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Hadamard Well-Posed Vector Optimization Problems

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Abstract

In this paper, two kinds of Hadamard well-posedness for vector-valued optimization problems are introduced. By virtue of scalarization functions, the scalarization theorems of convergence for sequences of vector-valued functions are established. Then necessary and sufficient conditions for efficient solutions are given, sufficient conditions of Hadamard well-posedness for vector optimization problems are obtained by using the scalarization theorems.

Keywords: Vector optimization, Variational convergence, Γ_c-convergence, Efficient solutions, Hadamard well-posedness.

1. Introduction

For well-posed optimization problems, there are concepts of two main types: Tykhonov well-posedness and Hadamard well-posedness. In 1966, Tykhonov [6] first introduced a concept of well-posedness imposing convergence of every minimizing sequence to the unique minimum point, which is called Tykhonov well-posedness. The concept of Hadamard well-posedness is inspired by the classical idea of Hadamard, which goes back to the beginning of the last century. It requires existence and uniqueness of the optimal solution together with continuous dependence on the problem data. In this paper, we further investigate Hadamard well-posedness for vector-valued optimization problems. By using the definition of variational convergence for vector-valued sequences of functions introduced by Oppezzi and Rossi [5] very recently, we define two different notions of Hadamard well-posedness for vector-valued optimization problems, i.e., extended Hadamard well-posedness and generalized Hadamard well-posedness. Finally, based on scalarization theorems we derived, we extend some basic results of Hadamard well-posedness of scalar optimization problems to the case of vector-valued optimization problems, and then get sufficient conditions for Hadamard well-posedness of vector-valued optimization problems. The paper is organized as follows. In Sect. 2, we present the concepts of two kinds of Hadamard well-posedness for vector-valued optimization problems and give examples to illustrate them. In Sect. 3, we present the necessary and sufficient conditions for efficient solution and $\varepsilon q -$ efficient solution. In Sect. 4, we consider scalarization theorems for convergence of sequences of vector-valued functions. In Sect.5, we extend Hadamard well-posedness results of scalar optimization problems to those of vector-valued optimization problems.

2. Preliminaries and notations

Let X be a topological vector space and Y be a topological vector space ordered by a convex closed and pointed cone $C \subset Y$ with its topological interior $\text{int}C = \emptyset$. For $y, y' \in Y$, we write

$y \leq y'$ if $y' - y \in C$.

Let us consider the scalar-valued functions I_n, $I : X \to [-\infty, +\infty]$.

Definition 2.1(see [1]) We say that I_n converges variationally to I, and write $\text{var} - \lim I_n = I$, iff $x_n \to x$ implies $\liminf_n I_n(x_n) \geq I(x)$ and for every $u \in X$ there exists $u_n \in X$ such that $\lim\sup_n I_n(u_n) \leq I(u)$.

Proposition 2.1(see [4]) If I_n, $I: X \to [-\infty, +\infty]$ satisfy that for every $x \in X$,

$$\sup_{U \in u(x)} \lim\sup\inf I_n(U) \leq I(x) \leq \sup_{U \in u(x)} \lim\sup\inf I_n(U) \qquad (1)$$

(Where u(x) is the system of neighborhoods of x), then $\text{var} - \lim I_n = I$.

Definition 2.2(see [4]) Let u(x) be the family of neighborhoods of $x \in X$, f_n, $f : X \to Y$ ($n \in N$) be given functions. We say that (f_n) $n \in N$ Γ_c-convergence to f and we shall write $f_n \overset{\Gamma_c}{\to} f$, if for every $x \in X$:

(a) $\forall U \in u(x)$, $\forall q_0 \in \text{int}C$, $\exists n_{q_0}, U \in N$ s.t $\forall n \geq n_{q_0}.U$, $\exists x_n \in U$ s.t $f_n(x_n) \leq f(x) + q_0$;

(b) $\forall q_0 \in \text{int}C$, $\exists U_{q_0} \in u(x)$, $k_{q_0} \in N$ s.t $f_n(x') \geq f(x) - q_0$, $\forall x' \in U_{q_0}$, $\forall n \geq k_{q_0}$;

Definition 2.3. We say that $f : X \to Y$ is strongly lower (upper) C-semicontinuousat the point $x_o \in X$ if for any $q_0 \in \text{int}C$ there exists U_{x_o, q_0}, a neighborhood of x_o, such that $\forall x \in U_{x_o, q_0}$ we have:

$$f(x) \in f(x_o) - q_0 + \text{int}C \quad (f(x_o) \in f(x) - q_0 + \text{int}C).$$

Definition 2.4. A function $f: X \to Y$ is said to be C-lower semicontinuous iff $f^{-1}(Y - \text{cl } C)$ is closed in X, $\forall y \in Y$.

