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Homotopy Analysis Method: A fresh view on Benjamin-Bona-Mahony-Burgers equation

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Abstract

In this paper, an analytic method, namely the homotopy analysis method (HAM) is applied to obtain approximations to the analytic solution of special form of the generalized nonlinear Benjamin-Bona-Mahony-Burgers equation (BBMB). This approximate solution, which is obtained as a series of exponentials, has a reasonable residual error. The results reveal that the presented method is very effective and convenient.

Keywords: Homotopy analysis method, Auxiliary parameter, Benjamin-Bona-Mahony-Burgers equations (BBMB), Nonlinear equation

1 Introduction

It is difficult to solve nonlinear problems, especially by analytic technique. In recent years, the homotopy analysis method has been devoted by scientists for solving nonlinear problems. This new method has been introduced by Liao [1] and applied to many nonlinear problems in engineering and science, such as nonlinear oscillations [2, 3, 4], Blasius viscous flow problems [5, 6], boundary-layer flows over an impermeable stretched plate [7], exponentially decaying boundary layers [8], nonlinear model of combined convective and radiative cooling of a spherical body [9], and many other problems [10, 11, 12, 13, 14, 15]. The goal of the this paper is to extend the homotopy analysis method has been introduced by Liao to finding analytic solution of special form of the generalized nonlinear Benjamin-Bona-Mahony-Burgers equations. These equations are in the form of

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + g(u)_x = 0,\ x \in \Re, t \geq 0, \tag{1}$$

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where $u(x,t)$ represents the fluid velocity in the horizontal direction x, α is a positive constant, $\beta \in \Re$ and finally, $g(u)$ is a $C2$-smooth nonlinear function [16].
The paper is organized as follows. In Section 2, the homotopy analysis method is applied to solve a special form of the generalized nonlinear Benjamin-Bona-Mahony-Burgers equations. Numerical results are given in section 3. We conclude the paper in the last section.

2 Homotopy analysis method

Taking some considerations as $\alpha = 0$, $\beta = 1$ and $g(u) = \frac{u^2}{2}$ into account changes BBMB equation into

$$u_t - u_{xxt} + u_x + (\frac{u^2}{2})_x = 0, \ x \in \Re, t \geq 0, \tag{2}$$

Since $u(x,t)$ in (2) is a complex function we assume that solutions of Eq. 2 is of the form

$$u(x,t) = \gamma U(\eta), \quad \zeta = \lambda x - \theta t, \tag{3}$$

where γ is amplitude which will determined later, and $\lambda > 0$, θ is arbitrary given constant.
substituting $u(x,t)$ into (2) yields:

$$(\lambda - \theta)U^{'} + \gamma\lambda U U^{'} + \theta\lambda^2 U^{'''} = 0, \tag{4}$$

where the prime denotes differentiation with respect to ζ.
Suppose $U \to \exp(-\zeta)$ as $\zeta \to \infty$, substituting $exp(-\zeta)$ into (4) we have

$$-(\lambda - \theta)\exp(-\zeta) - \gamma\lambda\exp(-2\zeta) - \theta\lambda^2\exp(-\zeta) = 0, \tag{5}$$

$\lambda > 0$ can be determined by equating the coefficient of $exp(-\zeta)$ to be zero

$$\theta\lambda^2 + \lambda - \theta = 0, \tag{6}$$

Assume that the solution $U(\zeta)$ arrives its maximum at the origin. Due to the continuity, the boundary conditions of the dimensionless solutions are

$$U(0) = 1, U^{'}(0) = 0, U(\infty) = 0. \tag{7}$$

According to the boundary conditions (7), it is natural to express (4) by the set of base function

$$\{\exp(-m\zeta) | m \geq 1\}, \tag{8}$$

in the form:

$$U(\zeta) = \sum_{m=1}^{\infty} c_m \exp(-m\zeta). \tag{9}$$

We choose the auxiliary linear operator as

$$L[U] = [U^{'''} + 3U^{''} + 2U^{'}]. \tag{10}$$

with the property

$$L[C_1 \exp(-\zeta) + C_2 \exp(-2\zeta) + C_3] = 0 \tag{11}$$

Now, we define a nonlinear operator as

$$N[\phi(\zeta;q), A(q)] = (\lambda - \theta)\frac{d\phi(\zeta;q)}{d\zeta} + \lambda\gamma(q)\phi(\zeta;q)\frac{d\phi(\zeta;q)}{d\zeta} + \lambda^2\theta\frac{d\phi^3(\zeta;q)}{d\zeta^3}. \tag{12}$$

