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Bayesian Estimation of Generalized Auto Regressive Conditionally Heteroscedastic Model with an Application to Foolad Mobarakeh Stock Returns

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Abstract

Problems in economics and finance have recently motivated the study of the volatility of a time series data setting. Several time series models to concern the volatility of such data have been considered. Although the Auto Regressive Moving Average (ARMA) models assume a constant variance, models such as the Auto Regressive Conditionally Heteroscedastic (ARCH) models are developed to the model changes in volatility. In this paper, we indicate that the generalized ARCH (GARCH) models which have been proposed are useful in many economics and financial studies. We thus develop both probabilistic properties and the Bayesian estimation method of a GARCH (1, 1) model. We then illustrate the model on Foolad Mobarakeh (F.M) daily returns from 2007 to 2012. Further we forecast future values of conditional variance of returns.

Keywords: ARCH, GARCH, Heteroscedastic, Volatility, Metropolis-Hasting algorithm.

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1. Introduction

Financial markets react nervously to political disorders, economic crises, wars and natural disasters. In such stress periods prices of financial assets tend to fluctuate very much. That is, it means the conditional variance for the given past,

$$\mathrm{var}(y_t | y_{t-1}, y_{t-2}, \dots),$$

is not constant over each time t and the process y_t is conditionally heteroskedastic [3]. Econometricians usually postulate that volatility,

$$\sigma_t = \sqrt{\mathrm{var}(y|y_{t-1}, y_{t-2}, \dots)},$$

changes over each time t. The model incorporating the possibility of a non-constant variance is often called heteroscedastic model. Understanding the nature of such time dependence is very important for many macroeconomic and financial applications. Models of conditional heteroskedasticity for time series have a very important role in today's financial risk management. it attempts to make financial decisions on the basis of the observed price.

The ARCH model is introduced by Engle in 1982 [6]. They were later extended to GARCH models by Bollerslev in 1986 [4]. There are many very good surveys covering the mathematical and statistical properties of GARCH models; see, for example, Shumway et al. [13], Silvennoinen [14] and Terasvirta [16]. There are also several comprehensive surveys that focus on the forecasting performance of GARCH models including Andersen et al. [2], Poon et al. [11] and Poon et al. [12]. Until recently GARCH models have mainly been estimated using the classical Maximum Likelihood technique. Moreover several R packages provide functions for their estimation; see, e.g. fGarch (Wuertz et al.) [17], rgarch (Ghalanos) [8] and tseries (Trapletti et al.) [15].

The Bayesian approach offers an attractive alternative which enables small sample results, robust estimation, model discrimination, model combination, and probabilistic statements on functions of the model parameters. The approach, based on the work of Nakatsuma [10], consists of a Metropolis-Hastings (M-H) algorithm where the proposal distributions are constructed from auxiliary ARMA processes on the squared observations. The M-H algorithm is a simulation scheme which allows to generate draws from any density of interest whose normalizing constant is unknown. The algorithm consists of the following steps.

1. Initialize the iteration counter to $j = 1$ and set an initial value $\boldsymbol{\Theta}^{[0]}$;
2. Move the chain to a new value $\boldsymbol{\Theta}^{*}$ generated from a proposal (candidate) density $q(\cdot\,|\boldsymbol{\Theta}^{[j-1]})$;
3. Evaluate the acceptance probability of the move from $\boldsymbol{\Theta}^{[j-1]}$ to $\boldsymbol{\Theta}^{*}$ given by:
$$\min\left\{\frac{p(\boldsymbol{\Theta}^{*}|y)q(\boldsymbol{\Theta}^{[j-1]}|\boldsymbol{\Theta}^{*})}{p(\boldsymbol{\Theta}^{[j-1]}|y)q(\boldsymbol{\Theta}^{*}|\boldsymbol{\Theta}^{j-1}}, 1\right\}.$$
If the move is accepted, set $\boldsymbol{\Theta}^{[j]} = \boldsymbol{\Theta}^{*}$, if not, set $\boldsymbol{\Theta}^{[j]} = \boldsymbol{\Theta}^{[j-1]}$ so that the chain does not move;
4. Change counter from j to $j + 1$ and go back to step 2 until convergence is reached.

