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Sigma Ideal Amenability of Banach Algebras

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Abstract

Let A be a Banach algebra and let I be a closed two-sided ideal in A. A is I-weakly amenable if $H^1(A,I^*) = \{0\}$. Further, A is ideally amenable if A is I-weakly amenable for every closed two-sided ideal I in A. In this paper we introduce σ-ideal amenability for a Banach algebra A, where σ is an idempotent bounded endomorphism of A.

Key words: σ-ideally amenable, σ-weakly amenable, closed two-sided ideal, σ–derivation.
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1. Introduction

Let A be a Banach algebra and let X be a Banach A-bimodule. Then X^*, the dual space of X, with the following module actions is a Banach A-bimodule,

$$\langle x, a\cdot x^*\rangle = \langle x\cdot a, x^*\rangle, \qquad \langle x, x^*\cdot a\rangle = \langle a\cdot x, x^*\rangle, \qquad (a\epsilon A, x\epsilon X, x^*\epsilon X^*).$$

In particular, if I is a closed ideal in A, then I and I^* will be a Banach A-bimodule and a dual Banach A-bimodule respectively.

Suppose that A is a Banach algebra and X is a Banach A-bimodule. A linear map $D: A \to X$ is called a derivation if

$$D(ab) = aD(b) + D(a)b, \qquad (a, b\ \epsilon\ A).$$

Given $x\epsilon\ X$, the map $\delta_x(a) = ax - xa$ is a derivation on A which is called an inner derivation. The set of all bounded derivations and inner derivations are denoted by $Z^1(A,X)$ and $N^1(A,X)$, respectively. $H^1(A,X) = Z^1(A,X)/N^1(A,X)$ is called the first cohomology of A with coefficients in X.

A Banach algebra A is called amenable if $H^1(A,X^*) = \{0\}$ for every Banach A-bimodule X.

Let A be a Banach algebra and σ be a bounded endomorphism of A, i.e. a bounded Banach algebra homomorphism from A into A. A σ-derivation from A into a Banach A-bimodule X is a bounded linear map $D: A \to X$ satisfying

$$D(ab) = \sigma(a) \cdot D(b) + D(a) \cdot \sigma(b), \qquad (a, b \in A).$$

For each $x \in X$, the mapping

$$\delta_x^\sigma : A \to X$$

defined by $\delta_x^\sigma(a) = \sigma(a) \cdot x - x \cdot \sigma(a)$, for all $a \in A$, is a σ-derivation called an inner σ-derivation.

Let A be a Banach algebra and I be a closed two-sided ideal in A. Then A is called I-weakly amenable if $H^1(A, I^*) = \{0\}$. Furthermore A is called ideally amenable if $H^1(A, I^*) = \{0\}$, for every closed two-sided ideal I of A. This definitions was introduced by Eshaghi Gordji and Yazdanpanah in [1]. In this paper we introduce σ -I-weakly amenability and σ-ideal amenability of Banach algebras.

2. σ -ideal amenability

We start this section with the following definition which is the basic definition for the present paper.

Definition 1 Let A be a Banach algebra and σ be a bounded endomorphism of A. Let I be a closed two-sided ideal of A, then A is said to be σ-I-weakly amenable if every σ-derivation from A into I^* is σ-inner. Furthermore A is σ-ideally amenable if A is $\sigma - I$-weakly amenable for every closed two-sided ideal I of A.

Proposition 2 Let A be a Banach algebra and σ be an idempotent epimorphism of A. If A is σ-weakly amenable, then A is essential.

