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Simulating Nonhomogeneous Poisson Point Process Based on Multi Criteria Intensity Function and Comparison with Its Simple Form

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Abstract

In this paper we first study the general nonhomogeneous Poisson point process based on strict form of an intensity function and its algorithm for generating it. Then, we employ multi criteria intensity function instead of simple form and establish a new algorithm, then we compare the efficiency of our new algorithm based on this modified intensity function.

Keywords: Intensity function, Nonhomogeneous Poisson point process, Simulation.

1. Introduction

It is well known that one of the most important and famous point processes is nonhomogeneous Poisson process. Let us first study this process:

1.1. The nonhomogeneous Poisson process

From a modeling point of view the major weakness of the Poisson process is its assumption that events are just as likely to occur in all intervals of equal size. A generalization, which relaxes this assumption, leads to the nonhomogeneous or non-stationary process. If "events" are occurring randomly in time, and $N(t)$ denotes the number of events that occur by time t , then we say that $\{N(t)\ ,\ t \geq 0\}$ constitutes a nonhomogeneous Poisson process with intensity function $\lambda(t), t \geq 0$,if

(a) $N(0) = 0$,

(b) The numbers of events that occur in disjoint time intervals are

Independent,

(c) $\lim_{h\to 0}\frac{P\{exactly\ 1\ event\ between\ t\ and\ t+h\}}{h}=\lambda(t)$,

(d) $\lim_{h\to 0}\frac{P\{2\ or\ more\ events\ between\ t\ and\ t+h\}}{h}=0.$

In this step, we bring a brief review of simulating the nonhomogeneous Poisson point process [1] that we need it in this paper. Then, we discuss on main method which we desire to use it.

Theorem 1: Suppose that events are occurring according to a Poisson process having rate λ , and suppose that, independently of anything that came before, an event that occurs at time t is counted with probability $p(t)$. Then the process of counted events constitutes a nonhomogeneous Poisson process with intensity function $\lambda(t)=\lambda p(t)$ [4] .

1.2. Generating a nonhomogeneous Poisson process (NHPP)

Suppose $N(t)$ is a nonhomogeneous Poisson process with intensity $\lambda(t)$ and that there exists a λ such that $\lambda(t)\le\lambda$ for all $t\le T$. Then we can use the following algorithm, based on above theorem, to simulate $N(t)$.

Algorithm 1: The thinning algorithm for simulating T time units of a NHPP

Step 1: $t=0$ $,I=0$.

Step 2: Generate a random number U .

Step 3: $t=t-\frac{1}{\lambda}\log U$. If $t>T$, stop.

Step 4: Generate a random number U .

Step 5: If $U\le\frac{\lambda(t)}{\lambda}$, set $I=I+1$, $S(I)=t$.

Step 6: Go to Step 2.

In the above $\lambda(t)$ is the intensity function and λ is such that $\lambda(t)\le\lambda$. The final value of I represents the number of events time T , and $S(1),\dots,S(I)$ are the events times. The above procedure, referred to as the thinning algorithm (because it “thins” the nonhomogeneous Poisson points), is clearly most efficient, in the sense of having the fewest number of rejected events times, when $\lambda(t)$ is near λ throughout the interval [5].

Thus, an obvious improvement is to break up the interval into subintervals and then use the procedure over each subinterval.

Now, with regards to the above explanations we establish a new algorithm based on multi criteria intensity function and we try to investigate on the efficiency of this algorithm.

2. Method

Here, we generate first *T* time units of nonhomogeneous Poisson process based on multi criteria intensity function.

That is, determine appropriate values k , $0=t_0<t_1<t_2<\dots<t_k<t_{k+1}=T$, $\lambda_1,\dots,\lambda_{k+1}$ such that

$$\lambda(s) \le \lambda_i \qquad if \;\; t_{i-1} \le s < t_i \;,\;\; i = 1,\ldots,k+1 \tag{1}$$

Now, we generate the nonhomogeneous Poisson process over the interval (t_{i-1}, t_i) by generating exponential random variables with rate λ_i, and accepting the generated event occurring at time s, $s \in (t_{i-1}, t_i)$, with probability $\frac{\lambda(s)}{\lambda_i}$. Because of the memoryless property of the exponential and the fact that the rate of an exponential can be changed upon multiplication by a constant, it follows that there is no loss of efficiency in going from one subinterval to the next. That is, if we are at $t \in (t_{i-1}, t_i)$ and generate X, an exponential with rate λ_i, which is such that $t + X > t_i$, then we can use $\frac{\lambda_i [X - (t_i - t)]}{\lambda_{i+1}}$ as the next exponential with rate λ_{i+1}.We thus have the following algorithm for generating the first T time units of a nonhomogeneous Poisson process with intensity function $\lambda(s)$ when the relations **(1)** are satisfied. In the algorithm t represents the present time, J the present interval $(i.e.,\; J = j \;\; when \; t_{j-1} \le t < t_j)$, I the number of events so far, and $S(1),\ldots,S(I)$ the events times.