The program is written in R with some subroutines implemented in C in order to speed up the simulation procedure.

In this paper, we indicate that the generalized ARCH (GARCH) models which have been proposed are useful in many economics and financial studies. We thus develop both probabilistic properties and the Bayesian estimation method of a GARCH (1, 1) model. We then illustrate the model on Foolad Mobarakeh (F.M) daily returns from 2007 to 2012. Further we forecast future values of conditional variance of returns.

2. Generalized ARCH models

A linear GARCH (p, q) model is defined as follows. Let $\mathrm{y_t}$ denote a real-valued discrete-time stochastic process, and let $\mathbf{Y}_t$ denote the information set (σ-field) of all information through time t. The GARCH (p, q) process is then given by,

$$y_t = \sqrt{\frac{\nu-2}{\nu}\varpi_t}\, \sigma_t\, \varepsilon_t, \qquad (1)$$

$$\varpi_t \sim I\mathcal{G}\left(\frac{\nu}{2}, \frac{\nu}{2}\right),$$

where $\{\varepsilon_t\}$ are iid random variables with zero mean and unit variance, $\nu > 2$ and $I\mathcal{G}(.,.)$ denotes the Inverted Gamma distribution. The conditional variance of the process, $\sigma_t^2 = E\{\, y_t^2 | \boldsymbol{Y}_{t-1}\, \}$, is defined as,

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j y_{t-j}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2\,, \qquad (2)$$

where

$$p > 0, \quad q \geq 0,$$

$$\alpha_0 > 0, \quad \alpha_j \geq 0, \quad i = 1, \dots, p,$$

$$\beta_j \geq 0, \quad j = 1, \dots, q.$$

For $q = 0$, the process reduces to the ARCH (p) process, and for $p = q = 0$, the process $\mathrm{y_t}$ is simply a white noise. The non-negativity of the parameters $\alpha_j, i = 1, \dots, p$, and $\beta_j,\ j = 1, \dots, q$, together with $\alpha_0 > 0$, are sufficient to ensure a positive conditional variance, σ_t^2.

The necessary and sufficient condition for (1) and (2) to define a wide-sense stationary process $\{y_t, t = 0, \pm 1, \pm 2, \dots\}$ with $E(y_t^2) < \infty$ is that,

$$\sum_{j=1}^{p} \alpha_j + \sum_{j=1}^{q} \beta_j < 1. \qquad (3)$$

Furthermore, for such a stationary solution, $E(y_t) = 0$ and $var(y_t) = \frac{\alpha_0}{1-\sum_{j=1}^{p}\alpha_j - \sum_{j=1}^{q}\beta_j}$; See Giraitis et al. [9], and also theorem 4.4 of Fan et al. [7].

Let $a_t = y_t^2 - \sigma_t^2$ so that $\sigma_t^2 = y_t^2 - a_t$. We can rewrite (2) in terms of y_t^2 and a_t as follows:

$$y_t^2 = \alpha_0 + \sum_{j=1}^{m} (\beta_j + \alpha_j)\, y_{t-j}^2 + a_t - \sum_{j=1}^{q} \beta_j a_{t-j} \qquad (4)$$

or

$$(1 - \gamma_1 B - \cdots - \gamma_m B^m) y_t^2 = \alpha_0 + \left(1 - \beta_1 B - \cdots - \beta_q B^q\right) a_t, \qquad (4.a)$$

where $m = \max(p, q)$, $\alpha_j = 0$ for $j > p$, $\beta_j = 0$ for $j > q$,

$$\gamma_j = (\beta_j + \alpha_j),$$

$$a_t = y_t^2 - \sigma_t^2.$$

a_t is the associated white noise process of y_t^2; and therefore, (4) or $(4.a)$ is a proper ARMA model, which follows, because $E_{t-1}(y_t^2) = \sigma_t^2$, σ_t^2 is the one-step ahead forcast of y_t^2, and a_t is the corresponding one-step ahead forecast error.