Proof. Assume towards a contradiction that $\overline{A^2} \neq A$. So there exists $b_0 \in A$ such that $b_0 \notin \overline{A^2}$. By Hahn-Banach theorem there exists $a_0^* \epsilon A^*$ with $a_0^*|A^2 = 0$ and $\langle b_0, a_0^* \rangle = 1$. Since σ is surjective, there exists $a_0 \in A$ such that $b_0 = \sigma(a_0)$ and so $\langle \sigma(a_0), a_0^* \rangle = 1$ Define $D : A \to A^*$ with

$$\langle b, D(a) \rangle = \langle \sigma(a), a_0^* \rangle \langle \sigma(b), a_0^* \rangle, \qquad (a, b \in A)$$

So D is a continuous linear map. We show that D is a σ–derivation. First note σ is an epimorphism and $a_0^*|A^2 = 0$ so $a_0^*|\sigma(A^2) = 0$. Thus for each $a, b \in A$, we have

$$\langle c, D(ab) \rangle = \langle \sigma(ab), a_0^* \rangle \langle \sigma(c), a_0^* \rangle = 0, \qquad (c \in A).$$

On the other hand for each $c \in A$

$$\begin{aligned}\langle c, \sigma(a) \cdot D(b) \rangle + \langle c, D(a) \cdot \sigma(ab) \rangle &= \langle c\sigma(a), D(b) \rangle + \langle \sigma(b)c, D(a) \rangle \\ &= \langle \sigma(c\sigma(a)), a_0^* \rangle \langle \sigma(b), a_0^* \rangle \\ &\quad + \langle \sigma(\sigma(b)c), a_0^* \rangle \langle \sigma(a), a_0^* \rangle \\ &= \langle \sigma(c)\sigma(a)), a_0^* \rangle \langle \sigma(b), a_0^* \rangle \\ &\quad + \langle \sigma(b)\sigma(c), a_0^* \rangle \langle \sigma(a), a_0^* \rangle \\ &= 0,\end{aligned}$$

because $a_0^*|\sigma(A^2) = 0$. Thus D is a σ–derivation. Now

$$\begin{aligned}\langle \sigma(a_0), D(a_0) \rangle = \langle \sigma(a_0), D(a_0) \rangle &= \langle \sigma(\sigma(a_0)), a_0^* \rangle \langle \sigma(a_0), a_0^* \rangle \\ &= \langle \sigma(a_0), a_0^* \rangle \langle \sigma(a_0), a_0^* \rangle \\ &= 1.\end{aligned}$$

But

$$\begin{aligned}\langle \sigma(a_0), \delta_{a^*}^\sigma(a_0) \rangle &= \langle \sigma(a_0), \sigma(a_0) \cdot a^* - a^* \cdot \sigma(a_0) \rangle \\ &= \langle \sigma(a_0)\sigma(a_0), a^* \rangle - \langle \sigma(a_0)\sigma(a_0), a^* \rangle \\ &= 0 \qquad (a^* \in A^*)\end{aligned}$$

So D is not σ–inner. and it is a contradiction of the fact that A is σ–weakly amenable.

Remark 3 Let A be a non-unital Banach algebra. We denote by $A^\#$ the Banach algebra formed by adjoining an identity to A, so that $A^\# = A \oplus Ce$, with the product

$$(a,\alpha)\,(b,\beta) = (ab+\alpha b+\beta a, \alpha\beta) \qquad (\alpha,\beta\epsilon\mathbb{C}, a,b\epsilon A)$$

Let A be a non-unital Banach algebra with adjoined identity e, take $e' \in (A^{\#})^*$ with $\langle e, \acute{e}\rangle = 1$ and $\acute{e}|A = 0$, and extend $a^* \in A^*$ to an element of $(A^{\#})^*$ by setting $(e, a^*) = 0$. Then $(A^{\#})^* = A^* \oplus \mathbb{C}\acute{e}$ as a Banach space, and $(A^{\#})^*$ is an $A^{\#}$-bimodule with the following module actions,

$$(a+\alpha e)\cdot(a^*+\beta\acute{e}) = aa^*+\alpha a^*+(\langle a,a^*\rangle+\alpha\beta)\acute{e},$$
$$(a^*+\beta\acute{e})\cdot(a+\alpha e) = a^*a+\alpha a^*+(\langle a,a^*\rangle+\alpha\beta)\acute{e}.$$

Proposition 4 Let A be a Banach algebra and σ be an idempotent epimorphism of A. If A is σ-weakly amenable, then $A^{\#}$ is $\hat{\sigma}$-weakly amenable, where $\hat{\sigma}$ is the endomorphism of $A^{\#}$ induced by σ, i.e. $\hat{\sigma}(a+\alpha) = \sigma(a)+\alpha$.