Algorithm 2: Simulation the first T time units of a NHPP

Step 1: $t = 0, J = 1, I = 0.$

Step 2: Generate a random number U and set $X = \frac{-1}{\lambda_J}\log U$.

Step 3: If $t + X > t_J$, go to Step 8.

Step 4: $t = t + X$.

Step 5: Generate a random number U .

Step 6: If $U \le \frac{\lambda(t)}{\lambda_J}$, set $I = I + 1, S(I) = t$.

Step 7: Go to Step 2.

Step 8: If $J = k + 1$, stop.

Step 9: $X = \frac{(X - t_J + t)\lambda_J}{\lambda_{J+1}}, t = t_J, J = J + 1.$

Step 10: Go to Step 3.

Suppose now that over some subinterval (t_{i-1}, t_i) we have that $\lambda_i > 0$, where

$$\lambda_i \equiv inf\, \{\lambda(s) : t_{i-1} \le s < t_i\}$$

In such a situation we should not use the thinning algorithm (**algorithm1**) directly but rather should first simulate a Poisson process with rate λ_i over the desired interval and then simulate a nonhomogeneous Poisson process with the intensity function $\lambda(s) = \lambda(s) - \lambda_i$ when $s \in (t_{i-1}, t_i)$. (The final exponential generated for the Poisson process, which carries one beyond the desired boundary, need not be wasted but can be suitably transformed so as to be reusable.) The superposition (or merging) of the two processes yields the desired process over the interval. The reason for doing it, this way is that it saves the need to generate uniform random variables for a Poisson distributed number, with mean $\lambda_i (t_i - t_{i-1})$, of the event times. For example, consider the case where

$$\lambda(s)=10+s \quad , \quad 0<s<1$$

Using the thinning method with $\lambda=11$ would generate an expected number of 11 events, each of which would require a random number to determine whether or not it should be accepted. On the other hand, to generate a Poisson process with rate 10 and then merge it with a nonhomogeneous Poisson process with rate $\lambda(s)=s, 0<s<1$ (generated by the thinning algorithm with $\lambda=1$), would yield an equally distributed number of event times but with the expected number needing to be checked to determine acceptance being equal to 1 .

3. Numerical result

Now, we bring an example based on algorithm 2 and present its results in the following figures.

Example: We write a program that uses **algorithm 2** to generate the first 10 time units of a nonhomogeneous Poisson process with intensity function $\lambda(t)$ such as:

$$\lambda(t)=\begin{cases} \frac{t}{5} & 0<t<5 \\ 1+5(t-5) & 5<t<10 \end{cases}$$

We consider this example for two individual cases:

1- Directly based on the intensity function $\lambda(t)$.

2- Based on probability function $p(t)$.

In the first case the following graph has been concluded.

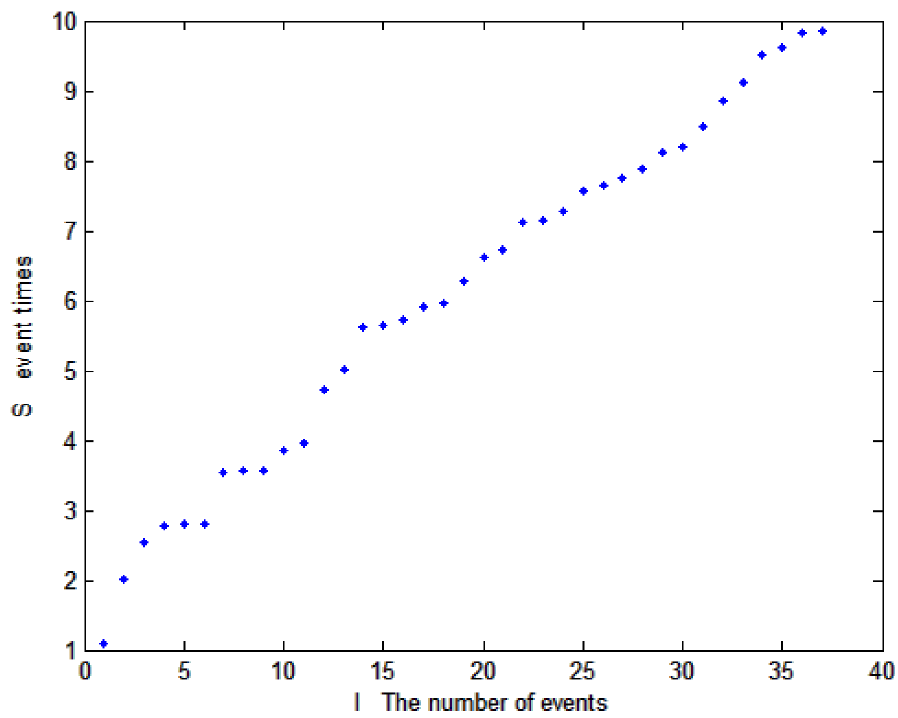

Figure 1. $(\lambda=[\frac{5}{2},\frac{135}{2}]$, $t_J=[5,10]$, $\lambda(t)=[\frac{t}{5},1+5(t-5)])$