In other words,

$$E_{t-1}(a_t) = E_{t-1}(y_t^2 - \sigma_t^2) = 0 = E(a_t)$$

and

$$\begin{aligned} E(a_i a_j) &= E(y_i^2 - \sigma_i^2)(y_j^2 - \sigma_j^2) \\ &= E[\sigma_i^2 \sigma_j^2 (\varepsilon_i^2 - 1)(\varepsilon_j^2 - 1)] \\ &= 0, \qquad for\ i \neq j, \end{aligned}$$

where, we note that ε_i^2 are i.i.d. $\chi^2(1)$. Thus, the GARCH(p,q) model in (1) and (2) implies that y_t^2 follows an ARMA(m,q) model in $(4.a)$ with the AR order being $m = \max(p,q)$.From (4) or $(4.a)$, we see that this process will have a unit root if $(1 - \gamma_1 B - \cdots - \gamma_m B^m) = 0$, i.e., if $\sum_{j=1}^{m} \gamma_j = \sum_{j=2}^{p} \alpha_j + \sum_{j=1}^{q} \beta_j = 1$. In this case, the model will be called an integrated GARCH (IGARCH) process.

3. Bayesian estimation of GARCH (1, 1) model

Since the GARCH (1, 1) model is one of the most common model for financial data and it is typically sufficient for investigation volatility, here we use GARCH (1, 1) model for which σ_t is given by,

$$y_t = \sqrt{\frac{\nu-2}{\nu}\varpi_t}\, \sigma_t\, \varepsilon_t, \quad t = 1, \dots, n \tag{1}$$

$$\varpi_t \sim I\mathcal{G}(\tfrac{\nu}{2}, \tfrac{\nu}{2}),$$

$$\sigma_t^2 = \alpha_0 + \alpha_1\, y_{t-1}^2 + \beta \sigma_{t-1}^2. \tag{5}$$

Usually parameters in the models are inferred by the maximum likelihood estimation or the generalized method of moments. Bayesian inference can be also applied to the GARCH model. Bayesian inference is commonly performed by MCMC algorithms which sample model parameters. The estimates of the model parameters are given by averaging over the sampled parameters. Popular MCMC algorithms in the Bayesian estimations are the Gibbs sampler and the Metropolis-Hastings algorithm. Here we apply the M-H algorithm to the Bayesian estimation of the GARCH (1, 1) model.

In order to write the likelihood function, we define the following vectors,

$$\boldsymbol{y} = (y_1, \dots, y_n)',$$

$$\boldsymbol{\varpi} = (\varpi_1, \dots, \varpi_n)',$$

$$\boldsymbol{\alpha} = (\alpha_0, \alpha_1)',$$

$$\boldsymbol{\Theta} = (\boldsymbol{\alpha}, \beta, \nu, \boldsymbol{\varpi}).$$

In addition, we define the $(n \times n)$ diagonal matrix $\Sigma = \Sigma(\boldsymbol{\Theta}) = diag(\left\{\frac{\nu-2}{\nu}\varpi_t\sigma_t^2(\boldsymbol{\alpha},\beta)\right\}_{t=1}^{n})$ where

$$\sigma_t^2(\boldsymbol{\alpha},\beta) = \alpha_0 + \alpha_1 y_{t-1}^2 + \beta\sigma_{t-1}^2(\boldsymbol{\alpha},\beta).$$

