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STARLIKENESS AND SHARPNESS RESULTS OF SPECIAL SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract

In the present paper, we investigate starlikeness of certain operators which are defined here by means of convolution. Also, by using the technique of finite Blaschke product (see [3,8]), we prove the sharpness of those results which were obtained earlier by authors in [1].

Keywords and phrases: Starlike function, sharpness, Blaschke product.

1. Introduction

Let $H = H(\mathbb{D})$be the class of all analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$. For $n \in \mathbb{N}$let $\mathcal{A}_n$ denote the subclass of H containing the functions $f(z)$ of the form

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots ; (z \in \mathbb{D})$$

with $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{A}$ is said to be starlike if it is univalent and $f(\mathbb{D})$ is starlike domain (withrespect to the origin). The class of starlike functions is denoted by S^*. A special subclass of S^* is the class of starlike functions of order γ with $0 \leq \gamma < 1$, given by

$$S^*(\gamma) = \left\{ f \in \mathcal{A}: Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma, z \in \mathbb{D} \right\}.$$

It is well known that $S^*(0) = S^*$,(see [2]).

For functions $f, g \in H$ given by

$$f(z) = \sum_{k=0}^{\infty} a_k\, z^k, g(z) = \sum_{k=0}^{\infty} b_k\, z^k$$

the Hadamard product (or convolution)of f, g in $\mathbb{D}$ is defined by

$$(f*g)(z)=\sum_{k=0}^{\infty} a_k\, b_k z^k=(g*f)(z).$$

For $n\in\mathbb{N}, 0<\alpha\leq 1, 0\leq\mu<\alpha n$ and $\lambda>0$ let $U_n(\alpha,\mu,\lambda)$ be defined as follows

$$U_n(\alpha,\mu,\lambda)=\left\{f\in\mathcal{A}_n:\left|(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu}+\alpha\left(\frac{z}{f(z)}\right)^{\mu+1}f'(z)-1\right|<\lambda, z\in\mathbb{D}\right\}$$

with $U_1(\alpha,\mu,\lambda)=U(\alpha,\mu,\lambda)$. The special case of this class has been studied in [5].

For $f\in U_n(\alpha,\mu,\lambda)$ we define the operator $G(z)$ by

$$G(z)=z\left(\frac{1}{\left(\frac{z}{f(z)}\right)^{\mu}*\Phi(a;c;z)}\right)^{\frac{1}{\mu}} \quad (1)$$

where $a,c\in\mathbb{C}, c\neq 0,-1,-2,\cdots,\left(\frac{z}{f(z)}\right)^{\mu}*\Phi(a;c;z)\neq 0$ and

$$\Phi(a;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}z^k,\ (z\in\mathbb{D}) \quad (2)$$

with $(a)_k=a(a+1)(a+2)\cdots(a+k-1)$ and $(a)_0=1$. Also, let

$$H(z)=z\left(\frac{1}{\left(\frac{z}{f(z)}\right)^{\mu}*\Psi(m,\gamma,z)}\right)^{\frac{1}{\mu}},(f\in U_n(\alpha,\mu,\lambda)) \quad (3)$$

where $m<1,\gamma\neq 0, Re\gamma\geq 0,\left(\frac{z}{f(z)}\right)^{\mu}*\Psi(m,\gamma,z)\neq 0$ and

$$\Psi(m,\gamma,z)=1+(1-m)\sum_{k=1}^{\infty}\frac{z^k}{k\gamma+1},(z\in\mathbb{D}). \quad (4)$$

In [1] certain sufficient conditions in terms of $\alpha,\mu,\lambda,\gamma$ and n were obtained, so that functions in $U_n(\alpha,\mu,\lambda)$ belong to $S^*(\gamma)$. Similarly, other conditions for these parameters were obtained such that the analytic functions $G(z)$ and $H(z)$ be in S^*. In all these cases, the sharpness part was not proved. In this paper, by using the same techniques as in [8] we prove the sharpness part. Also, we give another proof for the starlikeness of $G(z)$ and $H(z)$.In order to prove our results we need the following lemmas.

Lemma 1.1. ([3])Let $\varphi,\psi\in\mathbb{R}$. There exists a sequence $\{b_n\}$ of finite Blaschke products such that $b_n(0)=0, b_n(1)=e^{i\varphi}$ and $b_n(z)\longrightarrow e^{i\psi}z$ uniformly on compact subsets of $\mathbb{D}$.