Using the embedding parameter q, we construct the zero-order deformation equation

$$(1-q)L[\phi(\zeta;q) - U_0(\zeta)] = q\hbar N[\phi(\zeta;q), \gamma(q)],\ q \in [0,1], \tag{13}$$

such that

$$\phi(0;q) = 1, \quad \frac{\partial\phi(\zeta;q)}{\partial\zeta}|_{\zeta=0} = 0, \quad \phi(\infty;q) = 0, \tag{14}$$

Using Taylors theorem, we expand $\phi(\zeta;q)$ and $\gamma(q)$ in the power series of q as follows

$$\phi(\zeta;q) = U_0(\zeta) + \sum_{j=1}^{\infty} U_j(\zeta)q^j, \quad \gamma(q) = \gamma_0 + \sum_{j=1}^{\infty}\gamma_j q^j, \tag{15}$$

where

$$U_j(\zeta) = \frac{\partial^j\phi(\zeta;q)}{j!\partial q^j}\ |_{q=0}\ . \tag{16}$$

$$\gamma_j = \frac{\partial^j\gamma(q)}{j!\partial q^j}\ |_{q=0}\ . \tag{17}$$

Differentiating (13) and (14) m times with respect to q then setting $q = 0$ and finally dividing them by $m!$, we gain the mth-order deformation equation

$$\begin{cases} L[U_m(\zeta) - \chi_m U_{m-1}(\zeta)] = \hbar R_m(U_0, A_0, \ldots, U_{m-1}, \gamma_0, \gamma_1, \ldots, \gamma_{m-1}), \\ U_{m-1}(0) = 0, \quad U^{'}_{m-1}(0) = 0, \quad U_{m-1}(\infty) = 0, \quad m \geq 1. \end{cases} \tag{18}$$

Where

$$R_{m-1}(U_0, \gamma_0, \ldots, U_{m-1}, \gamma_{m-1}) = \frac{\partial^{m-1}N[\phi(\zeta;q), \gamma(q)]}{(m-1)!\partial q^{m-1}}\ |_{q=0} \tag{19}$$

According to property of the auxiliary linear operator L, the solution of the deformation equation contains the so-called term $\tau\exp(-2\zeta)$, if the right-hand side of (18) involves the term $exp(-2\zeta)$. Thus, we enforce the coefficient of the term $exp(-2\zeta)$ to be zero.

The general solution (18) is

$$U_m(\zeta) = U^*_m(\zeta) + C_1\exp(-\zeta) + C_2\exp(-2\zeta) + C_3$$

Order	$[m, m]$	γ
4	$[2, 2]$	0.927050980
6	$[3, 3]$	1.716857731
8	$[4, 4]$	1.816126344
10	$[5, 5]$	1.850144855
12	$[6, 6]$	1.857303598
14	$[7, 7]$	1.857392587
16	$[8, 8]$	1.857375067

Table 1: Results for [m,m] Homotopy-Pade technique ($\theta = 1$).

where C_1, C_2 and C_3 are constants and $U_m^*(\zeta)$ is a special solution of (18). The unknown C_1, C_2 and C_3 are obtained by solving the linear algebraic equation

$$C_1 = -2U_m^*(0) - U_m^{*'}(0), \quad C_2 = U_m^*(0) + U_m^{*'}(0), \quad C_3 = 0.$$

Theorem (Convergence) If the solution series $U_0(\zeta) + \sum_{m=1}^{\infty} U_m(\zeta)$ and $\gamma_0 + \sum_{j=1}^{\infty} \gamma_m$ are convergent, then they must be the exact solution of equation (2).
Proof: [1]

