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Obtain Multiplicative Resupinate Eigenvalue with use Hermitian Matrices

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Abstract

Benefiting from Schur theorem and temple's theories, we exposure new enough conditions for obtaining multiplicative resupinate eigenvalue with use Hermitian matrices.

1. Introduction

Let H_n be the set of Hermitian matrices of order n.:

$$(MH)LetA=(a_{ij})\in H_n$$

be a positive semi-definite matrix and $\lambda=(\lambda_1,\lambda_2...,\lambda_n)\in R^n$ be a nonnegative vector. The problem is to find a nonnegative diagonal matrix C such that the matrix CA has eigenvalues $\lambda_1,\lambda_2...,\lambda_n$. We assume in the problem that $a_{ii}=1(i-1,2,...n)$.

(GH) Let $A(a_{ij})$, $A_t=(a_{ij}^{(t)})\in H_n(t=1,...,n)$, and $\lambda=(\lambda_1,...,\lambda_n)\in R^n$. The problem is to find $c=(c_1,...c_n)\in R^n$ such that the matrix $A+\sum_{t=1}^{n}c_tA_t$ has eigenvalues $\lambda_1,...,\lambda_n$. We assume in the problem that $a_{ii}^{(i)}=\delta_{it}(i,t=1,...,n)$.

In this section, main results are introduced. Section 2 contains the proofs.

For $B=(b_{ij})\in H_n and b=(b_1,...,b_n)\in R^n, define:$

$$d(b)=\min_{i\neq j}\{|b_i-b_j|\}, \qquad \|b\|=\|b\|_\infty,$$

$$k_2(B)=\max_j\left\{\left(\sum_{i\neq j}|b_{ij}|^2\right)^{1/2}\right\}, \qquad m(B)=\min_{i\neq j}\{|b_{ij}|\}.$$

THEOREM 1. Let $A\in H_n$ be positive semi-definite with $a_{ii}=1(i=1,...,n) and\ 0\le\lambda_1<\lambda_2<...<\lambda_n$. Define

$$\phi\lambda=\begin{cases}\sqrt{\lambda_n(\lambda_{n-1}+...+\lambda_1)}.\\ \lambda_n\le\lambda_{n-1}+...+\lambda_1,\\ (\lambda_n+\lambda_{n-1}+...+\lambda_1)/2.\\ 0\le\lambda_n-(\lambda_{n-1}+...+\lambda_1)\le d(\lambda)/3,\\ \sqrt{[\lambda_n-d(\lambda)/6][\lambda_{n-1}+...+\lambda_1+d(\lambda)/6]},\\ \lambda_n\ge\lambda_{n-1}+...+\lambda_1+d(\lambda)/3.\end{cases}$$

Suppose

$$d(\lambda)\ge 2\sqrt{3}m(A)\phi(\lambda). \qquad (1.1)$$

Then (MH) is solvable.

THEOREM 2. Let A and $\lambda_i^{'}s$ be the same as in Theorem 1. Suppose

$$d(\lambda)\ge\sqrt{3}(\lambda_n+\lambda_{n-1})k_2(A). \qquad (1.2)$$

Then (MH) is solvable.

REMARK 1. Theorem 5 in [1] is contained in our Theorem 1 in the case when $\lambda_n\le\lambda_{n-1}+...+\lambda_1$, and in our Theorem 2.

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Conditions in [1,2, 8] show that λ_n the largest component of λ, plays a role in the solvability of (MH). In Theorems 1 and 2 we go further to show the effects of the smaller components of λ.

In problem (GH), let

$$a=(a_1,...,a_n)=(a_{11},...,a_{nn}),$$

$$A^{(0)} = A - diag(a_1, ..., a_n), \qquad A_t^{(0)} = A_t - diag(a_{11}^{(t)}, ..., a_{nn}^{(t)}),$$

$$\tilde{A} = A^{(0)} - \sum_{t-1}^{n} a_t A_t^{(0)}, \qquad S = \sum_{t=1}^{n} |A_t|;$$

here, for $B = (b_{ij})$, by $|B|$ we denote the matrix $(|b_{ij}|)$. Define

$$k = \|\lambda - a\| k_2(S) + k_2(A), \qquad k' = \|\lambda\| k_2(S) + k_2(\tilde{A}).$$

THEOREM 3. Let $A, A_t \in H_n$ with $a_{ii}^{(t)} = \delta_{it} (i, t = 1, ..., n)$ and $\lambda_1 < \lambda_2 ... < \lambda_n$. Suppose

(1.3) $$d(\lambda) \geq 2\sqrt{3} k'.$$

Then (GH) is solvable.