Here a finite Blaschke product is a function as the type

$$b(z)=e^{i\gamma}\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a}_k z},(\{a_k\}\subseteq\mathbb{D},\gamma\in\mathbb{R}).$$

Lemma 1.2. ([7]) If f and g are analytic and F and G are convex (univalent) such that $f\prec F$ and $g\prec G$, then $f*g\prec F*G$, where $\prec$ denotes the usual subordination, (see [2]).

Lemma 1.3. ([6]) Let $c\in\mathbb{C}$ with $Re\ c<1$ and $F_c(z)=\sum_{n=1}^{\infty}\frac{1-c}{n-c}z^{n-1}\in H$. Then

$$\sup_{z\in\mathbb{D}}|f(z)*F_c(z)|\leq\sup_{z\in\mathbb{D}}|f(z)|,(f\in H).$$

2.Main Results

We begin with the following lemma that will be used in the next theorems.

Lemma 2.1. For fixed real numbers $n \in \mathbb{N}, 0 < \alpha \leq 1, 0 < \mu < \alpha n$ and $\lambda > 0$ let $f \in U_n(\alpha, \mu, \lambda)$. There exists an analytic function $w(z)$ in $\mathbb{D}$ where $|w(z)| < 1$ and $w(0) = w'(0) = \cdots = w^{(n-1)}(0) = 0$, such that

$$\frac{zf'(z)}{f(z)} = \frac{1}{\alpha}\left(\frac{1 + \lambda w(z)}{1 - \frac{\lambda\mu}{\alpha}\int_0^1 \frac{w(tz)}{t^{\frac{\mu}{\alpha}+1}}\, dt} + \alpha - 1\right), (z \in \mathbb{D}).$$

Proof. For $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{A}_n$, we can write

$$\left(\frac{z}{f(z)}\right)^{\mu+1} = \frac{1}{1 + (\mu+1)a_{n+1}z^n + \cdots} = 1 - (\mu+1)a_{n+1}z^n + \cdots.$$

So, we obtain

$$\begin{aligned}(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) \\ &= (1 - (\mu+1)a_{n+1}z^n + \cdots)(1 + (1+\alpha n)a_{n+1}z^n + \cdots) \\ &= 1 + (\alpha n - \mu)a_{n+1}z^n + \cdots \\ &= 1 + \lambda\left(\frac{\alpha n - \mu}{\lambda} a_{n+1}z^n + \cdots\right).\end{aligned}$$

Therefore, there exists an analytic function $w(z)$ in $\mathbb{D}$ with$|w(z)| < 1$ and $w(0) = w'(0) = \cdots = w^{(n-1)}(0) = 0$, such that

$$(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) = 1 + \lambda w(z). \tag{5}$$

Let $p(z) = \left(\frac{z}{f(z)}\right)^{\mu}$. Then $p(z)$ is analytic in $\mathbb{D}$ and $p(0) = 1$. Differentiating$p(z)$ we obtain

$$p(z) - \frac{\alpha}{\mu} zp'(z) = 1 + \lambda w(z). \tag{6}$$

Solving the first order differential equation (6) we conclude that

$$p(z) = 1 - \frac{\lambda\mu}{\alpha}\int_0^1 \frac{w(tz)}{t^{\frac{\mu}{\alpha}+1}}\, dt$$

or equally

$$\left(\frac{f(z)}{z}\right)^{\mu} = \frac{1}{1 - \frac{\lambda\mu}{\alpha}\int_0^1 \frac{w(tz)}{t^{\frac{\mu}{\alpha}+1}}\, dt}. \tag{7}$$

Using (5) and (7) we obtain the required result. This completes the proof.■

Now, we restate the sharp version ofTheorem 2.1 in [1] and prove the sharpness part.

Theorem 2.2 [1. Theorem 2.1]**.** Let $n \in \mathbb{N}, n \geq 2, \frac{n+1}{2n} < \alpha \leq 1$ and $n(1-\alpha) < \mu < \alpha n$. If $f \in U_n(\alpha, \mu, \lambda)$, then $f \in S^*(\gamma)$ for $0 < \lambda \leq \lambda(\alpha, \mu, n, \gamma)$, where

$$\lambda(\alpha,\mu,n,\gamma) = \begin{cases} \dfrac{(\alpha n-\mu)\sqrt{2\alpha(1-\gamma)-1}}{\sqrt{(\alpha n-\mu)^2+\mu^2(2\alpha(1-\gamma)-1)}}; & 0 \leq \gamma \leq \dfrac{\mu-n(1-\alpha)}{\mu(1+n)} \\ \dfrac{(\alpha n-\mu)(1-\gamma)}{n+\mu(\gamma-1)}; & \dfrac{\mu-n(1-\alpha)}{\mu(1+n)} < \gamma < 1, \end{cases}$$

also, all bounds for λ are the best possible.