We can express the likelihood function of parameter $\boldsymbol{\Theta}$ as follows,

$$l(\Theta|\boldsymbol{y}) \propto (\det\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp[-\tfrac{1}{2}\boldsymbol{y}'\boldsymbol{\Sigma}^{-1}\boldsymbol{y}],$$

where, for convenience, we use the first observation as an initial condition and the initial variance is fixed to α_0. This likelihood refers to the conditional likelihood of the GARCH process given in (1) and (5). We propose the following proper priors on the parameters of the preceding model,

$$p(\boldsymbol{\alpha}) \propto N_2(\boldsymbol{\alpha}|\boldsymbol{\mu}_\alpha \,.\, \boldsymbol{\Sigma}_\alpha)I_{[\boldsymbol{\alpha}>0]},$$

$$p(\beta) \propto N(\beta|\mu_\beta \,.\, \Sigma_\beta)I_{[\beta>0]},$$

where $\mu_.$ and $\Sigma_.$ are the hyperparameters, $I_{[.]}$ is the indicator function and N_d is the d-dimensional Normal density.

The prior distribution of vector $\boldsymbol{\varpi}$ conditional on ν is found by noting that the components ϖ_t are independent and identically distributed from the Inverted Gamma density, which yields:

$$p(\boldsymbol{\varpi}|\nu) = (\frac{\nu}{2})^{\frac{n\nu}{2}}[\Gamma\left(\frac{\nu}{2}\right)]^{-n}(\prod_{t=1}^{n}\varpi_t)^{-\frac{\nu}{2}-1}exp\left[-\frac{1}{2}\sum_{t=1}^{n}\frac{\nu}{\varpi_t}\right].$$

We follow Deschamps [5] in the choice of the prior distribution on the degrees of freedom parameter. The distribution is a translated Exponential with parameters $\lambda > 0$ and $\delta \geq 2$:

$$p(\nu) = \lambda\exp[-\lambda(\nu-\delta)]I_{[\nu>\delta]}.$$

For large values of λ, the mass of the prior is concentrated in the neighborhood of δ and a constraint on the degrees of freedom can be imposed in this manner. The Normality of the errors is obtained when δ becomes large. Finally, we assume prior independence between α, β and $(\boldsymbol{\varpi}, \nu)$ which yields the following joint prior:

$$p(\boldsymbol{\Theta}) = p(\boldsymbol{\alpha})p(\beta)p(\boldsymbol{\varpi}|\nu)p(\nu).$$

By combining the likelihood function and the joint prior, we construct the joint posterior distribution via Bayes' rule,

$$p(\boldsymbol{\Theta}|\boldsymbol{y}) \propto l(\boldsymbol{\Theta}|\boldsymbol{y})p(\boldsymbol{\Theta}).$$

The recursive nature of the variance equation does not allow for conjugacy between the likelihood function and the prior density. Therefore, we rely on the M-H algorithm to draw samples from the joint posterior distribution. The algorithm in this section is a special case of the algorithm described by Nakatsuma [10].

4. Illustration

We apply our Bayesian estimation methods to daily observations of the Foolad Mobarakeh (F.M) returns. The data source is the daily prices of the Foolad Mobarakeh Stock. The sufficient sources of data can be accessed on the Tehran Stock Exchange (TSE) website (http://www.irbourse.com). The sample period is from March 11, 2007, to October 14, 2012, for a total of 1163 observations. The returns are plotted in Figure 1. From Figure 1, we clearly observe clusters of high and low variability in the time series data setting. This phenomenon is well known in financial data and is referred to as volatility clustering. In fact, the data show volatility clustering; that is, highly volatile periods tend to be clustered together. A problem in the analysis of these types of financial data is to forecast the volatility of future returns. Models such as ARCH and GARCH models and stochastic volatility models have been

