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# An optimal regulator design for fractional order linear systems with input time-delay

Saman Zahiri Rad[1], Saeed Balochian[1*]

[1]*Department of Electrical Engineering, Islamic Azad University, Gonabad branch, Iran.*

*saman.zahiri.rad@gmail.com, saeed.balochian@gmail.com*

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## *Abstract*

In this paper, an optimal regulator design for fractional order linear systems with input time-delay is developed. Fractional systems are very sensitive to delay so delay is a critical factor. Here the input time-delay in linear fractional-order system is applied and the response shows an improvement of less than 0.001 ms. The input time-delay is considered at the beginning of the control design and no approximation and estimation are used in the control system Thus the system performance and stability can be guaranteed. Instability in responses might occur if a system with input time-delay is controlled by an optimal regulator design for fractional linear system that was designed with no consideration of input time-delay. The transformation model which is first presented, change the optimal regulator design for fractional order linear systems with input time-delay into a system without delay formally. Simulation graphs demonstrate better performance of the proposed optimal regulator design by fractional order with consideration of the value criterion.

***Keywords***: fractional calculus, optimal regulator design, time delay systems, right –sided, fractional equation

## 1. Introduction

Time delay systems (shortly, TDS) are also called systems with aftereffect or dead-time, hereditary systems, equations with deviating argument or differential-difference equations. The reference books on time delay systems [1-3] note that finding an explicit form of a particular optimal control function might still remain difficult. We know the optimal control problem for linear systems with delays is a continuing study and dealing with delay type, specific system equations, cost function, etc. Thus the study of the optimal regulator control problem for systems with delay has a long history of over 20 years.

On the other hand, nowadays, the control of fractional-order system is one of the most popular topics in control theory. The fractional order calculus can be used in various systems such as biological reactive systems [4], physical [5] and electrochemical processes [6], mechatronics [7] and many others. There are several books which provide a good source of references on fractional calculus and its applications [8-10]. Furthermore, the fractional order controllers are also developed such as sliding mode control (SMC) [11], CRONE controllers [12], the fractional-order PID controller [13], the fractional-order robust PID controller [14-15].

However, there are fewer contributions available in the problem of an optimal regulator design for fractional order linear system with input time-delay. In [16], some results are obtained without using input time-delay. We develop in this work, new results on the optimal regulator design for fractional order linear system with input time-delay.

In this paper, an optimal regulator design is proposed for fractional order linear systems with input time-delay, where the input time-delay is considered in the fractional dynamic system .In the proposed controller approach, through a particular transformation, the fractional dynamic system with the input time-delay control system is first reformulated to a standard form of a fractional-order differential equation that contains no input time-delay, then a fractional order optimal regulator design is designed according to a classical optimal regulator design fractional order theory. Since an integral term appears in the expression of the obtained an optimal regulator design fractional order, in order to implement the proposed control method, a numerical algorithm is developed. In the proposed controller, the system performance and stability can be guaranteed since using the input time-delay can affect in the fractional dynamic system throughout derivation of the control algorithm. Finally, the performances of the controller order-fractional have been evaluated with three conditions (initial conditions, pulse and both). The rest of the paper is organized as follows. In section 2, gives a brief summary of the basic definitions related to fractional derivative. In section 3, properties of fractional integral are given. In section 4, the algorithms for the fractional system with input time-delay are given. Then the simulation examples of application are given in section 5. In section 6, three examples are presented to illustrate the theoretical developments. In section 7, we draw conclusions on the new results.

## 2. Basic Definitions

We briefly recall the definitions of Riemann–Liouville, Caputo and Riesz potential fractional derivatives, as well as their main properties [10, 21].

