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The use of Maple platform for the study of geodesics in the Schwarzschild de Sitter space-time

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Abstract

The article illustrates the graphical study of geodesic motion on curved space-time using the symbolic and graphical computation facilities of Maple platform.

Keywords: general relativity, computer algebra.

1. Introduction

Learning to solve the geodesic equations is an integral part of a first course on general relativity. However, most texts on general relativity do not contain a sufficient number of solvable examples that illustrate the behavior of geodesics. This lack is largely due to the fact that the geodesic equations are not easy to solve exactly. The use of Maple in general relativity and in the study of geodesic motion has a long history see [2] for list of articles using Maple and GrTensorII package.

The article illustrates the graphical study of geodesic motion on Schwarzschild de Sitter space-

time using the symbolic and graphical computation facilities of Maple platform + GrTensorII.

2. Geodesics in the Schwarzschild de Sitter space-time

In the standard Schwarzschild (t, r, θ, ϕ) , and geometric system of units, the Schwarzschild-de Sitter space times are determined by the line element

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Where M is the mass parameter of these space-time, And $\Lambda = \frac{3}{l^2}$ is positive cosmological constant, and the coordinates are defined such that $-\infty \leq t \leq \infty, r \geq 0, 0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. We can enter this metric directly in this form using ([1], [2]) make $g(\)$, giving it the name 'Sds':

```
> restart; grtw( ); makeg(Sds):
Make 2.0: GRTensor metric/basis entry utility
To quit makeg, type 'exit' at any prompt.
Do you wish to enter a 1) metric [g(dn,dn)],
                                2) line element [ds],
                                3) non-holonomic basis [e(1)...e(n)], or
                                4) NP tetrad [l,n,m,mbar]?

> 2:
Enter coordinates as a LIST (eg. [t,r,theta,phi]):
> [t,r,theta,phi]:
Enter the line element using d[coord] to indicate differentials.
(for example,      r^2*(d[theta]^2 + sin(theta)^2*d[phi]^2)
[Type 'exit' to quit makeg]
ds^2 =
> -(1-2*M/r-r^2/l^2)*d[t]*d[t]+d[r]*d[r]/(1-2*M/r-r^2/l^2)+r^2*
(d[theta]*d[theta]+((sin(theta))^2)*d[phi]*d[phi]);
> makeg completed;
```

The Christofel symbols in coordinates (t, r, θ, ϕ) are

$$\Gamma_{\tau\tau}^r = \frac{(-rl^2 + 2Ml^2 + r^3)(Ml^2 - r^3)}{r^3l^4}$$

$$\Gamma_{\tau\tau}^t = \Gamma_{\tau\tau}^r = \frac{Ml^2 - r^3}{r(-rl^2 + 2ml^2 + r^3)}$$

$$\Gamma_{r\theta}^\theta = \Gamma_{r\phi}^\phi = \frac{1}{r}$$

$$\Gamma_{\theta\theta}^r = \frac{-rl^2 + 2ml^2 + r^3}{l^2}$$

$$\Gamma_{\theta\phi}^\phi = \cot(\theta)$$

$$\Gamma_{\phi\phi}^r = -\sin(\theta)\cos(\theta)$$

Next, we shall define the 4-velocity and 4-acceleration, namely $v^i = \frac{dx^i}{d\tau}$; $a^i = \frac{dv^i}{d\tau}$ where τ is the proper reference frame of the particle. Similar to [2] we shall use an intermediate coordinate set $te(\tau)$, $er(\tau)$, $ph(\tau)$, $th(\tau)$ instead of the coordinates (t, r, θ, ϕ) fixes by GrTensorII, till we shall come back to Maple, where there is no possibility of confusion. We done as a series of GrTensorII definitions, we use some ideas from [1], [2] in our calculation.

```
> grdef('v^{i}:=[diff(te(tau),tau),diff(er(tau),tau),
diff(th(tau),tau),diff(ph(tau),tau)]');
> grcalc(v(up)); grdisplay(v(up));
> grdef('accel^{i}:=[diff(te(tau),tau,tau),diff(er(tau),tau,tau)
,diff(th(tau),tau,tau),diff(ph(tau),tau,tau)]');
> grcalc(accel(up)); grdisplay(accel(up));
> grdef('geo^{i}:=[diff(v^{i},tau)+Chr{j k}^{i}*v^{j}*v^{k}');
> grcalc(geo(up)); grdisplay(geo(up));
```

