Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci., 17 (2017), 465–476
Research Article
Online: ISSN 2008-949x
Journal of Mathematics and Computer Science
Journal Homepage: www.tjmcs.com - www.isr-publications.com/jmcs

Integral inequalities via generalized convex functions

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Abstract

In this paper, we introduce and investigate a new class of generalized convex functions, called generalized log-convex function. We establish some new Hermite-Hadamard integral inequalities via generalized log-convex functions. Our results represent refinement and improvement of the previously known results. Several special cases are also discussed. The concepts and techniques of this paper may stimulate further research in this field.

Keywords: Generalized convex functions, generalized log-convex functions, Hermite-Hadamard type inequalities.
2010 MSC: 26D15, 26D10, 90C23.

1. Introduction

In recent years, the theory of convex analysis has turned into one of the most interesting and useful field of mathematics due to its elegance in shape and numerous applications. The concept of convexity has been extended and generalized in several directions using new and innovative techniques to study different problems in a unified frame work. Consequently, several new classes of convex functions and convex sets have been introduced and investigated, see [1, 2, 9, 11–13, 16, 17]. Consequently many new inequalities related to these new classes of convex functions have been derived by many researchers. It is well-known that a function is convex, if and only if it satisfies Hermite-Hadamard inequality. Such type of integral inequalities are useful in finding the upper and lower bounds, see [4, 5, 8, 14, 15, 21, 23].

Gordji et al. [9] introduced and investigated a new class of convex functions involving the bifunction, which is called φ-convex function. These φ-convex functions are non-convex functions. For recent developments, see [6, 7, 9, 10, 18–20] and the references therein.

It is well-known that log-convex functions are of interest in many areas of mathematics and science. They play an important role in mathematical statistics and the theory of special functions, see [3]. Inspired by this ongoing research, we introduce a new class of generalized convex function, which is called generalized log-convex functions. We derive some new Hermite-Hadamard integral inequalities for generalized log-convex functions. Our results include a wide class of known and new ones as special cases. The techniques of this paper may motivate further research in this fascinating area.

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doi:10.22436/jmcs.017.04.03

Received 2017-05-17

2. Preliminaries

Let I be an interval in real line $\mathbb{R}$. Let $f : I = [x, y] \to \mathbb{R}$ and $\eta(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. We recall the following known concepts.

Definition 2.1 ([9]). Let I be an interval in real line $\mathbb{R}$. A function $f : I = [a, b] \to \mathbb{R}$ is said to be generalized convex with respect to a bifunction $\eta(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if

$$f((1-t)a + tb) \leqslant f(a) + t\eta(f(b), f(a)), \quad \forall a, b \in I, \quad t \in [0, 1].$$

If $\eta(b, a) = b - a$, then we will obtain the classical definition of a convex functions.

We now introduce the generalized log-convex functions and derive some integral inequalities, which is the main motivation of this paper.

Definition 2.2. A function $I = [a, b] \to \mathbb{R}$ is said to be generalized log-convex with respect to a bifunction $\eta(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, if

$$f((1-t)a + tb) \leqslant [f(a)]^{1-t}[f(a) + \eta(f(b), f(a))]^{t}, \quad \forall a, b \in I, t \in [0, 1]. \tag{2.1}$$

If $t = \frac{1}{2}$ in (2.1), then

$$f(\frac{a+b}{2}) \leqslant \sqrt{[f(a)][f(a) + \eta(f(b), f(a))]}, \quad \forall a, b \in I, \tag{2.2}$$

which is known as generalized Jensen log-convex function.

From Definition 2.1, we have

$$\begin{aligned} f((1-t)a + tb) &\leqslant [f(a)]^{1-t}[f(a) + \eta(f(b), f(a))]^{t} \\ &= f(a) + t(\eta(f(b), f(a)), \quad \forall a, b \in I, \quad t \in [0, 1]. \end{aligned}$$

This means that the generalized log-convex functions are generalized convex functions. However the converse is not true, see [7].

Example 2.3 ([7]). Consider $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geqslant 0, \\ e^{x} & \text{if } x < 0, \end{cases}$$

and define a bifunction $\eta = -x - y$ for all $x, y \in \mathbb{R}^{-} = (-\infty, 0)$. Then f is generalized log-convex, but the converse is not true.

If $\eta(f(a), f(b)) = f(a) - f(b)$, then Definition 2.2 reduces to:

Definition 2.4 ([22]). A function $f : I \to [0, \infty)$ is said to be log-convex or multiplicatively convex if log (f) is convex, or equivalently if for all $x, y \in I$ and $t \in [0, 1]$, one has the inequality

$$f((1-t)a + tb) \leqslant [f(a)]^{1-t}[f(b)]^{t}, \quad \forall a, b \in I, \quad t \in [0, 1].$$

We will use the following notations throughout this paper.