**Definition 2.1:** Let $f$ be a continuous and integrable function in the interval $[a,b]$ (for all $t \in [a,b]$).
The left Riemann–Liouville fractional derivative is defined by:

$$ {}_aD^{\alpha} f\left(t\right) = \frac{1}{\Gamma\left(n - \alpha\right)} \left(\frac{d}{dt}\right)^n \int_a^t \left(\tau - t\right)^{n-\alpha-1} f\,(s)\,ds \qquad (1)$$

$\alpha$ is the order of the derivative such that $n-1 \leq \alpha < n$ and $\Gamma$ the Euler gamma function.

**Remark 2.1:** If $\alpha$ is an integer, then $D_t^{\alpha} f\,(t) = \left(\dfrac{d}{dt}\right)^{\alpha} f\,(t)$.

**Definition 2.2:** Let us denote the Riemann–Liouville fractional derivative as ${}_a^{RL}D_t^{\alpha} f\,(t)$ and the caputo definition as $\binom{c}{a} D_t^{\alpha} f\,(t)$ and then the relationship between them is:

$$ {}_a^{RL}D_t^{\alpha} f\left(t\right) = {}_a^{RL}D_t^{\alpha} f\left(t\right) + \sum_{k=0}^{n-1} \frac{\left(t-a\right)^{k-a}}{\Gamma\left(k-a+1\right)} f^{\,(k)}\left(a\right) \qquad (2).$$

**Definition 2.3:** The fractional Riesz potential (for $\alpha \in (0,1)$ and based on R-L definition) is given as follows

$$ R_b^{\alpha} f\left(t\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^b \frac{f\left(\tau\right)}{\left|t-\tau\right|^{\alpha}}\, d\tau \qquad (3).$$

## 3. Properties of fractional integral

In this section the main definitions and some of its properties used in this paper are presented. More information about them can be found in [15], [22] and [23]. At the beginning the definitions of fractional (non-integer) order derivatives are given.

**Property 3.1:** For the scalar functions $f\,(t)$ and $g\,(t)$,

$$ \int_a^b f\left(t\right)\left[{}_0D_t^{-\alpha}\, g\left(t\right)\right]dt = \int_a^b g\left(t\right)\left[{}_tD_b^{-\alpha}\, f\left(t\right)\right]dt \qquad \alpha \geq 0 \qquad \left(4\right).$$

**Property 3.2:** Denoting $k-\alpha=\nu(k-1<\nu<k)$ yields

$$\int_a^b f(t)\,{}_aD_t^{\nu} g(t)\,dt=\sum_{r=0}^{k-1}(-1)^r[f^{(r)}(t)\ \ {}_aD_t^{\nu-1-r} g(t)]_{t=a}^{t=b}+(-1)^k\int_a^b g(t)\left[{}_t^C D_b^{\nu} f(t)\right]dt \quad (5)$$

the integration by parts formula is another important property, which is given as follows:

$$\int_a^b f(t)\left[{}_t^C D_b^{\alpha} g(t)\right]d(t)=\left[f(t)\,{}_tD_b^{\alpha-1} g(t)\right]_{t=a}^{t=b}+\int_a^b g(t)\left[{}_aD_t^{\alpha} f(t)\right]dt \quad (6).$$

## 4. Fractional system with input time-delay

Why we use the time-delay of the input only in optimal control for fractional order linear system?

The behavior features and the structural characteristics of delay systems are particularly enough to justify specific techniques. There are most of the papers dealing with optimal control for fractional order linear systems with free terminal time [17], several state delay [18], infinite delay [19] and delayed arguments [20] but none of them discussed thoroughly the input time-delay.