Proof. Let $D: A^{\#} \to (A^{\#})^*$ be a $\hat{\sigma}$-derivation. Since $D(e) = 0$ we can look at D as a map from A into $(A^{\#})^*$. Also since $(A^{\#})^* = A^* \oplus \mathbb{C}\acute{e}$, there exists two bounded linear maps $\leftthreetimes: A \longrightarrow \mathbb{C}$ and $d: A \to A^*$ such that $D(a) = d(a) + \leftthreetimes(a)\acute{e}, (a\epsilon A)$. Since $\hat{\sigma}|A = \sigma$, by previose Remark for each $a, b \in A$ we have,

$$\begin{aligned} d(ab) + \leftthreetimes(ab) &= D(ab) \\ &= \sigma(a)\cdot D(b) + D(a)\cdot \sigma(b) \\ &= \sigma(a)\cdot [d(b) + \leftthreetimes(b)] + [d(a) + \leftthreetimes(a)]\cdot \sigma(b) \\ &= \sigma(a)d(b) + \langle\sigma(a), d(b)\rangle + d(a)\sigma(b) \\ &+\langle\sigma(b), d(a)\rangle. \end{aligned}$$

So

$$d(a) = \sigma(a)\cdot a^* + a^*\cdot\sigma(a), \qquad (a\epsilon A).$$

and

$$\leftthreetimes(ab) = \langle\sigma(a), d(b)\rangle + \langle\sigma(b), d(a)\rangle.$$

Therefore $d: A \to A^*$ is a σ-derivation. So there exists $a^* \in A^*$ such that

$$d(a) = \sigma(a)\cdot a^* + a^*\cdot\sigma(a) \qquad (a\epsilon A).$$

Now we show that $\leftthreetimes\, |A^2 = 0$. let $a, b\, \epsilon\, A$, we have,

$$\begin{aligned} \langle\sigma(a)\sigma(b), \leftthreetimes\rangle &= \langle\sigma(\sigma(a)), d(\sigma(b))\rangle + \langle\sigma(\sigma(b)), d(\sigma(a))\rangle \\ &= \langle\sigma(a), \sigma(b)\cdot a^* - a^*\cdot\sigma(b)\rangle + \langle\sigma(b), \sigma(a)\cdot a^* - a^*\cdot\sigma(a)\rangle \\ &= 0 \end{aligned}$$

Which shows that $\leftthreetimes\, |\sigma(A^2) = 0$. Since σ is an epimorphism, so $\leftthreetimes\, |A^2 = 0$. Also since A is σ-weakly amenable by Proposition 2, $(\overline{A^2}) = A$. Thus $\leftthreetimes = 0$ on A. Therefore $D = d$ is a σ-inner derivation.

Proposition 5 Let σ be a bounded endomorphism of Banach algebra A. If $A^{\#}$ is $\hat{\sigma}$-weakly amenable, Then A is σ-weakly amenable.

Proof. Let $D: A \to A^*$ be a continuous $\hat{\sigma}$-derivation. Note A is a Banach $A^{\#}$-bimodule with the following module actions:

$$(a+\alpha)\cdot b \;=\; a\cdot b \;+\alpha b, \quad b\cdot(a+\alpha) = b\cdot a + \alpha b,$$

for all $a, b \in A, \alpha\epsilon\mathbb{C}$. Define $\widetilde{D}: A^{\#} \to A^*$ with $\widetilde{D}(a+\alpha) = D(a)$. Clearly $\widetilde{D}$ is continuous σ-derivation and we can look at it as a function into $(A^{\#})^*$.

Since $A^{\#}$ is $\hat{\sigma}$-weakly amenable, so there exists $a^* \in A^*$ such that $\widetilde{D} = \delta^{\sigma}_{a^*}$. Hence for each a$\in A$ we have

$$D(a) = \widetilde{D}(a+\alpha) = \hat{\sigma}(a+\alpha)a^* - a^*\hat{\alpha} = \sigma(a)a^* - a^*\sigma(a).$$

Which shows that D is σ-inner and so A is σ-weakly amenable.