Based on our new procedure we obtain λ as supremum of the case study and since $k=1$ then λ_1,λ_2 are equal to $\frac{5}{2},\frac{135}{2}$, also we have t_1,t_2 equal to $5,10$, respectively.

With regards to the above graph during 60 seconds the final value of I is equal to 37 and $S(I) = 9.6881$.

In the case 2, the corresponding figure is given below:

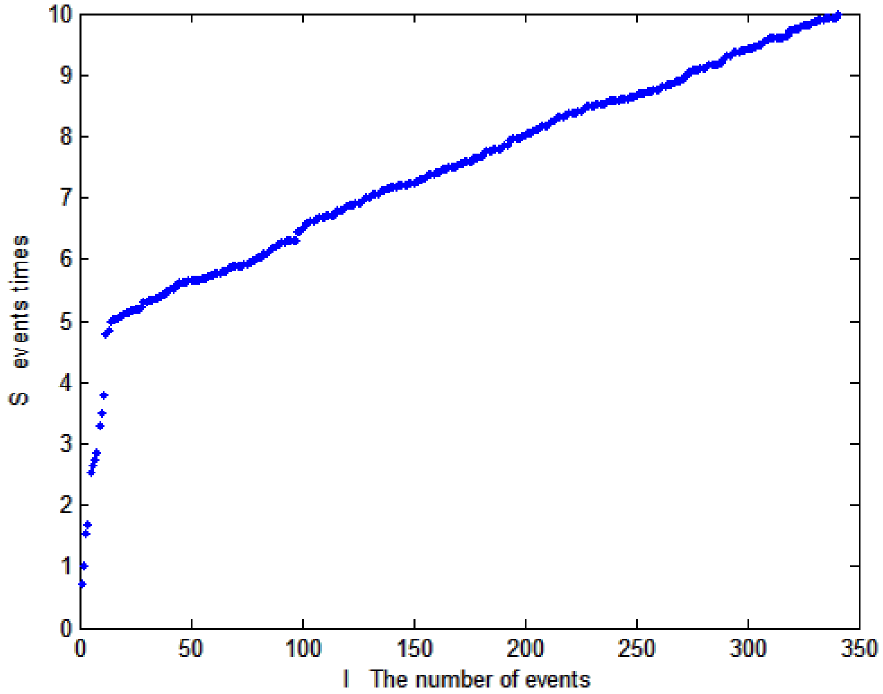

Figure 2. $(\lambda = [\frac{5}{2}, \frac{135}{2}]$, $t_J = [5,10]$, $p(t) = [\frac{2}{5}(\frac{t}{5}), \frac{2}{135}(1+5(t-5))])$

In this case based on **theorem 1**, $p_1(t), p_2(t)$ are obtained as below:

$$\lambda_1(t) = \frac{t}{5} = \underbrace{\frac{5}{2}}_{\lambda_1} \underbrace{(\frac{2}{5}(\frac{t}{5}))}_{p_1(t)} \qquad 0 < t < 5$$

$$\lambda_2(t) = 1+5(t-5) = \underbrace{\frac{135}{2}}_{\lambda_2} \underbrace{(\frac{2}{135}(1+5(t-5)))}_{p_2(t)} \qquad 5 < t < 10$$

As we can see from the above graph, the final value of I is equal to 341 and $S(I) = 9.9991$ during 15 seconds.

As we can see in Fig 2, there is increasing relationship between the number of events I and the events times $S(I)$. Moreover, based on the presented graph, comparing the rate of convergence of the above cases, we conclude that in case 2 we has much better smoothness in convergence, accuracy and it has higher speed to convergence. By the other word, when we employ the probability function the incrementing the events times behavior more regular and has faster convergence to tends to the desired events.

Now, if we compare the efficiency of the new algorithm with the thinning or random sampling method and its corresponding algorithm as presented in [1], we conclude that when we employ multi criteria intensity function the events times have grown better than the case we use the simple form intensity function.

4. Conclusion

We have presented performances of the new algorithm for simulating the nonhomogeneous Poisson point processes based on multi criteria $\lambda(t)$. We concluded that this algorithm has provided more smoothness and speed to converge to desired events which we have desired to reach it.

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