In order to study the fractional order system with considering input time-delay ،The system dynamics can be written as:

$$D^{\propto}X(t)=AX(t)+BU(t)+B_dU(t-\tau) \quad (7)$$

$$y(t)=CX(t) \quad (8)$$

the input time-delay $\tau\in[0,\infty)$ is considered to be constant then

$$J=\int_0^{tf}\frac{1}{2}\left[X^TQX+U^TRU\right]dt \quad (9).$$

**Lemma 1*:*** consider the state transformation

$$Z(t)=X(t)+\int_{t\text{-}\tau}^{t} e_\alpha^{A(t\text{-}s\text{-}\tau)}B_dU(s)ds \quad (10).$$

The system dynamics in Eq. (7) can be rewritten into a standard form of first-order differential equation without any explicit input time-delay term as

$$ {}_0^cD_t^{\alpha}Z(t)=AZ(t)+\hat{B}U(t) \quad (11)$$

$$y(t)=CZ(t) \quad (12)$$

where

$$\hat{B}=B+e_\alpha^{-A\tau}B_d \quad (13).$$

It is obvious that the control system specified by Eq. (11) is stable and controllable if Eq. (7) is a stable and controllable system.

## 5. Formulation of the optimal regulator design for fractional order linear systems

The cost function for the control system specified by Eq. (11) can be defined as

$$J=\int_0^{tf}\frac{1}{2}\left[Z^TQZ+U^TRU\right]dt \quad (14)$$

where $Q$ is positive-semi definite coefficient matrix, and $R$ is positive-definite coefficient matrix and the relative importance of the system response vector $Z(t)$ and the control input vector $U(t)$ respectively. To minimize the objective function $J$ the optimal controller can be obtained as

$$U(t) = -R^{-1}\hat{B}^T P\, Z(t) \qquad (15),$$

where P is the positive definite solution of the Riccati equation:

$$A^T P + PA - P\hat{B}R^{-1}\hat{B}^T P + Q = 0 \qquad (16)$$

Substituting *Eq. (10)* into *Eq. (15)*, the control input obtains where P is the positive definite solution to *Eq. (16).*

$$U(t) = -R^{-1}\hat{B}^T P\left[X(t) + \int_{t-\tau}^{t} e_{\alpha}^{A(t-s-\tau)} B_d\; U(s)\, ds\right] \qquad (17)$$

**Theorem 1:** In this case, we use an optimal regulator design for fractional order linear systems without considering input time delay (11,) (12). In Eq. (15) the cost function is minimized.

**Proof:** by using Lagrange multiplier method, we

$$W = \int_0^{tf} \frac{1}{2}[Z^T QZ + U^T RU] + \lambda^T [AZ + \hat{B}U - {}_0^C D_b^{\alpha} Z]\, dt \qquad (18)$$

and $\lambda(tf) = 0$, ${}_0D_t^{\alpha\text{-}1}\lambda(t)|_{t=0} = 0$ that

$$W = \int_0^{tf} \frac{1}{2}[Z^T Q\, Z + U^T R\, U] + ({}_tD_{Tf}^{\alpha}\, \lambda^T)Z + \lambda^T\left[A\, Z + \hat{B}U\right] dt \qquad (19).$$

The necessary conditions for optimal control are given by

$$\frac{dH}{du} = 0 \qquad (20)$$

$${}_tD_{tf}^{\alpha}\, \lambda^T(t) = \frac{dH}{dt} \qquad (21)$$

from the condition (20) we obtain

$$U = -R^{-1}\hat{B}^T \lambda(t) \qquad (22)$$

in addition, the costate trajectory (21) is

$${}_tD_{tf}^{\alpha}\, \lambda^T = {}_tD_{tf}^{\alpha}\,(P(t)Z(t)) = -A^T\, \lambda(t) - QZ(t) \qquad (23).$$

We consider

$$\lambda(t) = P(t)Z(t) \qquad (24)$$

*Z (t)* is state vector and *λ (t)* is costate vector.
And this assumption gives the following relation:

$$-A^T \lambda(t) - QZ(t) = P\, {}_tD_{tf}^{\alpha} Z(t) \qquad (25).$$

Finally gives relations:

$$-A^T P\, Z(t) - Q\, Z(t) = P\left(A\, Z(t) + \hat{B}\, R^{-1}\hat{B}^T P\, Z(t) - R_l\, R_{tf}^{\alpha}\, Z(t)\right) \qquad (26)$$

by solving this equation we obtain the time independent state feedback controller