1. Arithmetic mean:
$$A(a, b) = \frac{a+b}{2}, \quad \forall a, b \in \mathbb{R}_{+}.$$
2. Geometric mean:
$$G(a, b) = \sqrt{ab}, \quad \forall a, b \in \mathbb{R}_{+}.$$
3. Logarithmic mean:
$$L(a, b) = \frac{b-a}{\log b - \log a}, \quad \forall a, b \in \mathbb{R}_{+}, \quad a \neq b.$$
4. Quadratic mean:
$$K(a, b) = \sqrt{\frac{a^2 + b^2}{2}}, \quad \forall a, b \in \mathbb{R}_{+}.$$

3. Main results

In this section, we establish several new Hermite-Hadamard type integral inequalities for generalized log-convex functions.

Theorem 3.1. *Let* $f, g : I = [a, b] \to (0, \infty)$ *be generalized log-convex function on* I *and* a, $b \in I$ *with* $a < b$. *Then*

$$\begin{aligned} &2f(\frac{a+b}{2}) - \exp\left[\frac{1}{(b-a)}\int_a^b \log[f(x)+\eta(f(a+b-x), f(x))]dx\right] \\ &\leqslant \exp\left[\frac{1}{(b-a)}\int_a^b \log f(x)dx\right] \\ &\leqslant \left[\sqrt{f(a)f(b)[f(a)+\eta(f(b), f(a))][f(b)+\eta(f(a), f(b))]}\right]. \end{aligned}$$

Proof. Let f be generalized log-convex function on I. Then for all $a, b \in I, t \in [0, 1]$, we have

$$f((1-t)a+tb) \leqslant [f(a)]^{1-t}[f(a)+\eta(f(b), f(a))]^t.$$

This implies that

$$\log f((1-t)a+tb) \leqslant (1-t)\log[f(a)] + t\log[f(a)+\eta(f(b), f(a))]. \tag{3.1}$$

Integrating (3.1) with respect to t on [0,1], we have

$$\begin{aligned} \int_0^1 \log f((1-t)a+tb)dt &\leqslant \int_0^1 (1-t)\log[f(a)] + t\log[f(a)+\eta(f(b), f(a))]dt \\ &= \log\sqrt{[f(a)][f(a)+\eta(f(b), f(a))]}. \end{aligned}$$

Thus

$$\frac{1}{b-a}\int_a^b \log f(x)dx \leqslant \sqrt{[f(a)][f(a)+\eta(f(b), f(a)]}.$$

Using (2.2) and substituting $x = (1-t)a+tb$ and $y = ta+(1-t)b$, we have

$$f(\frac{a+b}{2}) \leqslant \sqrt{[f((1-t)a+tb)][f((1-t)a+tb)+\eta(f((1-t)b+ta), f((1-t)a+tb)]}.$$

From this it follows that

$$\begin{aligned} \log f(\frac{a+b}{2}) \leqslant \frac{1}{2}\Big\{ &\log[f((1-t)a+tb)] \\ &+ \log[f((1-t)a+tb)+\eta(f((1-t)b+ta), f((1-t)a+tb))]\Big\}. \end{aligned} \tag{3.2}$$

Integrating (3.2) with respect to t on [0,1], we have

$$\log f(\frac{a+b}{2}) \leqslant \frac{1}{2(b-a)}\int_a^b \log f(x) + \log[f(x)+\eta(f(a+b-x), f(x))]dx.$$

Thus

$$2\log f(\frac{a+b}{2}) - \frac{1}{(b-a)}\int_a^b \log[f(x)+\eta(f(a+b-x), f(x))]dx \leqslant \frac{1}{(b-a)}\int_a^b \log f(x)dx. \tag{3.3}$$

Again consider

$$f((1-t)a+tb) \leqslant [f(a)]^{1-t}[f(a)+\eta(f(b), f(a))]^t. \tag{3.4}$$

$$f((1-t)b+ta) \leqslant [f(b)]^{1-t}[f(b)+\eta(f(a),f(b))]^t. \tag{3.5}$$

Taking log on both sides of (3.4) and (3.5) and then adding, we have

$$\begin{aligned}\log f((1-t)a+tb)+\log f((1-t)b+ta) \leqslant (1-t)[&\log f(a)+\log f(b) \\ &+t\Big[\log[f(a)+\eta(f(b),f(a))] \\ &+\log[f(b)+\eta(f(a),f(b))]\Big].\end{aligned} \tag{3.6}$$

Integrating (3.6) over t on [0,1], we have

$$\begin{aligned}\frac{2}{b-a}\int_a^b \log[f(x)]dx &\leqslant \frac{\log f(a)+\log f(b)}{2}+\frac{1}{2}\Big[\log[f(a)+\eta(f(b),f(a))]+\log[f(b)+\eta(f(a),f(b))]\Big] \\ &= \log\sqrt{f(a)f(b)}+\log\sqrt{[f(a)+\eta(f(b),f(a))][f(b)+\eta(f(a),f(b))]} \\ &= \log\left[\sqrt{f(a)f(b)[f(a)+\eta(f(b),f(a))][f(b)+\eta(f(a),f(b))]}\right].\end{aligned} \tag{3.7}$$