When Y = R and C = R+, we have the usual notion of lower semicontinuity.

Remark 2.1. In the following example, we show that strong lower C- semicontinuity is more restrictive than C-lower semicontinuity.

Let $X = [0, \infty[$, $Y = R^2$, $C = \{(x, y) \in R^2 : x \geq 0, 0 \leq y \leq x\}$, and let $f : X \to Y$ be defined by

$$f(t) = \begin{cases} (0,0), & \text{if } t = 0 \\ \left(t, \frac{1}{t}\right), & \text{if } t > 0 \end{cases}$$

It follows that

$$f^{-1}\big((\bar{x},\bar{y}) - C\big) = \begin{cases} \emptyset\ , & \text{if } (\bar{x},\bar{y}) \in R^2 \backslash [0,\infty[^2 \\ \{0\}\,, & \text{if } (\bar{x},\bar{y}) \in [0,\infty[^2 \text{ and } \bar{x}\,\bar{y} < 1 \\ \{0\} \cup [a,b], & \text{if } \overline{xy} \geq 1\,, (\bar{x},\bar{y}) \in [0,\infty[^2 \end{cases}$$

with

$$b = \left(\frac{1}{2}\right)\left[\bar{x} - \bar{y} + \sqrt{(\bar{x}-\bar{y})^2 + 4}\right], \qquad a = 1/\bar{y}$$

Obviously, f is not strongly lower C-semicontinuous at t = 0.

Lemma 2.1. Letf_n, f: X → Y, n ∈ N. If $f_n \xrightarrow{\Gamma_c} f$, then f is stronglylower C-semicontinuous.

Proof .

By Definition $\Gamma_c -$ convergence if $\varepsilon \in \text{int}\, C$ there exist $k_\varepsilon \in N$, $U_\varepsilon \in u(x), U_\varepsilon$ open , such that

$f(x) - \frac{\varepsilon}{2} < f_n(y) \quad \forall\, y \in U_\varepsilon \quad \forall n \geq k_\varepsilon$. If $y \in U_\varepsilon$, then U_ε is a neighborhood of y , hence by Condition (a) of Definition $\Gamma_c -$ convergence there exists $\acute{k}_{\varepsilon,y} \in N$ such that for every $n \geq \acute{k}_{\varepsilon,y}$ there exists $y_n \in U_\varepsilon$ with $f_n(y_n) < f(y) + \frac{\varepsilon}{2}$. It follows clearly that $f(x) - \frac{\varepsilon}{2} < f(y) + \frac{\varepsilon}{2} \quad \forall y \in U_\varepsilon$.

Consider the following vector-valued optimization problem:

(S, f): min f (x)

x∈S

where f : S → Y and S is a nonempty subset of X . Let us recall that x_o is an efficient solution (resp. weak efficient solution) for problem (S, f) if $(f(x_o) - C \backslash\{0\}) \cap f(S) = \emptyset$ (resp. $(f(x_o) - \text{int}C) \cap f(S) = \emptyset$).

The set of efficient solutions (resp. weak efficient solutions) to problem (S,f) is denoted by Eff(f, S,C) (resp. WEff(f, S, C)). If Y = R and C = R+, then(S, f) is a scalar optimization problem. We denote the solution set for the scalar optimization problem by Inf(f, S) and we denote the minimizing value of the scalar optimization problemby val (S, f).

Let us consider Y = R and C = R+. It is said that x_o is an approximate solution for the scalar problem (S, f) if $f(x_o) - \varepsilon \leq f(x)$, ∀x ∈ S. The set of approximate solutions for the scalar problem (S, f) is denoted by Inf(f, S, ε). This notion can be extended to vector optimization problems by the following definition.

Definition 2.5. Let us consider $q \in \text{int}C$, $\varepsilon \geq 0$. It is said that x_o is an εq- efficient solution (resp. weak εq-efficient solution) for problem (S, f) if

$$(f(x_o) - \varepsilon q - C \setminus \{0\}) \cap f(S) = \emptyset \quad , \quad (\text{resp. } (f(x_o) - \varepsilon q - \text{int}C) \cap f(S) = \emptyset.)$$

The set of εq-efficient solutions (resp. weak εq-efficient solutions) is denoted by Eff(f, S, C, εq) (resp. WEff(f, S, C, εq)). It is obvious that Eff(f, S, C, 0q) = Eff(f, S, C)[resp. WEff(f, S, C, 0q) = WEff(f, S, C)].