Proposition 6 Let A be a Banach algebra and I be a closed two-sided ideal of A with a bounded approximate identity. Let σ be an idempotent endomorphism of A such that $\sigma(I) = I$. Then I is σ-weakly amenable if and only if $H^1_\sigma(A, I^*) = \{0\}$.

Proof. Suppose that I is σ-weakly amenable and let $D: A \to I^*$ be a σ- derivation and $i: I \to A$ be the embeding map. Then $d = D|_I: I \to I^*$ is a σ–derivation. So there exists $i^* \in I^*$ such that $d = \delta^\sigma_{i^*}$. Since I is σ-weakly amenable and $\sigma(I) = I$ by Proposition 2, $\overline{\sigma(I^2)} = \overline{I^2} = I$ .On the other hand for $i.j \in I$ we have,

$$\begin{aligned}\langle \sigma(ij), D(a)\rangle &= \langle \sigma(i)\sigma(j), D(a)\rangle \\ &= \langle \sigma(i)\sigma(j), D(a)\rangle \\ &= \langle \sigma(i)D(ja) - D(j)\sigma(a)\rangle \\ &= \langle \sigma(i), \sigma(ja)\cdot i^* - i^*\cdot \sigma(ja)\rangle \\ &\quad -\langle \sigma(a)\sigma(i), \sigma(j)\cdot i^* - i^*\cdot \sigma(j)\rangle \\ &= \langle \sigma(ij), \sigma(a)i^*\rangle - \langle \sigma(ij), i^*\sigma(a)\rangle \\ &= \langle \sigma(ij), \delta^\sigma_{i^*}(a)\rangle \qquad (a\epsilon A).\end{aligned}$$

Therefore $D = \delta^\sigma_{i^*}$, and so D is σ-inner.

Conversely, let $H^1_\sigma(A, I^*) = \{0\}$. and let $D: I \to I^*$ be a σ –derivation. Since I is σ–neo-unital Banach I-bimodule, i.e. $I = \sigma(I)\cdot I\cdot \sigma(I)$, by [2, Proposition 37], D has an extension $\widetilde{D}: A \to I^*$ such that $\widetilde{D}$ is also σ–derivation. Now by hypothesis $\widetilde{D}$ is σ-inner. Thus I is σ-weakly amenable.

Proposition 7 Let σ be a bounded endomorphism of Banach algebra A. If $A^\#$ is $\hat{\sigma}$-ideally amenable, then A is σ -ideally amenable.

Proof. Let I be a closed two-sided ideal of A and $D: A \to I$ be a σ–derivation. It is easy to see that I is a closed two-sided ideal of $A^\#$, and $\widetilde{D}: A^\# \to I^*$ with $D^\#(a+\alpha) = D(a)$, $(a \in A, \alpha\epsilon\mathbb{C})$ is a $\hat{\sigma}$-derivation. Hence there exists $i^*\epsilon I^*$ such that $D^\# = \delta^\sigma_{i^*}$. So for each $a \in A$ we have

$$D(a) = \widetilde{D}(a+\alpha) = \hat{\sigma}(a+\alpha)\cdot i^* - i^*\cdot\hat{\sigma}(a+\alpha) = \sigma(a)\cdot i^* - i^*\cdot\sigma(a)$$

Which shows that D is σ–inner. So A is σ-ideally amenable.

Proposition 8 Let A be a Banach algebra and σ be an idempotent epimorphism of A. Suppose A is σ-weakly amenable and for each closed two-sided ideal I such that $I = \overline{AI\ U\ IA}$, A is σ-I-weakly amenable. Then A is σ-ideally amenable.