In this way we defined and computed a GrTensorII 4-vector, geo (dn) containing as components the four geodesic equations. The following commands calculate the next four geodesic equations, thus we obtained the next four geodesic equations:

```
> four:=grcomponent(geo(up),[phi]);
> one:=grcomponent(geo(up),[t]);
> two:=grcomponent(geo(up),[r]);
> three:=grcomponent(geo(up),[theta]);
```

```

> one:=subs(r=r(tau),one);
> two:=subs(r=r(tau),two);
> three:=subs(r=r(tau),three);
> four:=subs(r=r(tau),four);
> one:=subs(diff(er(tau),tau)=diff(r(tau),tau),diff(th(tau),tau)
    =diff(theta(tau),tau),diff(ph(tau),tau)
    =diff(phi(tau),tau),diff(te(tau),tau)=diff(t(tau),tau),one);
> two:=subs(diff(er(tau),tau)=diff(r(tau),tau),diff(th(tau),tau)
    =diff(theta(tau),tau),diff(ph(tau),tau)
    =diff(phi(tau),tau),diff(te(tau),tau)=diff(t(tau),tau),two);
> three:=subs(diff(er(tau),tau)=diff(r(tau),tau),diff(th(tau),tau)
    =diff(theta(tau),tau),diff(ph(tau),tau)
    =diff(phi(tau),tau),diff(te(tau),tau)=diff(t(tau),tau),three);
> four:=subs(diff(er(tau),tau)=diff(r(tau),tau),diff(th(tau),tau)
    =diff(theta(tau),tau),diff(ph(tau),tau)
    =diff(phi(tau),tau),diff(te(tau),tau)=diff(t(tau),tau),four);

```

Thus we obtained the next four geodesic equations:

$$\frac{\left(\frac{d^2}{d\tau^2}t(\tau)\right)(-r(\tau)^2)l^2 + 2\left(\frac{d^2}{d\tau^2}t(\tau)\right)r(\tau)Ml^2 + \left(\frac{d^2}{d\tau^2}t(\tau)\right)(r(\tau))^4\left(\frac{d^2}{d\tau^2}t(\tau)\right)}{r(\tau)\left(-r(\tau)l^2 - 2Ml^2 + (r(\tau))^3\right)} - \frac{2\left(\frac{d}{d\tau}t(\tau)\right)\left(\frac{d}{d\tau}r(\tau)\right)Ml^2 - 2\left(\frac{d}{d\tau}t(\tau)\right)\left(\frac{d}{d\tau}r(\tau)\right)(r(\tau))^3}{r(\tau)\left(-r(\tau)l^2 - 2Ml^2 + (r(\tau))^3\right)} = 0$$

$two = 0$

$$\frac{\left(\frac{d^2}{d\tau^2}th(\tau)\right)r(\tau) + 2\left(\frac{d}{d\tau}er(\tau)\right)\frac{d}{d\tau}th(\tau) - \sin(\theta)\cos(\theta)\left(\frac{d}{d\tau}ph(\tau)\right)^2 r(\tau)}{r(\tau)} = 0$$

$$\frac{\left(\frac{d^2}{d\tau^2}ph(\tau)\right)r(\tau)\sin(\theta) + 2\left(\frac{d}{d\tau}er(\tau)\right)\left(\frac{d}{d\tau}ph(\tau)\right)\sin(\theta) + 2\cos(\theta)\left(\frac{d}{d\tau}th(\tau)\right)\left(\frac{d}{d\tau}ph(\tau)\right)r(\tau)}{r(\tau)\sin(\theta)} = 0$$