Combining (3.3) and (3.7), we have

$$\begin{aligned}&2\log f(\frac{a+b}{2})-\frac{1}{(b-a)}\int_a^b \log[f(x)+\eta(f(a+b-x),f(x))]dx \\ &\leqslant \frac{1}{(b-a)}\int_a^b \log f(x)dx \leqslant \log\left[\sqrt{f(a)f(b)[f(a)+\eta(f(b),f(a))][f(b)+\eta(f(a),f(b))]}\right].\end{aligned} \tag{3.8}$$

Taking antilog on both sides of (3.8), we have

$$\begin{aligned}2f(\frac{a+b}{2})-\exp&\left[\frac{1}{(b-a)}\int_a^b \log[f(x)+\eta(f(a+b-x),f(x))]dx\right] \\ &\leqslant \exp\left[\frac{1}{(b-a)}\int_a^b \log f(x)dx\right] \\ &\leqslant \left[\sqrt{f(a)f(b)[f(a)+\eta(f(b),f(a))][f(b)+\eta(f(a),f(b))]}\right].\end{aligned}$$

This completes the proof. □

Corollary 3.2. *If $\eta(f(b),f(a)) = f(b)-f(a)$, then under the assumption of Theorem 3.1, we have Hermite-Hadamard inequality for log convex function*

$$f(\frac{a+b}{2}) \leqslant \exp\left[\frac{1}{(b-a)}\int_a^b \log f(x)dx\right] \leqslant \sqrt{f(a)f(b)}.$$

Theorem 3.3. *Let $f,g : I = [a,b] \to (0,\infty)$ be generalized log-convex function on I and a, $b \in I$ with $a < b$. Then*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(a+b-x)dx &\leqslant \frac{g(a)\{f(a)+\eta(f(b),f(a))\}+f(a)\{g(a)+\eta(g(b),g(a))\}}{2} \\ &\leqslant \frac{1}{2}\Big\{A[f(a)+\eta(f(b),f(a)),f(a)]L[f(a)+\eta(f(b),f(a)),f(a)] \\ &\quad +A[g(a),g(a)+\eta(g(b),g(a))]L[g(a),g(a)+\eta(g(b),g(a))]\Big\} \\ &\leqslant \frac{[f(a)+(f(a)+\eta(f(b),f(a)))]^2}{16}+\frac{[g(a)+(g(a)+\eta(g(b),g(a)))]^2}{16} \\ &\quad +\frac{g(a)\{f(a)+\eta(f(b),f(a))\}+f(a)\{g(a)+\eta(g(b),g(a))\}}{4},\end{aligned}$$

where A and L are Arithmetic and Logarithmic means respectively.

Proof. Let f, g be generalized log-convex function on I. Then for all $a, b \in I$, $t \in [0,1]$, we have

$$f(ta+(1-t)b) \leqslant [f(a)]^t[f(a)+\eta(f(a),f(b))]^{1-t},$$

$$g((1-t)a+tb) \leqslant [g(a)]^{1-t}[g(a)+\eta(g(b),g(a))]^t.$$

Consider,

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)g(a+b-x)dx &= \int_0^1 f(ta+(1-t)b)g((1-t)a+tb)dt \\
&\leqslant \int_0^1 \Big[[\{f(a)\}^t\{f(a)+\eta(f(b),f(a))\}^{1-t}] \\
&\quad \times [\{g(a)\}^{1-t}\{g(a)+\eta(g(b),g(a))\}^t]\Big]dt \\
&= g(a)(f(a)+\eta(f(b),f(a)))\int_0^1 \left[\frac{f(a)\{g(a)+\eta(g(b),g(a))\}}{g(a)\{f(a)+\eta(f(b),f(a))\}}\right]^t dt \\
&= \frac{\big[f(a)\{g(a)+\eta(g(b),g(a))\}-g(a)\{f(a)+\eta(f(b),f(a))\}\big]}{\log\big[f(a)\{g(a)+\eta(g(b),g(a))\}\big]-\log\big[g(a)\{f(a)+\eta(f(b),f(a))\}\big]} \\
&\leqslant \frac{g(a)\{f(a)+\eta(f(b),f(a))\}+f(a)\{g(a)+\eta(g(b),g(a))\}}{2} \\
&\leqslant \frac{1}{2}\int_0^1 [\{f(ta+(1-t)b)\}^2+\{g((1-t)a+tb)\}^2]dt \\
&\leqslant \frac{1}{2}\int_0^1 \Big\{[\{f(a)\}^t\{f(a)+\eta(f(b),f(a))\}^{1-t}]^2 \\
&\quad + [\{g(a)\}^{1-t}\{g(a)+\eta(g(b),g(a))\}^t]^2\Big\}dt \\
&= \frac{1}{2}\Big\{[f(a)+\eta(f(b),f(a))]^2\int_0^1\left[\frac{f(a)}{f(b)+\eta(f(a),f(b))}\right]^{2t}dt \\
&\quad + [g^2(a)]\int_0^1\left[\frac{g(a)+\eta(g(b),g(a))}{g(a)}\right]^{2t}dt\Big\} \\
&= \frac{1}{4}\Big\{\left[\frac{f^2(a)-[f(a)+\eta(f(b),f(a))]^2}{\log f(a)-\log[f(a)+\eta(f(b),f(a))]}\right] \\
&\quad + \left[\frac{[g(a)+\eta(g(b),g(a))]^2-g^2(a)}{\log[g(a)+\eta(g(b),g(a))]-\log[g(a)]}\right]\Big\} \\
&\leqslant \frac{1}{2}\Big\{A[f(a)+\eta(f(b),f(a)),f(a)]L[f(a)+\eta(f(b),f(a)),f(a)] \\
&\quad + A[g(a),g(a)+\eta(g(b),g(a))]L[g(a),g(a)+\eta(g(b),g(a))]\Big\} \\
&\leqslant \frac{1}{4}\int_0^1 [f(ta+(1-t)b+g((1-t)a+tb))]^2 dt \\
&= \frac{1}{4}\int_0^1 \Big\{[\{f(a)\}^t\{f(a)+\eta(f(b),f(a))\}^{1-t}]^2+[\{g(a)\}^{1-t}\{g(a)+\eta(g(b),g(a))\}^t]^2 \\
&\quad + 2[\{f(a)\}^t\{f(a)+\eta(f(b),f(a))\}^{1-t}][\{g(a)\}^{1-t}\{g(a)+\eta(g(b),g(a))\}^t]\Big\}dt \\
&= \frac{1}{4}\Big\{[f(a)+\eta(f(b),f(a))]^2\int_0^1\left[\frac{f(a)}{f(a)+\eta(f(b),f(a))}\right]^{2t}dt
\end{aligned}$$