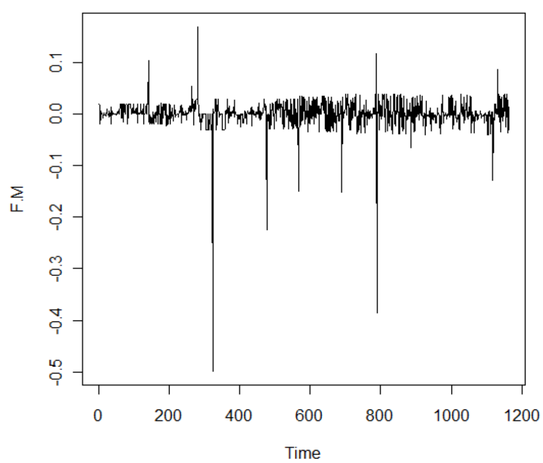

Figure 1: Returns of the Foolad Mobarakeh

to handle these problems. We fit a GARCH (1, 1) model to the series with the following results. We emphasize the fact that only positivity constraints are implemented in the MH algorithm; no stationary conditions are imposed in the simulation procedure. The results are summarized in Table 1. We generate two chains for 2000, 4000 and 5000 iteration in each passes. Here we give the results for the 4000 sampled data. The averages of α_0, α_1, β and ν are taken over 4000 sampled data. The results are summarized in table 1. Column 2 of table 1 shows the size effect of the parameters. Column 3 of table 1 shows the standard errors. Note that the value of estimation of α_1 is high. It can be implied that the high value of α_1 is due to volatilities are sensitive. It also shows that they have reaction with respect to market motivation rapidly.

Table 1 : Estimating coefficients and theirs standards errors in fitted GARCH (1, 1) model.

Coefficients	Estimate	S.E
α_0	0.001307	0.0007342
α_1	29.661821	6.8034365
β	0.435502	0.0174133
ν	2.393869	0.3160933

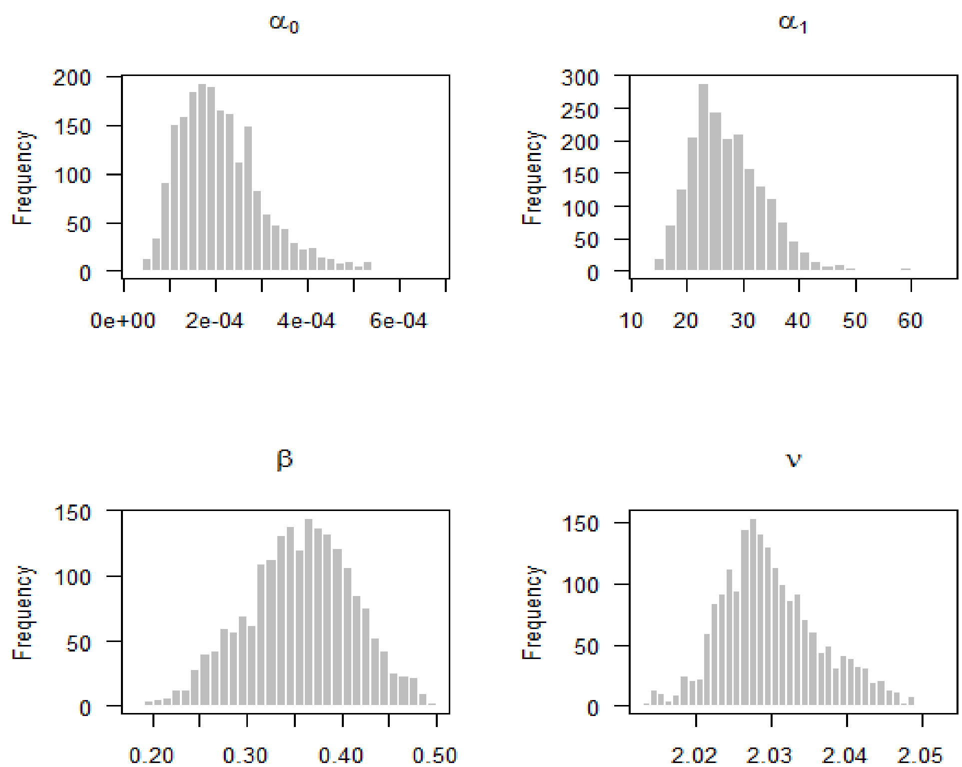

Figure 2: Marginal posterior distributions

The marginal distributions of the model parameters can be obtained by first transforming the output into a matrix and then using the function hist. Marginal posterior densities are displayed in Figure 2. We clearly notice the asymmetric shape of the histograms.