$$K = R^{-1}\hat{B}^T P \qquad (27)$$

in addition, $A_w = A - \hat{B}K$,

under this assumption we achieve the following algebraic Riccati equation

$$A_w^{\,T} P + PA_w - P\hat{B}R^{-1}\hat{B}^T P = -Q \qquad (29) \blacksquare$$

## 6. Simulation examples

In order to illustrate the value of the presented method, the simulation results are implemented in three given examples. The first example: The system state is considered initial conditions $D^{1-q}x_1(0)=1$, $D^{1-q}x_2(0)=1.5$, $D^{1-q}x_3(0)=2$. The second example: the initial conditions $D^{1-q}x_1(0)=0$, $D^{1-q}x_2(0)=0$, $D^{1-q}x_3(0)=0$. The third example: both of the examples I and II.

We consider the commensurate fractional system (10) with $\alpha = 0.5$, $\tau = 0.5$ and the following system parameters:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0.1 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 4 & 1 \end{bmatrix}, \; D = 0,$$

thus the matrix $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and R=5.

## 7. Example

**Example.1**

The system responses in initial conditions without input: $D^{1-q}x_1(0)=1$, $D^{1-q}x_2(0)=1.5$, $D^{1-q}x_3(0)=2$ are shown in Fig. 1.

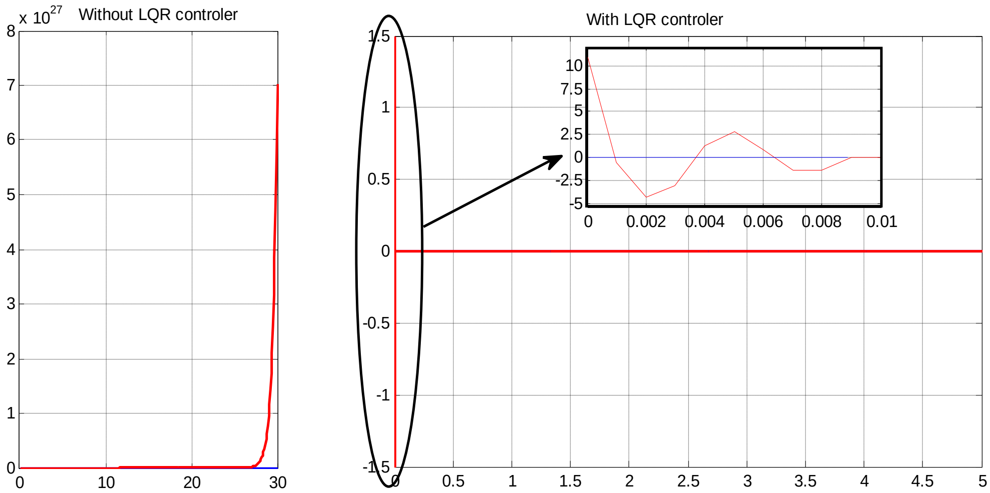


**Fig. 1.** The initial conditions without input, $D^{1-q}x_1(0)=1$, $D^{1-q}x_2(0)=1.5$, $D^{1-q}x_3(0)=2$ and the left side of figure is without controller and the right side is with controller

**Example.2**

In this section, we use the initial conditions: $D^{1-q}x_1(0)=0$, $D^{1-q}x_2(0)=0$, $D^{1-q}x_3(0)=0$ and symmetric input pulse in order to obtain the system responses with input considering. The simulation results are shown in Fig. 2.