The equation two is very tall so we replaced it by two. We can solve the equation one using the next Maple commands:

```
>bau:=diff(((1-2*M/r(tau)-(r(tau))^2/l^2))*diff(t(tau),tau),tau);
>expand(simplify(one*(-1+2*M/r(tau)+(r(tau))^2/l^2)+bau));
>one:=expand(simplify(one*(-1+2*M/r(tau)+(r(tau))^2/l^2)));
>ecuone:=(1-2*M/r(tau)-(r(tau))^2/l^2)*diff(t(tau),tau)-C1;
>diffte:=solve(ecuone,diff(t(tau),tau));
>ecuone:=expand(simplify(subs(diff(t(tau),tau)=diffte,ecuone)));
>one:=expand(simplify(subs(diff(t(tau),tau)=diffte,one)));
>one:=expand(simplify(one));
```

Obtaining for differ

$$\frac{dt}{d\tau} = -\frac{C_1 r(\tau) l^2}{-r(\tau) l^2 + 2M l^2 + (r(\tau))^3}$$

Where C_1 is a constant?

Now we will solve the four equations as:

```
>bau2:=diff(r(tau)^2*sin(theta(tau))^2*diff(phi(tau),tau),tau);
>expand(simplify(subs(sin(theta(tau))=sin(theta),cos(theta(tau))=
cos(theta),bau2/r(tau)^2/sin(theta(tau))^2)-four));
>ecufour:=r(tau)^2*sin(theta)^2*diff(phi(tau),tau)-C2;
>diffph:=solve(ecufour,diff(phi(tau),tau));
>ecufour:=expand(simplify(subs(diff(phi(tau),tau)=diffph,ecufour)));
>four:=expand(simplify(subs(diff(phi(tau),tau)=
subs(theta=theta(tau),diffph),subs(cos(theta)=
cos(theta(tau)),sin(theta)=sin(theta(tau)),four))));
>two:=subs(diff(t(tau),tau)=diffte,diff(phi(tau),tau)=diffph,two);
>three:=subs(diff(phi(tau),tau)=diffph,three);
```

Obtaining for diffph

$$\frac{d\phi}{d\tau} = \frac{C_2}{(r(\tau))^2 (\sin(\theta))^2}$$

where C_2 is a constant. To solve the last two remaining equation (two and three) is necessary to fix $\theta = \pi/2$, thus we have:

```
>bau3:=diff(r(tau)^2*diff(theta(tau),tau),tau)-
```

```

r(tau)^2*sin(theta)*cos(theta)*diffph^2;
>expand(simplify(bau3/r(tau)^2-three,trigsin));
>two;
>three;
>theta(tau):=Pi/2;
>two:=eval(subs(theta=Pi/2,eval(two)));
>three:=eval(subs(theta=Pi/2,eval(three)));

```

We observe that the second equation (three) is now canceled by the fixing of θ coordinate. Finally we have the equation two our purpose is from now one to split the above equation in two different ones, expressing the derivatives for r and t in terms of the angular coordinate ϕ .