$$
\begin{aligned}
&+ [g^2(a)] \int_0^1 \left[\frac{g(a)+\eta(g(b),g(a))}{g(a)}\right]^{2t} dt \\
&+ 2g(a)f(a)+\eta(f(b),f(a)) \int_0^1 \left[\frac{f(a)\{g(a)+\eta(g(b),g(a))\}}{g(a)\{f(a)+\eta(f(b),f(a))\}}\right]^t dt\Bigg\} \\
&= \frac{1}{8}\left\{\left[\frac{f^2(a)-[f(a)+\eta(f(b),f(a))]^2}{\log[f(a)]-\log[f(a)+\eta(f(b),f(a))]}\right]\right. \\
&\quad \left. + \left[\frac{[g(a)+\eta(g(b),g(a))]^2-g^2(a)}{\log[g(a)+\eta(g(b),g(a))]-\log g(a)}\right]\right\} \\
&\quad + \frac{1}{2}\left\{\frac{f(a)\{g(a)+\eta(g(b),g(a))\}-g(a)\{f(a)+\eta(f(b),f(a))\}}{\log[f(a)\{g(a)+\eta(g(b),g(a))\}]-\log\left[g(a)\{f(a)+\eta(f(b),f(a))\}\right]}\right\} \\
&\leqslant \frac{[f(a)+(f(a)+\eta(f(b),f(a)))]^2}{16} + \frac{[g(a)+(g(a)+\eta(g(b),g(a)))]^2}{16} \\
&\quad + \frac{g(a)\{f(a)+\eta(f(b),f(a))\}+f(a)\{g(a)+\eta(g(b),g(a))\}}{4}.
\end{aligned}
$$

This completes the proof. □

Corollary 3.4. *If* $\eta(f(b), f(a)) = f(b) - f(a)$, *then under the assumption of Theorem 3.3, we have*

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)g(a+b-x)dx &\leqslant \frac{g(a)f(b)+f(a)g(b)}{2} \\
&\leqslant \frac{1}{2}\Big\{A[f(b),f(a)]L[f(b),f(a)] + A[g(a),g(b)]L[g(a),g(b)]\Big\} \\
&\leqslant \frac{[f(a)+f(b)]^2}{16} + \frac{[g(b)+g(a)]^2}{16} + \frac{g(a)f(b)+f(a)g(b)}{4}.
\end{aligned}
$$

Theorem 3.5. *Let* $f, g : I = [a, b] \to (0, \infty)$ *be generalized log-convex functions on* I *and* $a, b \in I$ *with* $a < b$. *If* $\alpha + \beta = 1$, *then*