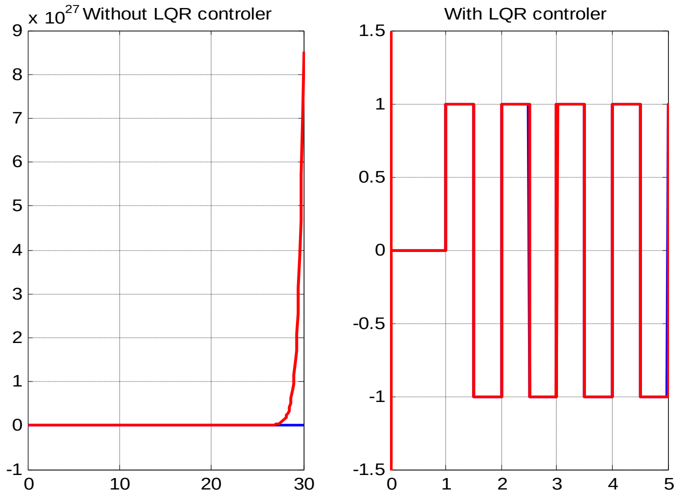


**Fig. 2.** The initial conditions with pulse input, $D^{1-q}x_1(0)=0$, $D^{1-q}x_2(0)=0$, $D^{1-q}x_3(0)=0$

**Examples.3**

We consider the initial conditions for system similar example I ($D^{1-q}x_1(0)=1, D^{1-q}x_2(0)=1.5, D^{1-q}x_3(0)=2$), then symmetric pulse input to applied system similar example II ($D^{1-q}x_1(0)=0, D^{1-q}x_2(0)=0, D^{1-q}x_3(0)=0$). As a result of these two examples, the following system is obtained which is shown in Fig. 3.

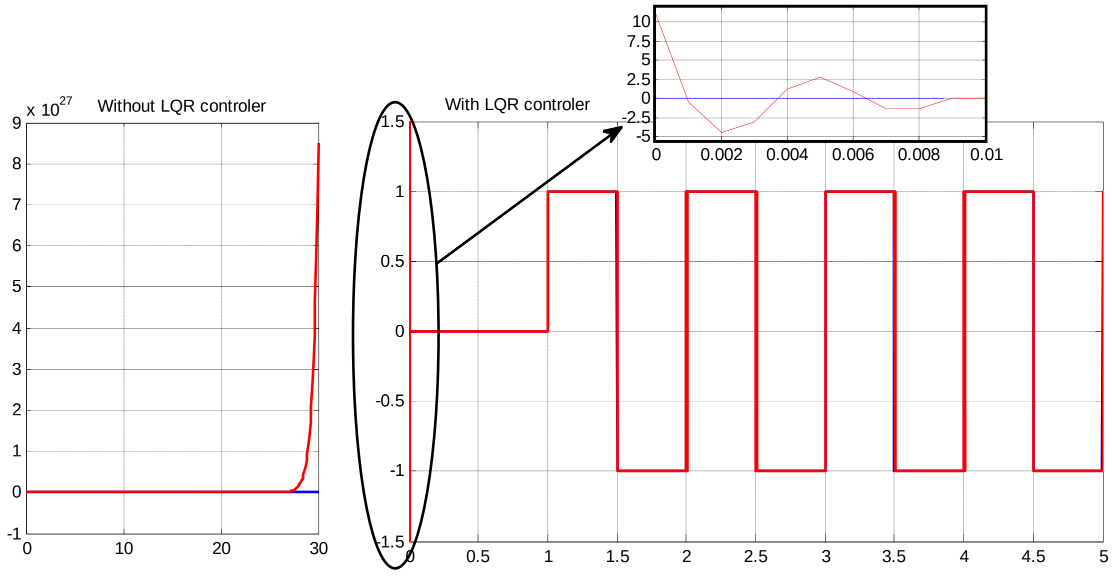


**Fig. 3.** As result the both of the example 1 and 2

## 8. Conclusions

A new method for calculating input time-delay in the optimal regulator design for fractional order linear system is proposed. The original system is fractional order linear systems only with input time-delay. Then, it is performed by fractional derivatives operations and Lemma 1 is applied in order to transformed the system to a without input time-delay system. Finally, we have solved the system without input time-delay with an optimal regulator design and desirable conclusion is obtained. Current work considers how are the best practices to implement the methodology developed, and simulation results are used to show their performance and the results seem promising.

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