THEOREM 4. Let A, A_t, and $\lambda_i^{'} s$ be the same as in Theorem 3. Suppose $a_{11} \geq a_{22} \geq ... \geq a_{nn}$ and

(1.4) $$d(\lambda) \geq 2\sqrt{3} k.$$

Then (GH) is solvable.

REMARK 2. Considering a suitable congruent permutation of A an A_t and reordering of $\{A_t\}$, we see that the condition $a_{11} \geq ... \geq a_{nn}$ can always be satisfied in problem (GH). Theorem 4 improves substantially Theorem 8 in [5].

2. PROOF OF THE THEOREMS

We need a lemma deduced from Krylov, Bogoljubov, and Weinstein's and Temple's theories.

LEMMA 1 (See [5, Lemma 5]). Let $B=(b_{ij})\in H_n$ with $b_{11}\le\ldots\le b_{nn}$. Let $k_2(B)>0, d\ge 2k_2(B)$, and $|b_{ij}-b_{ii}|\ge d(1-\delta_{ij})$ for $i,j=1,\ldots n$. Then for the eigenvalues $\lambda_1\le\lambda_2\le\ldots\le\lambda_n$ of B

$$|\lambda_i-b_{ii}|\le\frac{d-\left[d^2-4k_2(B)^2\right]^{1/2}}{2}.$$

We also need the concept of majorization and the following

LEMMA 2 (See [6,p. 193]). Let $B=(b_{ij})\in H_n$. Then for the eigenvalues $\lambda_1,\ldots,\lambda_n$ of B,

$$(b_{11},\ldots,b_{nn})\vdash(\lambda_1,\ldots,\lambda_n),$$

Where $u\longleftarrow v$ means that the real $vector\ v$ is majorized by the real $vector\ u$.

Some properties of quadratic functions are helpful in the proof.

LEMMA 3. Let $Q_1(x)=x^2-p_1x+q_1, Q_2(x)=x^2-p_2x+q_2$ be polynomials with $p_1\ge p_2\ge 0$ and $q_2\ge q_1\ge 0$. Suppose Q_1 and Q_2 have real roots $x_1\le x_2$ and $y_1\le y_2$, respectively. Then $x_1\le y_1, i.e.$

$$\frac{p_1-(p_1^2-4q_1)^{1/2}}{2}\le\frac{p_2-(p_2^2-4q_2)^{1/2}}{2}.$$

Proof. It suffices to show $Q_1(y_1)\le 0$. In fact, since $y_1\ge 0$ obviously, then

$$\begin{aligned}Q_1(y_1)&=y_1^2-p_1y_1+q_1\\&=p_2y_1-q_2-p_1y_1+q_1\\&=(p_2-p_1)y_1-q_2+q_1\\&\le 0,\end{aligned}$$

and we get the result.

LEMMA 4. Let $y(x)=x(a-x)$ be a quadratic function defined on the interval $x\in[c,d]$. Then

$$\max\{y(x)|c \le x \le d\} = \begin{cases} y(d), & d \le a/2, \\ y(a/2), & c \le a/2 \le d, \\ y(c), & a/2 \le c. \end{cases}$$

Proof of Theorem 1. Let

$$\varepsilon = \frac{d(\lambda) - \left[d(\lambda)^2 - 12m(A)^2\phi(\lambda)^2\right]^{1/2}}{6}.$$

By the assumption $d(\lambda) \ge 2\sqrt{3}m(A)\phi(\lambda)$ we have

$$\varepsilon \le \frac{m(A)\phi(\lambda)}{\sqrt{3}}, \qquad \varepsilon \le \frac{d(\lambda)}{6}. \tag{2.1}$$

Define

$$K(\varepsilon,\lambda) = \{x \in R^n \,|\, \|x-\lambda\| \le \varepsilon\}, \qquad D(\lambda) = \{x \in R^n \,|\, x \vdash \lambda\}.$$