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)g(a+b-x)dx &\leqslant \alpha^2\left[\frac{[f(a)+\eta(f(b),f(a))]^{\frac{1}{\alpha}}-[f(a)]^{\frac{1}{\alpha}}}{[\eta(f(b),f(a))]}\right] L[f(a),f(a)+\eta(f(b),f(a))] \\
&\quad + \beta^2\left[\frac{[g(a)]^{\frac{1}{\beta}}-[g(a)+\eta(g(b),g(a))]^{\frac{1}{\beta}}}{[\eta(g(b),g(a))]}\right] L[g(a)+\eta(g(b),g(a)),g(a)],
\end{aligned}
$$

where L *is the Logarithmic mean.*

Proof. Let f and g be generalized log-convex function on I. Then for all $a, b \in I$, $t \in [0, 1]$, we have

$$
f((1-t)a+tb) \leqslant [f(a)]^{1-t}[f(a)+\eta(f(b),f(a))]^t,
$$

$$
g(ta+(1-t)b) \leqslant [g(a)]^t[g(a)+\eta(g(b),g(a))]^{1-t}.
$$

We now recall the Young's inequality, that is,

$$
ab \leqslant \alpha a^{\frac{1}{\alpha}} + \beta b^{\frac{1}{\beta}}, \quad \forall \alpha,\beta > 0, \quad \alpha+\beta = 1.
$$

Consider

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)g(a+b-x)dx &= \int_0^1 f((1-t)a+tb)g(ta+(1-t)b)dt \\
&\leqslant \int_0^1 \left\{\alpha[f((1-t)a+tb)]^{\frac{1}{\alpha}} + \beta[g(ta+(1-t)b)]^{\frac{1}{\beta}}\right\}dt
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \int_0^1 \left\{\alpha\{[f(a)]^{1-t}[f(a)+\eta(f(b),f(a))]^t\}\right\}^{\frac{1}{\alpha}} \\
&\quad + \left\{\beta\{[g(a)]^t[g(a)+\eta(g(b),g(a))]^{1-t}\}\right\}^{\frac{1}{\beta}} dt \\
&= \alpha[f(a)]^{\frac{1}{\alpha}} \int_0^1 \left[\frac{f(a)+\eta(f(b),f(a))}{f(a)}\right]^{\frac{t}{\alpha}} dt \\
&\quad + \beta[g(a)+\eta(g(b),g(a))]^{\frac{1}{\beta}} \int_0^1 \left[\frac{g(a)}{g(a)+\eta(g(b),g(a))}\right]^{\frac{t}{\beta}} dt \\
&= \alpha^2[f(a)]^{\frac{1}{\alpha}} \left[\frac{(\frac{f(a)+\eta(f(b),f(a))}{f(a)})^u}{\log \frac{f(a)+\eta(f(b),f(a))}{f(a)}}\right]_0^{\frac{1}{\alpha}} \\
&\quad + \beta^2[g(a)+\eta(g(b),g(a))]^{\frac{1}{\beta}} \left[\frac{(\frac{g(a)}{g(a)+\eta(g(b),g(a))})^u}{\log \frac{g(a)}{g(a)+\eta(g(b),g(a))}}\right]_0^{\frac{1}{\beta}} \\
&= \alpha^2\left[\frac{[f(a)+\eta(f(b),f(a))]^{\frac{1}{\alpha}}-[f(a)]^{\frac{1}{\alpha}}}{\log[f(a)+\eta(f(b),f(a))]-\log f(a)}\right] \\
&\quad + \beta^2\left[\frac{[g(a)]^{\frac{1}{\beta}}-[g(a)+\eta(g(b),g(a))]^{\frac{1}{\beta}}}{\log[g(a)]-\log[g(a)+\eta(g(b),g(a))]}\right] \\
&= \alpha^2\left[\frac{[f(a)+\eta(f(b),f(a))]^{\frac{1}{\alpha}}-[f(a)]^{\frac{1}{\alpha}}}{[\eta(f(b),f(a))]}\right] L[f(a),f(a)+\eta(f(b),f(a))] \\
&\quad + \beta^2\left[\frac{[g(a)]^{\frac{1}{\beta}}-[g(a)+\eta(g(b),g(a))]^{\frac{1}{\beta}}}{[\eta(g(b),g(a))]}\right] L[g(a)+\eta(g(b),g(a)),g(a)].
\end{aligned}
$$

This completes the proof. □

Corollary 3.6. *If $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $\eta(f(b),f(a)) = f(b)-f(a)$, then under the assumption of Theorem 3.5, we have*

$$\frac{1}{b-a}\int_a^b f(x)g(a+b-x)dx \leqslant \frac{1}{2}\big\{A[f(b),f(a)]L[f(a),f(b)] + A[g(b),g(a)]L[g(a),g(b)]\big\}.$$

Corollary 3.7. *If $\alpha = \frac{1}{4}$, $\beta = \frac{3}{4}$ and $\eta(f(b),f(a)) = f(b)-f(a)$, then under the assumption of Theorem 3.5, we have*

$$\frac{1}{b-a}\int_a^b f(a+b-x)g(x)dx \leqslant \frac{1}{16}\left[\frac{f^4(b)-f^4(a)}{f(b)-f(a)}\right]L[f(a),f(b)] + \frac{9}{16}\left[\frac{g^{\frac{4}{3}}(a)-g^{\frac{4}{3}}(b)}{g(a)-g(b)}\right]L[g(a),g(b)].$$