3 Results and Discussion

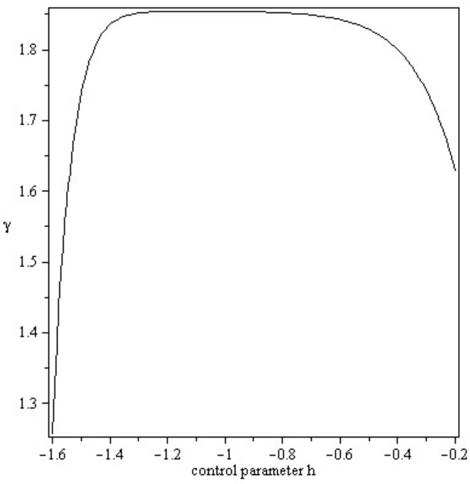

Figure 1: The curve of the amplitude a versus $\hbar$ for the 15th-order approximation ($\omega = 1$).

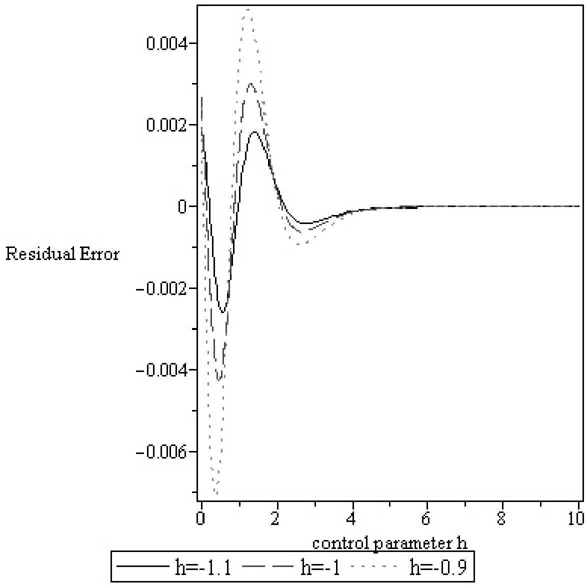

Figure 2: The residual error for the 15th-order approximation ($\omega = 1$). Solid curve: $\hbar = -1.1$; dashed curve: $\hbar = -1$; dotted curve: $\hbar = -0.9$.

The Nth-order approximation of the solutions $U(\zeta)$ and γ can be expressed as

$$U(\zeta) \approx S_N(\zeta) = U_0(\zeta) + \sum_{j=1}^{N} U_j(\zeta), \quad \gamma \approx Y_N = \gamma_0 + \sum_{j=1}^{N} \gamma_j \tag{20}$$

which are dependent upon the convergence-control parameters $\hbar$. We need only to concentrate on the convergence of the obtained results by proper choosing of $\hbar$.

Case $\theta = 1$:

Suppose $\theta = 1$, hence from (6), we have $\lambda = \frac{-1+\sqrt{5}}{2}$. We can investigate the influence of $\hbar$ on the convergence of γ by the γ-curves,as shown in Fig. 1. It is seen that convergent results can be obtained when $-1.3 < \hbar < -0.8$. Thus, we can choose an appropriate value for $\hbar$ in this range to get the convergent solution of the amplitude γ. Also, the residual error with HAM by 15th-order approximation, i.e.,

$$Residual\ Error \approx (\lambda - \theta)S'_{15}(\zeta) + Y_{15}\lambda S_{15}(\zeta)S'_{15}(\zeta) + \lambda^2 S'''_{15}(\zeta), \tag{21}$$

is plotted in Fig. 2.

Case $\theta = 2$:

Suppose $\theta = 2$, hence from (6), we have $\lambda = \frac{-1+\sqrt{17}}{4}$. We can investigate the influence of $\hbar$ on the convergence of γ by the γ-curves,as shown in Fig. 3. It is seen that convergent results can be obtained when $-0.4 < \hbar < -0.2$. Thus, we can choose an appropriate value for $\hbar$ in this range to get the convergent