Proof. Let I be a closed two-sided ideal in A. Put $J = \overline{AI\ U\ IA}$. It is easy to see that J is a closed two-sided ideal in A and $J = \overline{AJ\ U\ JA}$. Let $\iota: J \to I$ be the natural embeding and $D: A \to I^*$ be a σ-derivation. Then $\iota^*\circ D: A \to J^*$ is a derivation. So there exists $m \in J^*$ such that $\iota^*\circ D = \delta^\sigma_m$. Let x^* be the extension of m into I, by the Hahn-Banach theorem, for every $a, b \in A$ we have,

$$\begin{aligned}\langle i, D(ab)\rangle &= \langle i\sigma(a), D(b)\rangle + \langle \sigma(b)i, D(a)\rangle \\ &= \langle \iota(i\sigma(a)), D(b)\rangle + \langle \iota(\sigma(b)i), D(a)\rangle \\ &= \langle i\sigma(a), \iota^*\circ D(b)\rangle + \langle \sigma(b)i, \iota^*\circ D(a)\rangle \\ &= \langle i\sigma(a), \sigma(b)m - m\sigma(b)\rangle + \langle \sigma(b)i, \sigma(a)m - m\sigma(a)\rangle \\ &= \langle i, \sigma(ab)m - m\sigma(ab)\rangle \\ &= \langle i, \delta^\sigma_{x^*}(ab)\rangle \qquad (i\epsilon I).\end{aligned}$$

Hence $D(ab) = \delta^\sigma_{x^*}(ab)$. Since A is σ-weakly amenable, so $(\overline{A^2}) = A$ and therefore $D = \delta^\sigma_{x^*}$ and D is σ–inner.

Proposition 9 Let A be a Banach algebra and σ be a bounded idempotent endomorphism of A. Let I be a closed two-sided ideal in A with a bounded approximate identity. If A is σ-ideally amenable, then I is σ-ideally amenable.

Proof. Let J be a closed two-sided ideal in I. It is easy to see that J is an ideal in A. Let $D: I \to J^*$ be a σ-derivation. By [2, proposition 37], D can be extended to a σ-derivation, $\widetilde{D}: A \to J^*$. So there exists $m \in J^*$ such that $D = \delta^\sigma_{x^*}$. Thus $D(i) = \widetilde{D}(i) = \delta^\sigma_{x^*}$, for each $i \in I$. So D is σ- inner.

Definition 10 Let I be a closed two-sided ideal in Banach algebra A. We say that I has the trace extension property, if every $m \in I^*$ such that $am = ma$ for each $a \in A$, can be extended to $a^* \in A^*$ such that $aa^* = a^*a$ for each $a \in A$.

Proposition 11 Let A be a Banach algebra and σ be a bounded endomorphism of A with dense range. suppose A is a σ-ideally amenable and I be a closed two-sided ideal in A that has the trace extension property. Then $\frac{A}{I}$ is σ–ideally amenable.

Proof. Let $\frac{J}{I}$ be a closed two-sided ideal in $\frac{A}{I}$. Then J is a closed two-sided ideal in A. Suppose $\pi: J \to \frac{J}{I}$ and $q: A \to \frac{A}{I}$ are natural quotient maps and π^* is adjoint of π. Let $D: \frac{A}{I} \to \left(\frac{J}{I}\right)^*$ be a σ-derivaton. Then $\pi^* \circ D \circ q: A \to J^*$ is a σ–derivation. So there exists $x^* \in J^*$ such that $\pi^* \circ D \circ q = \delta^\sigma_{x^*}$. Let m be the restriction of x^* to I. Then $m \in I^*$ and for each $i \in I$ we have,

$$\begin{aligned}\langle i, \sigma(b)m - m\sigma\rangle &= \langle i\sigma(b) - \sigma(b)i, m\rangle \\ &= \langle i\sigma(b) - \sigma(b)i, x^*\rangle \\ &= \langle i, \sigma(b)x^* - x^*\sigma(b)\rangle \\ &= \langle i, \delta^\sigma_{x^*}\rangle = \langle i, \pi^* \circ D \circ q(b)\rangle \\ &= \langle \pi(i), D \circ q(b)\rangle \\ &= \langle I, D(b+I)\rangle = 0 \qquad (b \in A)\end{aligned}$$

Therefore $\sigma(b)m = m\sigma(b)$ for each $b \in A$. Since σ has a dense range, for each $a \in A$, there exists a net $(b_\alpha) \subseteq A$, such that $a = lim_\alpha \sigma(b_\alpha)$. So for each $a \in A$ we have,

$$am = \lim_\alpha \sigma(b_\alpha)m = \lim_\alpha m\sigma(b_\alpha)m = m \lim_\alpha \sigma(b_\alpha)m = ma$$