Theorem 3.8. *Let $f,g : I=[a,b] \to (0,\infty)$ be an increasing and generalized log-convex function on I and $a,b \in I$ with $a<b$. Then*

$$
\begin{aligned}
8L[g(a),g(a)+\eta(g(b),g(a))]f(\frac{a+b}{2}) \leqslant \frac{1}{b-a}\int_a^b & f^4(x)dx + K^2[g(a),g(a)+\eta(g(b),g(a))] \\
& A[g(a),g(a)+\eta(g(b),g(a))] \\
& L[g(a),g(a)+\eta(g(b),g(a))] + 8.
\end{aligned}
$$

Proof. Let f and g be generalized log-convex function on I. Then for all $a,b \in I$, $t \in [0,1]$, we have

$$f(ta+(1-t)b) \leqslant [f(a)]^t[f(a)+\eta(f(b),f(a))]^{1-t},$$

$$g((1-t)a+tb) \leqslant [g(a)]^{1-t}[g(a)+\eta(g(b),g(a))]^t.$$

Using the inequality,

$$8xy \leqslant x^4 + y^4 + 8, \quad \forall x, y \in \mathbb{R},$$

we have

$$\begin{aligned}&8f(ta+(1-t)b)[g(a)]^{1-t}t[g(a)+\eta(g(b),g(a))]^t\\&\qquad\leqslant f^4(ta+(1-t)b)+[g(a)]^{4(1-t)}[g(a)+\eta(g(b),g(a))]^t+8.\end{aligned}$$

Now integrating the above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}&8\int_0^1 f(ta+(1-t)b)[g(a)]^t[g(a)+\eta(g(b),g(a))]^{1-t}dt\\&\qquad\leqslant \int_0^1 f^4(ta+(1-t)b)dt+\int_0^1[g(a)]^{4t}[g(a)+\eta(g(b),g(a))]^{4-4t}dt+8.\end{aligned}$$

As f and g are increasing functions, then we have

$$\begin{aligned}&8\int_0^1 f(ta+(1-t)b)dt\int_0^1[g(a)]^t[g(a)+\eta(g(b),g(a))]^{1-t}dt\\&\qquad\leqslant \int_0^1 f^4(ta+(1-t)b)dt+\int_0^1[g(a)]^{4t}[g(a)+\eta(g(b),g(a))]^{4-4t}dt+8.\end{aligned}$$

From the above inequality, it is easy to observe that

$$\begin{aligned}\frac{8L[g(a),g(a)+\eta(g(b),g(a))]}{b-a}\int_a^b f(x)dx \leqslant{}& \frac{1}{b-a}\int_a^b f^4(x)dx+K^2[g(a),g(a)+\eta(g(b),g(a))]\\&A[g(a),g(a)+\eta(g(b),g(a))]\\&L[g(a),g(a)+\eta(g(b),g(a))]+8.\end{aligned}\tag{3.9}$$

Now using the L.H.S of Hermite Hadamard's inequality in (3.9), we have

$$\begin{aligned}8L[g(a),g(a)+\eta(g(b),g(a))]f(\frac{a+b}{2}) \leqslant{}& \frac{1}{b-a}\int_a^b f^4(x)dx+K^2[g(a),g(a)+\eta(g(b),g(a))]\\&A[g(a),g(a)+\eta(g(b),g(a))]\\&L[g(a),g(a)+\eta(g(b),g(a))]+8,\end{aligned}$$

where A, L and K are Arithmetic, Logarithmic and Quadratic means, respectively. □

Corollary 3.9. *If* $\eta(f(b),f(a)) = f(b)-f(a)$, *then under the assumption of Theorem* 3.8, *we have*

$$8L[g(a),g(b)]f(\frac{a+b}{2}) \leqslant \frac{1}{b-a}\int_a^b f^4(x)dx+K^2[g(a),g(b)]A[g(a),g(b)]L[g(a),g(b)]+8.$$

Theorem 3.10. *Let* $f,g : I=[a,b] \to (0,\infty)$ *be an increasing and generalized log-convex function on* I *and* a, $b \in I$ *with* $a<b$. *Then*

$$\begin{aligned}&f(\frac{a+b}{2})L[g(a),g(a)+\eta(g(b),g(a))]+g(\frac{a+b}{2})L[f(a),f(a)+\eta(f(b),f(a))]\\&\qquad\leqslant \frac{1}{b-a}\int_a^b f(x)g(x)dx + L\Big[f(b)g(a),[f(b)+\eta(f(a),f(b))][g(a)+\eta(g(b),g(a))]\Big].\end{aligned}$$

