities For The Nine Reynolds Numbers: 3050, 3300, 3550, 3800, 4050, 4300, 4550, 4800 And 5050), the author would like to suggest a proper tabulation of all the 27 million fluid velocities (v) in the x, y, z dimensions or axes obtained through the simulations in another table (or booklet), which may be titled "Velocity Results Of The 27 Million Simulations In The X, Y, Z Dimensions," which shows whole ranges of velocities (minimum velocity, maximum velocity, and, all the velocities between the minimum and the maximum, a total of 27 million velocities (v's), three million velocities (v's) for each of the nine Reynolds numbers), with their order of listing in this table (or booklet) exactly the same as the order they appeared in during the simulations, with indications whether they are positive (+) or negative (-) velocities, and, the 999,999 (q) moving averages (m) each for each of the 1 million v (x)'s, 1 million v (y)'s and 1 million v (z)'s for each of the nine Reynolds numbers (giving a total of 26,999,973 moving averages (m) for all the nine Reynolds numbers) included; the "velocity" diagrams described above, which would also be helpful as guides, should be incorporated, e.g., as an appendix, in this table (or booklet). With this additional table and its "velocity" diagrams acting as a further guide the task of approximating (including interpolating and extrapolating) the velocities for the various Reynolds numbers (for turbulence) would be made easier. With these two tables there are now more or less solid and realistic data to carry out the approximation (including interpolation and extrapolation) - for both fluid velocities, and, directions of fluid motions; the approximations could also include probabilities of occurrence (an example of which is presented below). For the approximation (including interpolation and extrapolation), however, sound common sense, good intuition and a sharp eye for patterns and details are important. Since turbulence is a much complex phenomenon the approximation (including interpolation and extrapolation) should be carried out with great care and patience. We have two choices here now. We could supplant the approximation method of the Navier-Stokes equations with the above-mentioned statistical method. Or, we could use this statistical method as a complement to the Navier-Stokes equations, whereby we might have the "best of both worlds". [Ref. 1, 12, 15, 20, 21, 22 & 23]

14. Conclusion

There is a possibility, however remote, that turbulence or chaos when viewed en masse, on a very large scale, would appear to be smooth, present some sort of pattern or appear to have some order. According to the precepts of fractal geometry, a relatively new branch of mathematics pioneered by Benoit Mandelbrot, phenomena which appear random, when viewed en masse, display some orderliness and pattern, which could be termed "fractal". Therefore, a new mathematical technique for describing the flow of an incompressible viscous fluid in three dimensions, i.e., in turbulence, a probably statistical one involving large samples of data, like the method described just above, is a logical step.

As for the case of computer simulation it is becoming more and more popular and simulation has been used in the physical sciences, as well as the engineering sciences, e.g., in aeronautical design, electronic circuit design and mechanical design. The author himself has considerable experience with computer simulation. Simulation has generally proven to be cost-saving, time-saving and effective. In engineering applications, e.g., simulation has made it unnecessary to produce or manufacture the prototypes for any new product designs for the purpose of feasibility studies (which could be costly affairs) - the feasibility studies are now carried out directly through the simulation exercises, whereby it is possible to quickly find out whether the designs would work or not. The only serious obstacle to simulation appears to be the cost of the simulation software itself, which could come up to many thousands of dollars, so that a cost/benefit analysis for using the software is necessary - the benefits and cost-savings for using the software should outweigh the cost of the software itself in order for its use to be justifiable. The other problem might be a technical one; some simulation software are complex and not user-friendly, requiring a long learning-curve, and these are usually the more powerful software with more built-in features - however, once these software have been mastered their use would bring about relatively greater and more effective results; some might feel that such powerful, and hence more expensive, software are an overkill and prefer to go for something at the lower end which is also cheaper. Such software usually have powerful features such as allowing us to have "walk-through" views and "inside-out" views of an object which would be physically impossible otherwise - such are the great powers of simulation software now. The simulation of turbulence with powerful software and computers would certainly prove to be very useful.

As nonlinear equations such as the Navier-Stokes equations have to rely on approximation and exact solutions are highly unlikely, especially for turbulent fluid movement, a statistical guide based on actual data collected such as the guides or tables described just above is definitely a boon. A statistical guide which is based on reliable data collected and which has been put to the test and fine-tuned would be in a better position to lead us to more accurate approximations than the Navier-Stokes equations.

From the data in the above-mentioned statistical tables it is now possible, e.g., to make approximations or estimates of various fluid velocities (as well as the directions of various fluid motions) with various probabilities of occurrence for the various Reynolds numbers, such as the following:-

1) x metres/second (accompanied by the appropriate "velocity" diagram): a percent probability of occurrence
2) y metres/second (accompanied by the appropriate "velocity" diagram): b percent probability of occurrence
3) z metres/second (accompanied by the appropriate "velocity" diagram): c percent probability of occurrence

Etc.