Now since I has the trace extension property, m can be extend to $a^* \in A^*$ such that $aa^* = a^*a$, for each $a \in A$. Let y^* be the restriction of a^* to J. Then $y^* \in J^*$ and $x^* - y^* = 0$ on I. Therefore $x^* - y^* \in \left(\frac{J}{I}\right)^*$ and we have,

$$\begin{aligned}\langle j + I, D(a+I)\rangle &= \langle \pi(j), D(q(a))\rangle \\ &= \langle j, \pi^* \circ D \circ q(a)\rangle \\ &= \langle j, \delta^\sigma_{x^*}(a)\rangle \\ &= \langle j, \delta^\sigma_{x^*-y^*}(a)\rangle \\ &= \langle j + I, \delta^\sigma_{x^*-y^*}(a+I)\rangle \qquad (j \in J)\end{aligned}$$

Hence$D = \delta^\sigma_{x^*-y^*}$ and therefore $\frac{A}{I}$ is σ-ideally amenable.

Proposition 12 Let A be a Banach algebra and let j be a closed two-sided ideal in A with a bounded approximate identity. Let σ be a bounded idempotent endomorphism of A. Then for every closed two-sided ideal I in J, J is σ-I- weakly amenable if and only if A is σ-I-weakly amenable.

Proof. Let $(e_\alpha)_{\alpha\in I}$ be a bounded approximate identity for J and $D: J \to I^*$ be a σ–derivation. In [2, Proposition 37], we showed that the map $\widetilde{D} : A \to I^*$defined by

$$\widetilde{D}(a) = w^* - \lim_\alpha\big(D\,(ae_\alpha)\big) \qquad (a \in A)$$

is a countinuous σ-derivation. If $D = 0$ then $\widetilde{D} = 0$, since $JI = IJ = I$. Therefore $H^1_\sigma(J, I^*) = H^1_\sigma(A, I^*)$ and this implies that A is σ-I-weakly amenable if and only if J is σ-I-weakly amenable.

Proposition 13 Let A be a Banach algebra whit a bounded approximate identity and σ be a bounded idempotent epimorphism of A. Let φ be a non-trivial character on A and $I = ker\,\varphi$. If $\sigma(I) \subseteq I$, then

1) Every σ-derivation $D: A \to A^*$ restricts to a σ-derivation $d: I \to I^*$ such that there exists a function $\psi: I \to \mathbb{C}$ satisfying

$$\psi(ij) = \langle\sigma(i), D(j)\rangle + \langle\sigma(j), D(i)\rangle \qquad (i, j \epsilon I)$$

2) Every σ-derivation $D: I \to I^*$ for which there exists such a function ψ extends to a σ-derivation $\widetilde{D}: A \to A^*$.

Proof. 1) Clearly every σ-derivation restricts to a σ-derivation. Let (e_α) be a bounded approximate identity in A. So the bounded net $(\widehat{e_\alpha}) \subseteq A^{**}$ with $\langle a^*, \widehat{e_\alpha}\rangle = \langle e_\alpha, a^*\rangle$, $(a^* \epsilon A^*)$ has a weak*-convergence subnet. Thus without loss of generality, we may suppose that for each $a \in A^*$, the limit of the net $\langle e_\alpha, a^*\rangle$ exists. Now define $\psi(i) = lim_\alpha\,\langle e_\alpha, D(i)\rangle$ then for each $i, j \epsilon I$ we have,

$$\psi(ij) = \lim_\alpha\langle e_\alpha, D(ij)\rangle$$

$$= \lim_{\alpha}\langle e_\alpha, \sigma(i)D(j) + D(i)\sigma(j)\rangle$$
$$= \lim_{\alpha}\langle e_\alpha \sigma(i), D(j)\rangle + \langle D(i)\sigma(j)\rangle$$
$$= \lim_{\alpha}\langle e_\alpha \sigma(i), D(j)\rangle + \langle \sigma(i)e_\alpha, D(i)\rangle$$
$$= \langle \sigma(i), D(j)\rangle + \langle \sigma(j), D(j)\rangle$$