Proof. Let f and g be generalized log-convex functions on I. Then for all $a, b \in I, t \in [0,1]$, we have

$$f(ta+(1-t)b) \leqslant [f(a)]^t[f(b)+\eta(f(a),f(b))]^{1-t},$$

$$g((1-t)a+tb) \leqslant [g(a)]^{1-t}[g(a)+\eta(g(b),g(a))]^t.$$

Using the inequality,

$$(a-b)(c-d) \geqslant 0, \quad \forall a,b,c,d \in \mathbb{R}, \quad a<b, \quad c<d,$$

we have

$$\begin{aligned}f(ta+(1-t)b)\big[[g(a)]^{1-t}[g(a)+\eta(g(b),g(a))]^t\big] + [g((1-t)a+tb)]\big[[f(b)]^{1-t}[f(b)+\eta(f(a),f(b))]^t\big]\\ \leqslant f(ta+(1-t)b)g((1-t)a+tb)\\ + \big[[g(a)]^{1-t}[g(a)+\eta(g(b),g(a))]^t[f(b)]^{1-t}[f(b)+\eta(f(a),f(b))]^t\big].\end{aligned}$$

Now integrating the above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}&\int_0^1 [f(ta+(1-t)b)]\big[[g(a)]^{1-t}[g(a)+\eta(g(b),g(a))]^t\big]dt\\ &\quad+\int_0^1 [g((1-t)a+tb)]\big[[f(b)]^{1-t}[f(b)+\eta(f(a),f(b))]^t\big]dt\\ &\qquad\leqslant \int_0^1 [f(ta+(1-t)b)]g((1-t)a+tb)dt\\ &\qquad\quad+\int_0^1 \big[[g(a)]^{1-t}[g(a)+\eta(g(b),g(a))]^t[f(b)]^{1-t}[f(b)+\eta(f(a),f(b))]^t\big]dt.\end{aligned}$$

As f and g are increasing functions, we have

$$\begin{aligned}&\int_0^1 [f(ta+(1-t)b)]dt\int_0^1 \big[[g(a)]^{1-t}[g(a)+\eta(g(b),g(a))]^t\big]dt\\ &\quad+\int_0^1 [g((1-t)a+tb)]dt\int_0^1 \big[[f(b)]^{1-t}[f(b)+\eta(f(a),f(b))]^t\big]dt\\ &\qquad\leqslant \int_0^1 [f(ta+(1-t)b)]dt\int_0^1 [g((1-t)a+tb)]dt\\ &\qquad\quad+\int_0^1 [f(b)g(a)]^{1-t}\Big[[f(b)+\eta(f(a),f(b))][g(a)+\eta(g(b),g(a))]\Big]^t dt.\end{aligned}$$

Now after some simple integration, we have

$$\begin{aligned}&f(\frac{a+b}{2})L[g(a),g(a)+\eta(g(b),g(a))]+g(\frac{a+b}{2})L[f(a),f(a)+\eta(f(b),f(a))]\\ &\quad\leqslant \frac{1}{b-a}\int_a^b f(x)g(x)dx + L\Big[f(b)g(a),[f(b)+\eta(f(a),f(b))][g(a)+\eta(g(b),g(a))]\Big].\end{aligned}$$

This completes the proof. □

Corollary 3.11. *If* $\eta(f(b),f(a)) = f(b)-f(a)$, *then under the assumption of Theorem 3.10, we have*

$$f(\frac{a+b}{2})L[g(a),g(b)]+g(\frac{a+b}{2})L[f(a),f(b)] \leqslant \frac{1}{b-a}\int_a^b f(x)g(x)dx + L\big[f(b)g(a),f(a)g(b)\big].$$

Theorem 3.12. *Let* $f,g : I=[a,b] \to (0,\infty)$ *be an increasing and generalized log-convex functions on* I *and* a, $b \in I$ *with* $a<b$. *Then*

$$\frac{1}{b-a}\int_a^b f^2(x)dx + A[f(b),f(b)+\eta(f(a),f(b))]L[f(b),f(b)+\eta(f(a),f(b))]$$

$$
\begin{aligned}
&+ A[g(a), g(a)+\eta(g(b), g(a))]L[g(a), g(a)+\eta(g(b), g(a))] \\
&\quad \geqslant f(\frac{a+b}{2})L[f(a)+\eta(f(b), f(a)), f(a)] \\
&\quad\quad + L[g(a)(f(a)+\eta(f(b), f(a))), f(a)(g(a)+\eta(g(b), g(a)))] \\
&\quad\quad + f(\frac{a+b}{2})L[g(a), g(a)+\eta(g(b), g(a))].
\end{aligned}
$$

Proof. Let f and g be generalized log-convex functions on I. Then for all $a, b \in I,\ t \in [0, 1]$, we have

$$
f(ta+(1-t)b) \leqslant [f(a)]^t[f(a)+\eta(f(b), f(a))]^{1-t},
$$

$$
g((1-t)a+tb) \leqslant [g(a)]^{1-t}[g(a)+\eta(g(b), g(a))]^t.
$$

Using the inequality,

$$
x^2+y^2+z^2 \geqslant xy+yz+zx, \quad \forall x, y, z \in \mathbb{R},
$$

we have

$$
\begin{aligned}
&f^2(ta+(1-t)b)+[f(a)]^{2t}[f(a)+\eta(f(b), f(a))]^{2(1-t)t} \\
&\quad + [g(a)]^{2(1-t)}[g(a)+\eta(g(b), g(a))]^{2t} \\
&\quad\quad \geqslant \Big[f(ta+(1-t)b)[f(a)]^t[f(a)+\eta(f(b), f(a))]^{1-t}\Big] \\
&\quad\quad\quad + \Big[[f(a)]^t[f(a)+\eta(f(b), f(a))]^{1-t}[g(a)]^{1-t}[g(a)+\eta(g(b), g(a))]^t\Big] \\
&\quad\quad\quad + \Big[[g(a)]^{1-t}[g(a)+\eta(g(b), g(a))]^t f(ta+(1-t)b)\Big].
\end{aligned}
$$