The Navier-Stokes equations do not have any allowance for such probabilistic approximations. These partial differential equations have been found to be solvable for the two-dimensional case, i.e., for each of the equations a function (or, solution) could be found that, when substituted for the dependent variable in the equation, leads to an identity. But for three dimensions finding such functions (or, solutions) has been a problem. In other words, though the formulas could be found to describe a two-dimensional fluid motion, such formulas are not available or obtainable for the three-dimensional case, e.g., the case for turbulent fluid motion. Evidently, the geometry of a three-dimensional fluid motion is rather complex, and, finding the formulas to describe this complex three-dimensional movement of fluid is a tough problem. To understand this difficulty consider the motion of a speck of particle carried along in a flowing fluid; the speck could be thrust first in one direction, then another, and another, and so on, sometimes moving in a fairly straight line, other times spiralling around as the current takes it along; this movement of the speck is also the movement of the fluid that carries it along; it is a three-dimensional movement that appears chaotic or turbulent - it is rather complicated and, hence, not amenable to a description by some formulas. The Navier-Stokes equation makes use of differentiation to obtain the rate of change of some changing quantity, and, in order to do this the value or position or path of that quantity has to be given by an appropriate formula; differentiation then acts upon this formula to produce another formula which gives the rate of change. Since the formulas for the three-dimensional case, e.g., turbulent fluid motion, are unobtainable the only recourse is approximation. Is there any hope of discovering these formulas in the future? The author is much pessimistic. Let us look again at the case of turbulent fluid motion, an essentially three-dimensional phenomenon. As mentioned earlier, a curve describing a fluid's movement under conditions of turbulence could be expected to be rough and nonlinear thus making it not possible to derive the differential equations (or formulas) for describing this curve, making predictions relating to the fluid's movement very difficult if not impossible. Moreover, turbulence or chaos implies disorder, irregularity, lack of discernable pattern, confusion and puzzlement. This implies that turbulence could never be described by formulas or differential equations, and, to be able to obtain the formulas or differential equations for turbulence and thus be able to make predictions relating to turbulence would mean that the so-called turbulence is not really turbulence at all, a contradiction. (Do the editors of our dictionaries then have to revise the meanings for turbulence and chaos? According to the Encyclopedic World Dictionary published by Paul Hamlyn, "turbulence" could be defined as "the haphazard secondary motion due to eddies within a moving fluid", or, "irregular motion of the atmosphere, as that indicated by gusts and lulls in the wind", "turbulent flow" is "fluid flow in which the motion at any point varies rapidly in magnitude and direction", and, "chaos" could be defined as "utter confusion or disorder, wholly without organization or order", or, "the infinity of space or formless matter supposed to have preceded the existence of the ordered universe".) It is therefore absurd for us to expect the Navier-Stokes equations to have the solution for a three-dimensional phenomenon such as turbulent or chaotic fluid motion. It is hence appropriate to have another mathematical technique to deal with this difficult situation instead, viz., the statistical method, taking into consideration the example of statistical mechanics in quantum theory; as the outcomes of turbulence or chaos could never be predicted with certainty and the predictions might not even be highly accurate it is logical and practical to consider the possible outcomes of this phenomenon in a statistical or probabilistic way. Much attempts have already been carried out to understand turbulence or chaos, which is now a hot subject. The author thinks that even if we understand the causes and mechanics or physics of turbulence or chaos it would be naïve to believe it is possible to obtain the formulas for describing such a phenomenon, for, to say that we know how an object that causes confusion and puzzlement (a chaotic object) would behave is to say that we are not confused and puzzled by this object, a self-contradiction. As described above, the logistic equation has been a popular model for chaos but, here again, the author thinks that it would be naïve, and, self-contradicting, to believe that the logistic equation is a sufficient formula for making predictions relating to chaos. Some others might state that only when we could not predict the outcomes of a phenomenon could we regard that phenomenon as chaotic but once we could get those predictions the phenomenon is no more chaotic, which would imply that chaos is transitional and subjective. These are possibly the ones who believe that there is a solution for the Navier-Stokes equations in the three-dimensional case. Simply put, if chaos is predictable it is not chaos, only when it is really unpredictable could it be chaos. Mathematically, and, objectively speaking too, it has never been found possible to derive the differential equations to describe a nonlinear phenomenon such as turbulence, as there is no regular pattern found in turbulence (which is fluid flow in which the motion at any point varies rapidly in magnitude and direction), and, mathematics, which, in a sense, is the study and analysis of patterns, simply found nothing possible to study or analyse in turbulence since it does not display any discernable, set pattern or regularity, except for the presence of eddies, ripples and whorls, which in the terms of fractal geometry could be described as a fractal characteristic. The only plausible solution appears to be a statistical one, wherein there is some hope of discovering some meaningful patterns or orderly features when large samples of data are analysed. (According to the precepts of fractal geometry, phenomena which appear random when viewed en masse display some orderliness and pattern which could be regarded as a fractal characteristic.) E.g., the prime numbers are very random and haphazard entities, yet, when viewed en masse they display a regularity in the way they thin out, whereby it is affirmed that the number of primes not exceeding a given natural number n is approximately $n/\ln n$, in the sense that the ratio of the number of such primes to $n/\ln n$ eventually approaches 1 as n becomes larger and larger, $\ln n$ being the natural logarithm (to the base e) of n (vide the prime number theorem proved in 1896 by Hadamard and Vallee-Poussin).