It can be verified that $V(\varepsilon,\lambda) = K(\varepsilon,\lambda) \cap D(\lambda)$ is a nonempty, bounded, convex, and closed set in R^n. For $x = (x_1,...,x_n) \in V(\varepsilon,\lambda)$ define the matrix $X = diag(x_1,...,x_n)$. Then X is nonnegative. Define $A(x) = X^{1/2}AX^{1/2}$. We know that XA and A(x) have the same set of eigenvalues, denoted by $\lambda_1(x) \le ... \le \lambda_n(x)$. Let $\lambda(x) = (\lambda_1(x),...,\lambda_n(x))$. Since $\varepsilon \le d(\lambda)/6$ and $\lambda_1 < ... < \lambda_n$, then $x_1 \le ... \le x_n$ for any vector $x \in V(\varepsilon,\lambda)$. With $x \vdash \lambda$ we have $x_1 + ... + x_n = \lambda_1 + ... + \lambda_n$ for $x \in V(\varepsilon,\lambda)$ and therefore

$$
\begin{aligned}
k_2(A(x))^2 &= \max_i \left\{ \sum_{j\neq 1} |a_{ij}|^2 x_i x_j \right\} \\
&\le m(A)^2 \max_i \left\{ x_i \sum_{j\neq 1} x_j \right\} \\
&= m(A)^2 x_n (x_1 + \ldots + x_{n-1}) \\
&= m(A)^2 x_n \left(\sum_{j=1}^{n} \lambda_j - x_n \right),
\end{aligned}
$$

Since $\lambda_n - d(\lambda)/6 \le x_n \le \lambda_n$, then from Lemma 4 we have

$$
x_n \left(\sum_j \lambda_j - x_n \right) \le \begin{cases} \lambda_n (\lambda_{n-1} + \ldots + \lambda_1), \\ \lambda_n \le (\lambda_1 + \ldots + \lambda_n)/2, \\ \dfrac{(\lambda_n + \ldots + \lambda_1)^2}{4}, \\ \lambda_n - \dfrac{d(\lambda)}{6} \le \dfrac{\lambda_1 + \ldots + \lambda_n}{2} \le \lambda_n, \\ \left(\lambda_n - \dfrac{d(\lambda)}{6} \right)\left(\lambda_{n-1} + \ldots + \lambda_1 + \dfrac{d(\lambda)}{6} \right), \\ \dfrac{\lambda_1 + \ldots + \lambda_n}{2} \le \lambda_n - \dfrac{d(\lambda)}{6}. \end{cases}
$$

Therefore

$$
k_2(A(x)) \le m(A)\phi(\lambda). \tag{2.2}
$$

By the assumption in Theorem 1 and (2.1), (2.2)

$$
\begin{aligned}
d(\lambda) &\ge 2\sqrt{3} m(A)\phi(\lambda) \\
&= 2m(A)\phi(\lambda) + (2 - 2/\sqrt{3})\sqrt{3} m(A)\phi(\lambda) \\
&\ge 2m(A)\phi(\lambda) + 2\varepsilon \\
&\ge 2k_2(A(x)) + 2\varepsilon.
\end{aligned} \tag{2.3}
$$

Besides, $d(x) \ge d(\lambda) - 2\varepsilon$ for $x \in K(\varepsilon, \lambda)$. Thus for $x \in V(\varepsilon, \lambda)$

$$
d(x) \ge d(\lambda) - 2\varepsilon \ge 2k_2(A(x)).
$$

Note that $x_1, \ldots x_n$ are diagonal elements of A(x). hence by Lemmas 1 and 3

$$\begin{aligned}\|x-\lambda(x)\| &\le \frac{d(x)-[d(x)^2-4k_2(A(x))^2]^{1/2}}{2}\\ &\le \frac{[d(\lambda)-2\varepsilon]-\{[d(\lambda)-2\varepsilon]^2-4m(A)^2\phi(\lambda)^2\}^{1/2}}{2}\\ &=\varepsilon. \qquad (2.4)\end{aligned}$$

To verify the late equality of (2.4). we note that ε satisfies

$$3\varepsilon^2-d(\lambda)\varepsilon+m(A)^2\phi(\lambda)^2=0,$$

Which is equivalent to

$$[d(\lambda)-4\varepsilon]^2=[d(\lambda)-2\varepsilon]^2-4m(A)^2\phi(A)^2.$$

Since $d(\lambda)-4\varepsilon\ge 0$ $[see(2.1): \varepsilon \le d(\lambda)/6 \le d(\lambda)/4]$. Then we have

$$d(\lambda)-4\varepsilon=\{[d(\lambda)-2\varepsilon]^2-4m(A)^2\phi(\lambda)^2\}^{1/2}.$$

Thus (2.4) can be verified.