Now integrating the above inequality with respect to t on $[0, 1]$, we have

$$
\begin{aligned}
&\int_0^1 f^2(ta+(1-t)b)dt+\int_0^1 \big[[f(a)]^{2t}[f(a)+\eta(f(b), f(a))]^{2(1-t)}\big]dt \\
&\quad + \int_0^1 \big[[g(a)]^{2(1-t)}[g(a)+\eta(g(b), g(a))]^{2t}\big]dt \\
&\quad\quad \geqslant \int_0^1 \big[f(ta+(1-t)b)[f(a)]^t[f(a)+\eta(f(b), f(a))]^t\big]dt \\
&\quad\quad\quad + \int_0^1 \big[[f(a)]^t[f(a)+\eta(f(b), f(a))]^{1-t}[g(a)]^{1-t}[g(a)+\eta(g(b), g(a))]^t\big]dt \\
&\quad\quad\quad + \int_0^1 \big[[g(a)]^{1-t}[g(a)+\eta(g(b), g(a))]^t f(ta+(1-t)b)\big]dt.
\end{aligned} \tag{3.10}
$$

To solve the integral in (3.10), let

$$
\begin{aligned}
\mathbb{A} &= \int_0^1 f^2(ta+(1-t)b)dt+\int_0^1 \big[[f(a)]^{2t}[f(a)+\eta(f(b), f(a))]^{2(1-t)}\big]dt \\
&\quad + \int_0^1 [[g(a)]^{2(1-t)}[g(a)+\eta(g(b), g(a))]^t]dt \\
&= \frac{1}{b-a}\int_a^b f^2(x)dx+A[f(a)+\eta(f(b), f(a)), f(a)] \\
&\quad L[f(a)+\eta(f(b), f(a)), f(a)]+A[g(a), g(a)+\eta(g(b), g(a))] \\
&\quad L[g(a), g(a)+\eta(g(b), g(a))],
\end{aligned}
$$

and

$$\begin{aligned}\mathbb{B} &= \int_0^1 \left[f(ta+(1-t)b)[f(a)]^t[f(a)+\eta(f(b),f(a))]^{1-t}\right]dt\\&\quad+\int_0^1 \left[[f(a)]^t[f(a)+\eta(f(b),f(a))]^{1-t}[g(a)]^{1-t}[g(a)+\eta(g(b),g(a))]^t\right]dt\\&\quad+\int_0^1 [g(a)]^{1-t}[g(a)+\eta(g(b),g(a))]^t f(ta+(1-t)b)dt\\&\geqslant f(\frac{a+b}{2})L[f(a)+\eta(f(b),f(a)),f(a)]\\&\quad+L[g(a)(f(a)+\eta(f(b),f(a))),f(a)(g(a)+\eta(g(b),g(a)))]\\&\quad+f(\frac{a+b}{2})L[g(a),g(a)+\eta(g(b),g(a))].\end{aligned}$$

Substituting the values of $\mathbb{A}$ and $\mathbb{B}$ in (3.10), we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f^2(x)dx+A[f(b),f(b)+\eta(f(a),f(b))]L[f(b),f(b)+\eta(f(a),f(b))]+A[g(a),g(a)\\&\quad+\eta(g(b),g(a))]L[g(a),g(a)+\eta(g(b),g(a))]\\&\qquad\geqslant f(\frac{a+b}{2})L[f(a)+\eta(f(b),f(a)),f(a)]\\&\qquad\quad+L[g(a)(f(a)+\eta(f(b),f(a))),f(a)(g(a)+\eta(g(b),g(a)))]\\&\qquad\quad+f(\frac{a+b}{2})L[g(a),g(a)+\eta(g(b),g(a))].\end{aligned}$$

This completes the proof. □

Corollary 3.13. *If* $\eta(f(b),f(a))=f(b)-f(a)$, *then under the assumption of Theorem* 3.12, *we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f^2(x)dx+A[f(b),f(a)]L[f(b),f(a)]+A[g(a),g(b)]L[g(a),g(b)]\\&\quad\geqslant f(\frac{a+b}{2})L[f(b),f(a)]+L[f(b)g(a),f(a)g(b)]+f(\frac{a+b}{2})L[g(a),g(b)].\end{aligned}$$

Acknowledgment

The authors would like to thank Rector, COMSATS Institute of Information Technology, Pakistan, for providing excellent research and academic environments.

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