2)Let (e_α) be a bounded pproximate identity for A. Then $lim_\alpha\ \varphi(e_\alpha) = 1$. Hence we may assume that $\varphi(e_\alpha) = 1$ for all α. Let $D: I \to I^*$ be a σ-derivation. Since for each $a \in A$, $a - \varphi(a)e_\alpha \in I$. So for each $i \in I$ every extension of $D(i) \in I$ to A^* is unique. Therefore we can extend D to $D_1 : I \to A^*$ with

$$\langle a, D_1(i)\rangle = \lim_{\alpha}\langle a - \varphi(a)e_\alpha, D(i)\rangle + \varphi(a)\psi(i)$$

We show that D_1 is a σ-derivation from I to A^*.

$$\langle a, D_1(ij)\rangle = \lim_\alpha\langle a - \varphi(a)e_\alpha, D(ij)\rangle + \varphi(a)\psi(ij)$$
$$= \lim_\alpha\langle a - \varphi(a)e_\alpha, D(i)\sigma(j) + \sigma(i)D(j)\rangle + \varphi(a)\psi(ij)$$
$$= \lim_\alpha\langle \sigma(j)(a - \varphi(a)e_\alpha, D(i)\rangle$$
$$+\langle a - \varphi(a)e_\alpha, \sigma(i), D(j)\rangle + \varphi(a)\psi(ij)$$
$$= \langle \sigma(j)a - \sigma(j)\varphi(a), D(i)\rangle$$
$$+\langle a\sigma(i) - \varphi(a)\sigma(i), D(j)\rangle$$
$$+\varphi(a)(\langle \sigma(i), D(j)\rangle) + \langle \sigma(j), D(i)\rangle$$
$$= \langle \sigma(j)a, D(i)\rangle + \langle a\sigma(i), D(j)\rangle$$
$$= \lim_\alpha\langle \sigma(j)a - \varphi(\sigma(j)a)e_\alpha, D(i)\rangle$$
$$= \lim_\alpha\langle a\sigma(i) - \varphi\big(a\sigma(i)\big)e_\alpha, D(j)\rangle$$
$$+\varphi(\sigma(j)a)\psi(i) + \varphi(a\sigma(i)\psi(j)$$
$$= \langle \sigma(j)a, D_1(i)\rangle + \langle a\sigma(i), D_1(j)\rangle$$
$$= \langle a, D_1(i)\sigma(j) + \sigma(i), D_1(j)\rangle$$

So $D_1: I \to A^*$ is a σ-derivation. Since A has a bounded approximate identity and σ is an epimorphism, A is a σ-neo-unital Banach I-module. Thus by [2; Proposition 4.14] D_1 extends to a σ–derivation $\widetilde{D}: A \to A^*$which extends D.

Proposition 14 Let A be a Banach algebra with a bounded approximate identity and σ be a bounded idempotent epimorphism of A. Let φ be a non-trivial character on A and $I = ker\ \varphi$ which $\sigma(I) \subseteq I$. Suppose that $D: I \to I^*$ be a σ-derivation. If $\left(\overline{I^2}\right) = I$, then there is at most one extension of D to a σ-derivation $D: A \to A^*$.

Proof. Let $\widetilde{D}_1$ and $\widetilde{D_2}$ be two extension of a σ-derivatin D. Then $\widetilde{D}_1 - D_2$ is a σ-derivation extending the zero derivation. Therefore we assume that $\widetilde{D}$ be any extension of $D = 0$. So for each $i, j\epsilon I$ and $a\epsilon\ A$ we have,

$$\langle a, \widetilde{D}(i,j\rangle = \langle a, \sigma(i)\widetilde{D}(j) + \widetilde{D}(i)\sigma(j)\rangle$$
$$= \langle a\sigma(i), \widetilde{D}(j)\rangle + \langle \sigma(j)a, \widetilde{D}(i)\rangle$$
$$= \langle a\sigma(i), D(j)\rangle + \langle \sigma(j)a, D(i)\rangle = 0$$