Now define a continuous map $f(x): V(\varepsilon,\lambda)\to R^n$ with

(2.5) $$f(x)=\lambda+x-\lambda(x).$$

For the proof of Theorem 1 it suffices to show that f(x) has a fixed point in $V(\varepsilon,\lambda)$. (See [5].)

The inequality (2.4) means for $x\in(\varepsilon,\lambda)$

$$\begin{aligned}\|f(x)-\lambda\| &= \|x-\lambda(x)\|\\ &\le \varepsilon\\ &\le d(\lambda)/6:\end{aligned}$$

Thus $f(x) \in (\varepsilon, \lambda)$ and $f_1(x) \le \ldots \le f_n(x)$, where $f_i(x)$ is the ith component of f(x). Since $x \vdash \lambda(x)$ (Lemma 2) and $\{f_i(x)\}$, $\{x_i\}$, and $\{\lambda_i(x)\}$ are all in increasing order, it can be verified that $f(x) \vdash \lambda, i.e.\ f(x) \in D(\lambda)$. Therefore $f(x) \in V(\varepsilon, \lambda)$. Applying Brouwer's fixed-point theorem, we conclude that there is a fixed point $c = (c_1, \ldots c_n) \in V(\varepsilon, \lambda)$ such that $f(c) = c, i.e. \lambda(c) = \lambda.$ The proof of Theorem 1 is completed.

Proof of Theorem 2. We just give an outline for conciseness. Define

$$\varepsilon_1 = \frac{d(\lambda) - [d(\lambda)^2 - 3k_2(A)^2(\lambda_n + \lambda_{n-1})^2]^{1/2}}{6},$$

$V(\varepsilon_1, \lambda) = K(\varepsilon_1, \lambda) \cap D(\lambda)$, and consider the map $f(x): V(\varepsilon_1, \lambda) \to R^n$ with (2.5). for $x \in (\varepsilon_1, \lambda)$ we have

$$k_2(A(x))^2 = \max_j \left\{ x_i \sum_{j \ne i} |a_{ij}|^2 x_j \right\}$$

$$\le x_n x_{n-1} \max_i \left\{ \sum_{j \ne 1} |a_{ij}|^2 \right\},$$

(2.6)
$$k_2(A(x)) = \sqrt{x_n x_{n-1}} k_2(A)$$

$$\le k_2(A) \frac{x_n + x_{n-1}}{2}$$

$$\le k_2(A) \frac{\lambda_n + \lambda_{n-1}}{2}.$$

The value $k_2(A)(\lambda_n + \lambda_{n-1})/2$ plays the same role as $m(A)\phi(\lambda)$ in Theorem 1. Replacing ε and (2.2) by ε_1 and (2.6) in the proof of Theorem 1. Respectively, we can get the result by similar arguments.

Proof of Theorem 3. Let

$$\varepsilon' = \frac{d(\lambda) - [d(\lambda)^2 - 12k'^2]^{1/2}}{6}$$

It can be verified that

(2.7) $$\varepsilon' \le \frac{d(\lambda)}{6}, \qquad \varepsilon' \le \frac{k'}{\sqrt{3}}.$$