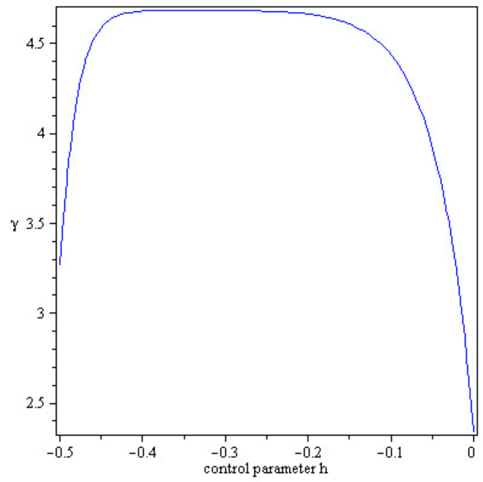

Figure 3: The curve of the amplitude a versus $\hbar$ for the 15th-order approximation ($\omega = 2$).

solution of the amplitude γ. Also, the residual error with HAM by 18th-order approximation, i.e.,

$$Residual\ Error \approx (\lambda - \theta)S_{18}^{'}(\zeta) + Y_{18}\lambda S_{18}(\zeta)S_{18}^{'}(\zeta) + \lambda^2 S_{18}^{'''}(\zeta), \tag{22}$$

is plotted in Fig. 4.

We can employ the so-called homotopy-pade technique to accelerate the convergence of the series

$$\gamma \approx Y_K = \gamma_0 + \sum_{j=1}^{K} \gamma_j \tag{23}$$

For the series solution (20), the corresponding $[m; m]$ pade approximate is expressed by

$$\gamma_0 + \sum_{j=1}^{2m} \gamma_j q^j = \frac{C_m(q)}{D_m(q)} = \frac{\sum_{j=0}^{m} C_j q^j}{1 + \sum_{j=1}^{m} D_j q^j} \tag{24}$$

where C_j and D_j are coefficients and are determined by the coefficients $\gamma_j, (j = 0, 1, , 2m)$.

Setting $q = 1$, we have the $[m; m]$ homotopy-pade approximant

$$\gamma_0 + \sum_{j=1}^{2m} \gamma_j = \frac{C_m(1)}{D_m(1)} = \frac{\sum_{j=0}^{m} C_j}{1 + \sum_{j=1}^{m} D_j} \tag{25}$$

The homotopy-pade approximation of γ, is listed in Table 1. Table 1 illustrate that the homotopy-pade technique can greatly enlarge the convergence rate of γ given by HAM.

4 Conclusions

In this paper, the HAM is applied to obtain the solution of special form of generalized nonlinear Benjamin-Bona-Mahony-Burgers equation (BBMB) which may contain high nonlinear terms. The success

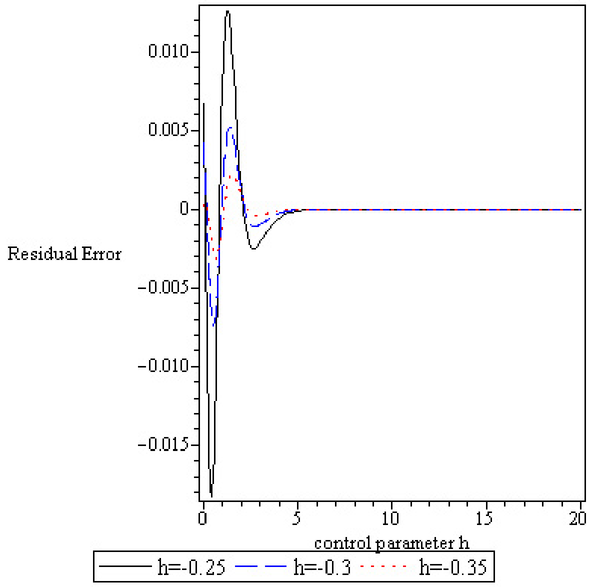

Figure 4: The residual error for the 15th-order approximation ($\omega = 2$). Solid curve: $\hbar = -0.25$; dashed curve: $\hbar = -0.3$; dotted curve: $\hbar = -0.35$.

of this method lies in the fact that provides a convenient way to control the convergence of approximation series. Our results illustrate that the best value for $\hbar$ is not -1. Finally by homotopy-pade technique, reasonable approximation for γ is obtained.

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