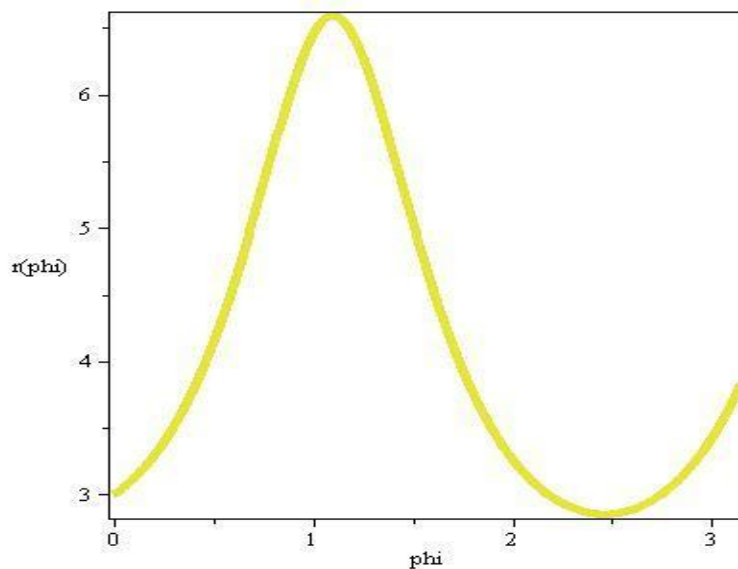


Figure 1:

```

>aba1:=C2*diff(r(phi),phi)/r(phi)^2;
>aba2:=C2^2/r(phi)^4*(diff(r(phi),phi,phi)-diff(r(phi),phi)^2/r(phi));

```

```
>two1:=subs(diff(r(tau),tau,tau)=aba2,diff(r(tau),tau)=aba1,r(tau)=r(phi),tw;
>two1:=expand(simplify(two1));
>two2:=diff(t(phi),phi)-C1*r(phi)^3/C2/(r(phi)-2*M);
```

Thus we obtained the next two differential equations:

$$\text{two1} = 0 \quad (1)$$

$$\text{two2} = 0 \quad (2)$$

We used the Maple commands sequence below for numerical integration of the system of equations (2, 3) we used $M = 1$, $C_1 = 1$, $C_2 = \pi$.

```
>with(DEtools);
>two1graph:=subs(C2=Pi,C1=1,M=1,l=10,two1);
>ini1:=r(0)=3,D(r)(0)=1;
>two2graph:=subs(M=1,C2=Pi,C1=1,two2);
>ini2:=t(0)=0;
>DEplot(two1graph,r(phi),phi=0..Pi,[[ini1]],method=classical,
axes=BOXED,thickness=5,stepsize=0.001);

>DEplot3d([two1graph,two2graph],[r(phi),t(phi)],-Pi/4..Pi,[[ini1,ini2]],
method=classical,stepsize=0.001,axes=BOXED,thickness=5);
```

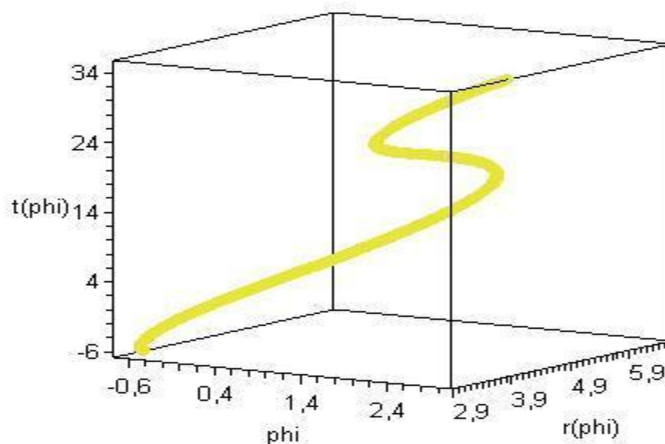


Figure 2:

References

- [1] Heal, K. M., Hansen, M., Rickard, K., Maple V Learning Guide for Release 5, Springer Verlag, 1997, ISBN 0-387-98397-X, see also the references from <http://www.maplesoft.com/maplebooks.html>
- [2] Musgrave P., Pollney D. and Lake K. ,“GRTensorII: A package for General Relativity” (To appear in the Proceedings of the 6th Canadian Conference in General Relativity and Relativistic Astrophysics), 1996, see also <http://grtensor.org>
- [3] Dumitru N. Vulcanov., Valentina D. Vulcanov., The use of Maple platform for the study of geodesic motion on curved spacetimes. 0 – 7695 – 2740 -X/0620.00 2006 IEEE.