With statistical methods, such as that described above, we could now evaluate turbulence on a probabilistic basis, which is a more practical and realistic way of looking at turbulence, whose outcomes are uncertain, irregular, haphazard and very difficult if not impossible to predict. As the Navier-Stokes equations fare very poorly in the three-dimensional case, i.e., in the case of turbulence, a new mathematical technique for making approximations for the three-dimensional case, such as the statistical one described above, should be given a chance to take over, to supplant the Navier-Stokes equations, or, at least, to complement them.

After all, the results of the differential equations, such as the Navier-Stokes equations, would still have to be confirmed by actual physical data, or, physical experiment. The more direct, faster or more efficient method of understanding and making forecasts pertaining to the various levels of turbulence or chaos is evidently to execute a well-planned computer simulation exercise (or, at least, a well-planned physical experiment) and apply the proper statistical technique in the interpretation of the data which are obtained through the computer simulation exercise; such a procedure has been described in detail above.

We should be realistic and refrain from expecting certainly true or almost certainly true forecasts from our "mathematisation" of turbulence or chaos. It is more reasonable or realistic to expect forecasts with only some degree of probability of being true, and, where the forecasts do indeed turn out to be totally true on occasions they should be regarded as rare phenomena not unlike the cases of some fortunate persons hitting the lotteries. In fact, if the day ever arrives when an equation is available for accurately forecasting the outcome of turbulence or chaos, turbulence or chaos would be obsolete. Evidently, with the lack of a discernable pattern or patterns in turbulence or chaos, such an equation would never be found. [Ref. 7, 10, 11, 13, 15, 16 & 17]

APPENDIX

The Self-Similarity Concept And Fractal Geometry

The formulation of the self-similarity concept has brought fame to Mitchell Feigenbaum, who has worked in the Los Alamos Laboratory in the early 1970s. This concept, upon which the method of renormalization in perturbation theory is based, postulates that there is a tendency of identical mathematical structures to recur on many levels. Within a given structure, there would be smaller copies of the same structure, their sizes being determined by the scaling factor. Feigenbaum found that at the utmost tips of the fig-tree, there is some mathematical structure which remains the same when its size is changed (enlarged) by a scaling factor of 4.669, which is found to be a constant like pi (3.142); this structure is the shape of the fig-tree itself; in other words, little whorls could be found within big whorls. Renormalization has been a well-established technique in chaos theory/fractal geometry and is a mathematical trick which functions rather like a microscope, zooming in on the self-similar structure, removing any approximations, and filtering out everything else. All this shows the universality of some features of chaos. I.e., some kind of order or pattern could be found in or is inherent in disorder, turbulence or chaos.

In Feigenbaum's famous fig-tree example, for instance, there is a self-similar mathematical pattern or structure (which is the shape of the fig-tree itself) in the various parts of the fig-tree, i.e., its trunk to bough section, bough to branch section, branch to twig section and twig to twiglet section. Such self-similar mathematical pattern or structure, or, fractal characteristic, could also be found in other aspects of nature, e.g., waves, turbulence or chaos, the structures of viruses and bacteria, polymers and ceramic materials, the universe and many others, even the movements of prices in financial markets, the growths of populations, the sound of music, the flow of blood through our circulatory system, the behaviour of people en masse, etc., which have all spawned a relatively new and important branch of mathematics with wide practical applications known as fractal geometry, which has been pioneered by Benoit Mandelbrot. As a matter of fact, self-similarity or fractal characteristic could be regarded as the fundamental mathematical aspect found in practically everything in nature, and, this new branch of mathematics, fractal geometry, besides having a great practical impact on us also gives us a deeper vision of the universe in which we live and our place in it. [Ref. 7, 10 & 11]

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