The volatility σ of a stock is a measure of uncertainty about the returns provided by stock. In financial market, volatility is often referred to standard deviation σ or variance σ^2. Once we have obtained satisfactory models, we forecast future values of conditional variance of return.

To explore the GARCH predictions, we calculated and plotted the middle of the data along with the one-step-ahead predictions of the corresponding volatility, σ_t^2. The results are displayed as $\pm\hat{\sigma}_t$ as a dashed line surrounding the data in Figure 3.

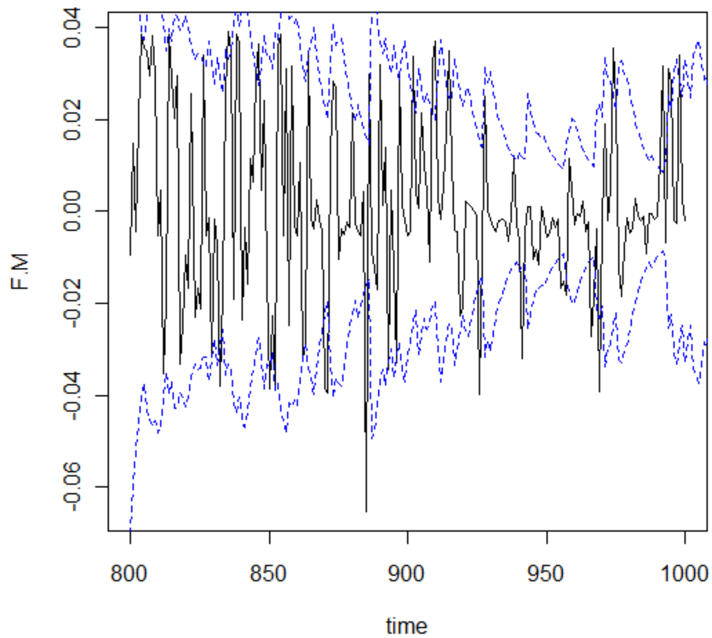

Figure 3: GARCH predictions of the F.M volatility, $\pm\hat{\sigma}_t$, displayed as dashed line.

5. Conclusion

The purpose of this paper has been to model volatility and estimate the model. There are times stock price is much more volatile than other times and often happens with financial assets. Clustering in volatility is often present in financial time series data. Volatility in a period depends on volatility in the previous period. Because of having been used extensively in the literature to model asset returns, we model volatility in the context of GARCH processes in each time t. Stock return volatility changes over time and can be well described by a GARCH-type model. In this paper, we investigate the behavior of stock return volatility and present some preliminary results and theoretical properties of GARCH model. Volatility has attracted increasing attention of many authors during the past three decades. Volatility is unobservable in financial market and it is measured by standard deviations or variance of return which can be directly considered as a measure of risk of assets. According to Akgiray [1], there are two reasons why forecasting volatility attracts interests of investors. Firstly, good forecast capability of volatility models provides a practical tool for stock market analysis. Secondly, as proxy for risk, volatility is related to expected returns, hence good forecast models enable investors give more appropriate securities pricing strategies. Therefore, performance of volatility forecast models is one of the main concerns for investors. The paper investigates the estimation and forecast ability of alternative univariate GARCH models for conditional variance in Foolad Mobarakeh stock returns over the 5 year period from 2007 to 2012. Since maximum likelihood estimation is not feasible due to path dependence, we used a reliable Bayesian estimation algorithm for this model. We applied the M-H algorithm to the Bayesian analysis of the GARCH (1, 1) model. The construction of the proposal density is performed using the data generated by MCMC methods. During the MCMC simulations the proposal density is updated adaptively. The MCMC scheme has been derived in order to simulate the joint posterior distribution for the model's parameters. Further research could be oriented in several directions. A direction could be to refine the specification by using existing extensions of the simple GARCH (1, 1) model, and allowing an ARMA structure for the conditional mean. Designing diagnostic tools for testing and improving GARCH equations may be one of the challenges for the future.

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