Proof. Suppose that $f(z) = z + a_{n+1}z^{n+1} + \cdots \in U_n(\alpha, \mu, \lambda)$. By Lemma 2.1 there exists an analytic function $w(z)$ in $\mathbb{D}$ with $|w(z)| < 1$ and $w(0) = w'(0) = \cdots = w^{(n-1)}(0) = 0$ such that

$$\frac{zf'(z)}{f(z)} = \frac{1}{\alpha}\left(\frac{1+\lambda w(z)}{1-\frac{\lambda\mu}{\alpha}\int_0^1 \frac{w(tz)}{t^{\frac{\mu}{\alpha}+1}}\,dt} + \alpha - 1\right),$$

and therefore

$$\frac{1}{1-\gamma}\left(\frac{zf'(z)}{f(z)} - \gamma\right) = \frac{\big((\alpha-1)-\alpha\gamma\big)\left(\alpha - \lambda\mu\int_0^1 \frac{w(tz)}{t^{\frac{\mu}{\alpha}+1}}\,dt\right) + \alpha(1+\lambda w(z))}{\alpha(1-\gamma)\left(\alpha - \lambda\mu\int_0^1 \frac{w(tz)}{t^{\frac{\mu}{\alpha}+1}}\,dt\right)}.$$

Now, we have to show that $Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma$. To do this, according to a well-known result in [6]and the last equation, it is sufficient to show that

$$\frac{\big((\alpha-1)-\alpha\gamma\big)\left(\alpha - \lambda\mu\int_0^1 \frac{w(tz)}{t^{\frac{\mu}{\alpha}+1}}\,dt\right) + \alpha(1+\lambda w(z))}{\alpha(1-\gamma)\left(\alpha - \lambda\mu\int_0^1 \frac{w(tz)}{t^{\frac{\mu}{\alpha}+1}}\,dt\right)} \neq -iT, (T \in \mathbb{R}),$$

which is equivalent to

$$\lambda\left(\frac{w(z) + \mu\left(\frac{\alpha\gamma+1-\alpha}{\alpha} - T(1-\gamma)i\right)\int_0^1 \frac{w(tz)}{t^{\frac{\mu}{\alpha}+1}}\,dt}{\alpha(1-\gamma)(1+Ti)}\right) \neq -1, (T \in \mathbb{R}).$$

Let

$$M = \sup_{z\in\mathbb{D}, w\in B_n, T\in\mathbb{R}} \left|\frac{w(z) + \mu\left(\frac{\alpha\gamma+1-\alpha}{\alpha} - T(1-\gamma)i\right)\int_0^1 \frac{w(tz)}{t^{\frac{\mu}{\alpha}+1}}\,dt}{\alpha(1-\gamma)(1+Ti)}\right|,$$

with

$$B_n = \{w \in H(\mathbb{D}): |w(z)| < 1 \text{ and } w^{(k)}(0) = 0, k = 0,1,2,\cdots,n-1\}.$$

Then $f \in S^*(\gamma)$if $\lambda M \leq 1$. This shows that it is sufficient to find M. By the general Schwarz lemma we have $|w(z)| \leq |z|^n$, so we see that

$$M \leq \sup_{T\in\mathbb{R}} \left\{ \frac{1+\frac{\mu\alpha}{n\alpha-\mu}\sqrt{\left(\frac{\alpha\gamma+1-\alpha}{\alpha}\right)^2+(1-\gamma)^2T^2}}{\alpha(1-\gamma)\sqrt{1+T^2}} \right\}. \quad (8)$$