Define

$$K(\varepsilon',\lambda,a) = \{x \in R^n \,\|\, \|x + a - \lambda\| \le \varepsilon'\}.$$

$$D(\lambda, a) = \{x \in R^n | x + a \vdash \lambda\}.$$

It can be verified that $V(\varepsilon',\lambda,a) = K(\varepsilon',\lambda,A) \cap D(\lambda,a)$ is a nonempty. Bounded, convex, and closed set in R^n. With (2.7) and $\lambda_1 < \ldots < \lambda_n$ we have $x_1 + a_1 \le \ldots \le x_n + a_n$ for $x = (x_1, \ldots x_n) \in V(\varepsilon',\lambda,a)$. Let $A(x) = A + \sum_{t-1}^{n} x_t A_t$ for $x = (x_1, \ldots, x_n)$ in $V(\varepsilon',\lambda,a)$. By $\lambda_1(x) \le \ldots \le \lambda_n(x)$ we donote the eigenvalues of $A(x)$ .let $\lambda(x) = (\lambda_1(x), \ldots, \lambda_n(x))$, Since $A(x) = \mathring{A} + \sum_{t=1}^{n} (x_t + a_t) A_t$ and $x + a \vdash \lambda$, we have $|x_i + a_i| \le \|\lambda\|$ and therefore

(2.8) $$\begin{aligned} k_2(A(x)) &\le k_2(\mathring{A}) + \|\lambda\| k_2(S) \\ &= k'. \end{aligned}$$

Define the continuous map $f(x) : V(\varepsilon',\lambda,a) \to R^n$ with (2.5). for the proof of Theorem 3 it suffices to show that f has a fixed point in $V(\varepsilon',\lambda,a)$ (see [5]). Similarly to (2.3) we have

$$\begin{aligned} d(\lambda) &\ge 2k' + 2\varepsilon' \\ &\ge 2k_2(A(x)) + 2\varepsilon' \end{aligned}$$

With the assumption in Theorem 3. On the other hand, for $x \in V(\varepsilon',\lambda,a)$ we have

$d(x + a) \ge d(\lambda) - 2\varepsilon'$. Thus

(2.9)
$$\begin{aligned} d(x+a) &\geq d(\lambda)-2\varepsilon' \\ &\geq 2k' \\ &\geq 2k_2(A(x)). \end{aligned}$$

By Lemmas 1 and 3 we have $\|(x+a)-\lambda(x)\| \leq \varepsilon'$ for $x \in V(\varepsilon',\lambda,a)$. The deduction is similar to (2.4). On the other hand $\|f(x)+a-\lambda\| = \|(x+a)-\lambda(x)\|$.

Thus $f(x) \in K(\varepsilon',\lambda,a)$. With $\varepsilon' \leq d(\lambda)/6$ and $\lambda_1 < \dots < \lambda_n$ we have $f_1(x)+a_1 \leq \dots \leq f_n(x)+a_n$. Since $x+a \vdash \lambda(x)$, we can verify that $f(x)+a \vdash \lambda$ and therefore $f(x) \in K(\varepsilon',\lambda,a) \cap D(\lambda,a) = V(\varepsilon',\lambda,a)$. Brouwer's fixed-point theorem implies that there is a vector $c=(c_1,\dots,c_n)$ such that $f(c)=c$, i.e. $\lambda(c)=\lambda$. In other words, $A(c)=A+\sum_{t=1}^{n} c_t A_t$ has eigenvalues $\lambda_1,\dots\lambda_n$. The proof is completed.

Proof of Theorem 4. Define

$$\varepsilon'' = \frac{d(\lambda)-\left[d(\lambda)^2-12k^2\right]^{1/2}}{6}$$

$V(\varepsilon'',\lambda,a) = K(\varepsilon'',\lambda,a) \cap D(\lambda,a)$ and consider the map $f(x): V(\varepsilon'',\lambda,a) \to R^n$ with (2.5). For $x=(x_1,\dots x_n) \in V(\varepsilon'',\lambda,a)$ we have $\lambda_1 \leq x_i + a_i \leq \lambda_n \ (i=1,\dots,n)$, since $x+a \vdash \lambda$. Thus with the assumption $a_1 \geq \dots \geq a_n$ we have $\|x\| \leq \|\lambda - a\|$ and

(2.10)
$$\begin{aligned} k_2(A(x)) &\leq k_2(A)+\|x\| k_2(S) \\ &\leq k. \end{aligned}$$

Then replacing ε' and k' by ε'' and k in the proof of Theorem 3, respectively, we can get the result by similar arguments.

3. NUMERICAL EXAMPLES

EXAMPLE 1. Let $\lambda = (0,1,2)$ and

$$A = I + 0.19\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Consider problem (MH).