Since $\left(\overline{I^2}\right) = I$, we have $\langle a, \widetilde{D}(i)\rangle = 0$ for all $i \in I$ and $a \in A$. Also for each $a, b \in A$ we have,

$$\langle \sigma(a)\sigma(b), \widetilde{D}(e_\alpha)\rangle = \langle \sigma(a), \widetilde{D}(be_\alpha)\rangle - \langle \sigma(e_\alpha)\sigma(a), \widetilde{D}(b)\rangle$$

Thus

$$\lim_\alpha\langle \sigma(a)\sigma(b), \widetilde{D}(e_\alpha)\rangle = 0$$

As σ is an epimorphism and A has a bounded approximate identity, by the Cohen Factorisation theorem, every element of A can be factorized. Hence

$$\langle b, \widetilde{D}(a)\rangle = \lim_\alpha\langle b, \widetilde{D}(a - \varphi(a)e_\alpha)\rangle = 0$$

because $a - \varphi(a)e_\alpha \in I$. Which shows that any extension of D is unique.

Next, assume that A is a complex Banach space which has dimension at least 2 and let $0 \neq \varphi \in Ball\ (A^*)$. Define a multiplication on A by

$$a \cdot b = \varphi(a)b \qquad (a, b \in A)$$

This multiplication evidently makes A into a Banach algebra denoted by A_φ. Which is called the ideally factored algebra associated to φ. It is easy to see that A_φ has left identity e which is that element in A such that $\varphi(e) = 1$, while it has not right approximate identity. Also if I is a proper ideal of A_φ, then $I \subseteq ker\ \varphi$. Suppose that $\sigma: A_\varphi \to A_\varphi$ be defined $by\ \sigma(a) = \varphi\ (a)e$. Then σ is the only idempotent endomorphism of A_φ.

Proposition 15 A_φ is σ-ideally amenable, where $\sigma(a) = \varphi(a)e$ is the only idempotent endomorphism of A_φ.

Proof. Let I be a closed two-sided ideal in A_φ and $D: A_\varphi \to I^*$ be a σ-derivation. Since $I \subseteq ker\ \varphi$, for each $a, b \in A_\varphi$ and $i \in I$ we have,

$$\begin{aligned}\varphi(a)\langle i, D\ (b)\rangle &= \langle i, D\ (a \cdot\ b)\rangle = \langle i, \sigma\ (a) \cdot\ D\ (b) + D\ (a) \cdot \sigma\ (b)\rangle \\ &= \langle i, \sigma\ (a) \cdot\ D\ (b)\rangle + \langle \sigma\ (b) \cdot i, D(a)\rangle \\ &= \varphi(b)\langle i, D(a)\rangle \qquad (1)\end{aligned}$$

In (1), set $b = e$. So we have,

$$\varphi(a)\langle i, D\ (e)\rangle = \varphi(e)\langle i, D(a)\rangle = \langle i, D\ (a)\rangle \qquad (2)$$

On the other hand, let $i^* \in I^*$, and let $\delta^\sigma_{i^*} : A \to I^*$ be the σ-inner derivation specified by i^*. Then for each $a \in A_\varphi$ and $i \in I$ we have,

$$\begin{aligned}\langle i, \delta^\sigma_{i^*}(a)\rangle &= \langle i, \sigma\ (a) \cdot i^* - i^* \cdot \sigma(a)\rangle \\ &= \langle i \cdot \sigma\ (a), i^*\rangle - \langle \sigma(a) \cdot i, i^*\rangle \\ &= \varphi(i)\langle \sigma\ (a), i^*\rangle - \varphi(\sigma(a))\langle i, i^*\rangle \\ &= -\varphi(a)\langle i, i^*\rangle \qquad (3)\end{aligned}$$

Set $i^* = -D(e)$. By (2) , (3) we have

$$\langle i, \delta^\sigma_{i^*}(a)\rangle = -\varphi\ (a)\langle i, -D\ (e)\rangle = \langle i\ , D\ (a)\rangle, \qquad (a \in A\varphi, i \in I)$$

Therefore, $D = \delta^\sigma_{i^*}$. Thus $A\varphi$ is σ-ideally amenable.

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