In fact, in the sequel, we prove that equality holds in the above relation, hence the sharpness is established. By Lemma 1.1 given $\psi, \varphi \in \mathbb{R}$ there exists a sequence of finite Blaschke products $\{w_k(z)\}$ such that $w_k(1) = e^{i\psi}$ and $w_k(z) \longrightarrow e^{i\varphi}z^n$uniformly on compact subsets of $\mathbb{D}$. Therefore, we have the following relation for each $T \in \mathbb{R}$:

$$\sup_{z\in\mathbb{D}, w\in B_n} \left| \frac{w(z) + \mu\left(\frac{\alpha\gamma+1-\alpha}{\alpha} - T(1-\gamma)i\right)\int_0^1 \frac{w(tz)}{t^{\frac{\mu}{\alpha}+1}}\,dt}{\alpha(1-\gamma)(1+Ti)} \right|$$

$$\leq \sup_{\psi,\varphi\in\mathbb{R}} \frac{\left| e^{i\psi} + \frac{\mu\alpha}{n\alpha-\mu}\sqrt{\left(\frac{\alpha\gamma+1-\alpha}{\alpha}\right)^2 + (1-\gamma)^2T^2}\, e^{i(\varphi+\theta_1)} \right|}{\alpha(1-\gamma)\sqrt{1+T^2}},$$

where $\theta_1 = Arg\left(\frac{\alpha\gamma+1-\alpha}{\alpha} - T(1-\gamma)i\right)$. Fixing φ and choosing $\psi = \varphi + \theta_1$, we get the required equality in (8). Thus the bound for M is sharp as a function of T. ■

By taking $\gamma = 0$ in Theorem 2.2 we obtain the following sharp result.

Corollary 2.3. Let $n \in \mathbb{N}, n \geq 2, \frac{n+1}{2n} < \alpha \leq 1$ and $n(1-\alpha) < \mu < \alpha n$. If $f \in U_n(\alpha,\mu,\lambda)$, then $f \in S^*$ for $0 < \lambda \leq \frac{(\alpha n-\mu)\sqrt{2\alpha-1}}{\sqrt{(\alpha n-\mu)^2+\mu^2(2\alpha-1)}}$, and the bound for$\lambda$ is sharp.

Theorem 2.4. Let $n \in \mathbb{N}, n \geq 2, \frac{n+1}{2n} < \alpha \leq 1$ and $n(1-\alpha) < \mu < \alpha n$. Also, let $\varphi(z) = 1 + b_1z + b_2z^2 + \cdots$ with $b_n \neq 0$ be convex (univalent) in $\mathbb{D}$. If $f(z) = z + a_{n+1}z^{n+1} + \cdots \in U_n(\alpha,\mu,\lambda)$ and $\Phi(a;c;z)$ defined by (2) satisfy the conditions

$$\left(\frac{z}{f(z)}\right)^\mu * \Phi(a;c;z) \neq 0, \Phi(a;c;z) \prec \varphi(z), (z \in \mathbb{D})$$

then the function $G(z)$defined by (1) has the following properties:

- $G \in U_n(\alpha,\mu,\lambda|b_n|)$,
- $G \in S^*$ for $0 < \lambda \leq \frac{(\alpha n-\mu)\sqrt{2\alpha-1}}{|b_n|\sqrt{(\alpha n-\mu)^2+\mu^2(2\alpha-1)}}$.

In the case 2 the bound for λ is sharp.

Proof. The definition of G shows that

$$\left(\frac{z}{G(z)}\right)^\mu = \left(\frac{z}{f(z)}\right)^\mu * \Phi(a;c;z).$$

Also, a simple calculation gives

$$\frac{z}{\mu}\left(\left(\frac{z}{G(z)}\right)^{\mu}\right)' = \left(\frac{z}{G(z)}\right)^{\mu} - \left(\frac{z}{G(z)}\right)^{\mu+1} G'(z).$$

Therefore, we obtain

$$(1-\alpha)\left(\frac{z}{G(z)}\right)^{\mu} + \alpha\left(\frac{z}{G(z)}\right)^{\mu+1} G'(z) = \left(\frac{z}{G(z)}\right)^{\mu} - \frac{\alpha z}{\mu}\left(\left(\frac{z}{G(z)}\right)^{\mu}\right)'$$

$$= \left(\frac{z}{f(z)}\right)^{\mu} * \Phi(a;c;z) - \frac{\alpha}{\mu}\left(\left\{z\left(\left(\frac{z}{f(z)}\right)^{\mu}\right)'\right\} * \Phi(a;c;z)\right)$$

$$= \left(\frac{z}{f(z)}\right)^{\mu} * \Phi(a;c;z) - \alpha\left(\frac{z}{f(z)}\right)^{\mu} * \Phi(a;c;z) + \alpha\left(\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right) * \Phi(a;c;z)$$

$$= \left((1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right) * \Phi(a;c;z).$$