Apply Theorem 1. Since

$$d(\lambda) = 1, \qquad \phi(\lambda) = \sqrt{\left(\lambda_3 - \frac{d(\lambda)}{6}\right)\left(\lambda_2 + \lambda_1 + \frac{d(\lambda)}{6}\right)} = \frac{\sqrt{77}}{6}$$
$$m(A) = 0.19,$$

and thus $2\sqrt{3}m(A)\phi(\lambda) = 0.9625833 < d(\lambda) = 1$, we know by Theorem 1 that problem (MH) in this example is solvable. In fact C= diag (0, 1.081552, 1.9184448) is a numerical solution. We also see that the vector c= diag (C) satisfies $\|c - \lambda\| \le \varepsilon = 0.1401566$ and c $\longleftarrow \lambda$. This agrees with our theoretical analysis.

In some cases, reordering of the rows and columns of matrix A may affect the question of solvability when sufficient conditions shown in [2] and [8] are applied. (See also [3].) The matrix in Example 1, hewever, does not change under arbitrary congruent permutation, For this we use this kind of matrices in our numerical tests.

EXAMPLE Let $\lambda = (2.5, 5, 7.5, 10, 12.4)$ and A=I+0.039 B. where

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Consider problem (MH).

Apply Theorem 1. Since $\phi(\lambda) = \sqrt{\lambda_5(\lambda_4 + \lambda_3 + \lambda_2 + \lambda_1)} = 17.606816, m(A) = 0.039$, and

$$2\sqrt{3}m(A)\phi(\lambda) = 2.3786803 < d(\lambda) = 2.4,$$

The problem is solvable. C= diag (2.5197887, 5.0386475, 7.5492414, 10.039076, 12.252945) is a numerical solution. The vector c=diag(C) satisfies $\|c - \lambda\| < \varepsilon = 0.3468022$ and $c \vdash \lambda$.

EXAMPLE 3. Let $\lambda = (0, 0.333, 0.666, 1, 7)$ and A = 1 + 0.012B. Where B is the same as in Example 2. Consider problem (MH).

Apply Theorem 2. With $k_2(A)$=0.024 and

$$\sqrt{3}(\lambda_5 + \lambda_4)k_2(A) = 0.3325537 < d(\lambda) = 0.333$$

we know the problem is solvable. The matrix C = diag(0, 0.3332137, 0.6662960, 0.9998153, 6.9996749) is a numerical solution. The vector c = diag (C) is in $V(\varepsilon_1, \lambda)$, where $\varepsilon_1 = 0.0526277$.

REMARK 3. Examples 1-3 show that our results are not contained in those of [1], [2], or [8]. The following examples, however. Show the converse.

Let

$$\lambda=(0.5,1) \qquad and \qquad A=\begin{bmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{bmatrix}.$$

This example satisfies [2, Theorem 3] and [8, Theorem 2], but does not satisfy our Theorem 1 or 2.

Let

$$\lambda=(5,6,7) \qquad and \qquad A=I+\frac{1}{42}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This example (note that $\lambda_3<\lambda_1+\lambda_2$) satisfies [1, Theorem 5],

but does not satisfy our Theorem 1.

Our results do not contain those in [3], which can be applied to non-symmetric matrices.

EXAMPLE 4. Let $\lambda=(0,0.4), A=diag(-1,1), A_1=e_1e_1^T+0.05B$, and $A_2=e_2e_2^T+0.1B$, where e_i is the ith column of I_2 and $B=[e_2,e_1]$. Consider problem (GH).

Since S= 0.15B, A = -0.05B, then

$$\begin{aligned} 2\sqrt{3}k' &= 2\sqrt{3}\left[k_2(\tilde{A})+\|\lambda\|k_2(S)\right] \\ &= 0.3810511 \\ &< 0.4 = d(\lambda) \end{aligned}$$

By Theorem 3 problem (GH) is solvable. c = (1.0002507, -0.6002507) is a numerical solution. For this example, assumptions in [5, Theorem 5] are not satisfied.

REMARK 4. Theorem 6 in [5] is not contained in our Theorem 3. See the numerical example shown in [5].

Our theorems 3-4 are not contained in the results of [7] and vice versa.

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