Since $1+\lambda z^n$ and $\varphi(z)$ are convex in $\mathbb{D}$ and by the assumption (also, see relation (5))

$$(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) \prec 1+\lambda z^n, \Phi(a;c;z) \prec \varphi(z)$$

so, by Lemma 1.2, we deduce that

$$(1-\alpha)\left(\frac{z}{G(z)}\right)^{\mu} + \alpha\left(\frac{z}{G(z)}\right)^{\mu+1} G'(z) \prec 1+\lambda b_n z^n.$$

Case 1 now follows from the last subordination, while 2 is a simple consequence of Corollary 2.3. ■

It is well-known that if $a > 0$ and $c \geq \max\{2, a\}$, then $\Phi(a;c;z)$ defined by (2) is convex in $\mathbb{D}$, (see [4]). So, if we take $\varphi(z) = \Phi(a;c;z)$ in Theorem 2.4, we obtain the following sharp result.

Corollary 2.5. Let $n \in \mathbb{N}, n \geq 2, a > 0$ and $c \geq \max\{2, a\}$. Also, let $\frac{n+1}{2n} < \alpha \leq 1$ and $n(1-\alpha) < \mu < \alpha n$. If $f \in U_n(\alpha,\mu,\lambda)$ and $\Phi(a;c;z)$ defined by (2) satisfy the condition $\left(\frac{z}{f(z)}\right)^{\mu} * \Phi(a;c;z) \neq 0$ for all $z \in \mathbb{D}$, then the function $G(z)$ defined by (1) has the following properties:

- $G \in U_n\left(\alpha,\mu,\frac{\lambda|(a)_n|}{|(c)_n|}\right)$,
- $G \in S^*$ where $0 < \lambda \leq \frac{|(c)_n|(\alpha n-\mu)\sqrt{2\alpha-1}}{|(a)_n|\sqrt{(\alpha n-\mu)^2+\mu^2(2\alpha-1)}}$.

Also, the bound for λ is sharp.

Theorem 2.6. For $n \in \mathbb{N}, n \geq 2, \frac{n+1}{2n} < \alpha \leq 1$ and $n(1-\alpha) < \mu < \alpha n$ let $f \in U_n(\alpha,\mu,\lambda)$. If $m < 1$, $Re\gamma > 0$ and $\Psi(m,\gamma,z)$ defined by (4) satisfy the condition $\left(\frac{z}{f(z)}\right)^{\mu} * \Psi(m,\gamma,z) \neq 0$ for all $z \in \mathbb{D}$, then then the function $H(z)$ given by (3) has the following properties:

- $H \in U_n(\alpha,\mu,\lambda(1-m))$,
- $H \in S^*$ where $0 < \lambda \leq \frac{(\alpha n-\mu)\sqrt{2\alpha-1}}{(1-m)\sqrt{(\alpha n-\mu)^2+\mu^2(2\alpha-1)}}$.

In the case 2 the bound for λ is best possible.

Proof. Using the same steps as in the proof of Theorem 2.4 we obtain

$$(1-\alpha)\left(\frac{z}{H(z)}\right)^{\mu}+\alpha\left(\frac{z}{H(z)}\right)^{\mu+1}H'(z)-1=$$

$$=\left((1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu}+\alpha\left(\frac{z}{f(z)}\right)^{\mu+1}f'(z)\right)*\Psi(m,\gamma,z)-1$$

$$=\left((1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu}+\alpha\left(\frac{z}{f(z)}\right)^{\mu+1}f'(z)-1\right)*\left(1+(1-m)\sum_{k=1}^{\infty}\frac{z^k}{k\gamma+1}\right)$$

$$=(1-m)\left((1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu}+\alpha\left(\frac{z}{f(z)}\right)^{\mu+1}f'(z)-1\right)*\left(\sum_{k=1}^{\infty}\frac{(1/\gamma)z^{k-1}}{k-(1-1/\gamma)}\right).$$

Now, by using Lemma 1.3 with $c=1-\frac{1}{\gamma}$, we conclude that

$$\left|(1-\alpha)\left(\frac{z}{H(z)}\right)^{\mu}+\alpha\left(\frac{z}{H(z)}\right)^{\mu+1}H'(z)-1\right|$$

$$\leq(1-m)\sup_{z\in\mathbb{D}}\left|(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu}+\alpha\left(\frac{z}{f(z)}\right)^{\mu+1}f'(z)-1\right|$$

$$\leq(1-m)\lambda.$$

This proves the case 1. Case 2 follows simply from Corollary 2.3.■

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