If $Y = R^p$ and $C = R_+^p$, then (S, f) is a Pareto optimization problem and if p = 1, then (S, f) is a scalar optimization problem, we denote by P_1 .

Assume that f: $S \to Y$, $q \in \text{int}C$ and for all $n \in N$, $f_n: S \to Y$. Let $\{A_n\}$ be a sequence of subsets of X. It is said that $z \in \text{Limsup}_n A_n$ (outer limit of $\{A_n\}$) if, there exista subsequence $\{A_{n_k}\}$ of $\{A_n\}$ and a sequence $\{z_{n_k}\}$ converging to z such that $z_{n_k} \in A_{n_k}$ for each $n_k \in N$.

Now we introduce two notions of Hadamard well-posedness for vector optimization problems.

Definition 2.6. Let $f_n \overset{\Gamma_c}{\to} f$, (S, f) is said to be generalized Hadamard well-posed with respect to $\{f_n\}$, if $\text{Lim sup}_n [\text{WEff}(f_n, S, C, \varepsilon_n q)] \subset \text{WEff}(f, S, C)$, for $\varepsilon_n \geq 0$ and $\varepsilon_n \to 0$.

Definition 2.7. Let $f_n \overset{\Gamma_c}{\to} f$, (S, f) is said to be extended Hadamard well-posed with respect to $\{f_n\}$, if there exists $\varepsilon_0 > 0$ such that $\text{Li m sup}_n [\text{WEff}(f_n, S, C, \varepsilon q)] \subset \text{WEff}(f, S, C, \varepsilon q)$, for all

$$0 \leq \varepsilon \leq \varepsilon_0.$$

Remark 2.2. If (S, f) is generalized Hadamard well-posed with respect to $\{f_n\}$, $\text{Li m sup}_n [\text{WEff}(f_n, S, C)] \subset \text{WEff}(f, S, C)$.

Let us illustrate these definitions by the following examples.

Example 2.1. Let X = R, Y =R^2, C = R+ and q = (1, 1).

Let $S = R$, $f_n : S \to R^2$ be defined for every $n \in N$ and $x \in R$ by

$$f_n(x) = \begin{cases} (x, x) & \text{if} \quad x \geq 0 \\ \frac{1}{n}(x, x) & \text{if} \quad 0 \geq x \geq -n \\ (-1,1) & \text{if} \quad -n \geq x \end{cases}$$

We can easily verify that $f_n \overset{\Gamma_c}{\to} f$ with

$$f(x) = \begin{cases} (x, x) & \text{if} \quad x \geq 0 \\ (0,0) & \text{if} \quad x \leq 0 \end{cases}$$

Then, $\forall \varepsilon_n \to 0$, $\varepsilon_n \geq 0$, $\text{WEff}(f_n, S, C, \varepsilon_n q) = (-\infty, -n(1 - \varepsilon_n)]$. We obtain Li m sup_n (WEff (f_n, S, C, $\varepsilon_n q$)) = $\emptyset$, which is included in WEff (f, S, C) = $(-\infty, 0]$.

Moreover, WEff (f_n, S, C, εq) = $(-\infty, -n(1-\varepsilon)]$, $\forall \varepsilon < 1$, and WEff(f, S, C, εq) = $(-\infty, \varepsilon] \supset \operatorname{Lim\,sup}_n$ (WEff(f_n, S, C, εq)). Therefore, (S, f) is both extended Hadamard well-posed with respect to { f_n} and generalized Hadamard well-posed with respect to { f_n}.

Proposition 2.2. (See [4]) Let f_n , f: S → Y, $f_n \xrightarrow{\Gamma_c} f$. If (S, f) is extended Hadamard well-posed with respect to { f_n}, then it is generalized Hadamard well-posed with respect to { f_n}.

3. Necessary and Sufficient conditions

Definition 3.1. Let us consider $\varphi : Y \to R$ and $y_0 \in Y$.

(a) φ is monotone with respect to y_0 if $y_0 - y \in C \Rightarrow \varphi(y) \le \varphi(y_0)$,

(b) φ is strongly monotone with respect to y_0 if $y_0 - y \in C\backslash\{0\} \Rightarrow \varphi(y) < \varphi(y_0)$,

(c) φ is strictly monotone with respect to y_0 if $y_0 - y \in \operatorname{int}(C) \Rightarrow \varphi(y) < \varphi(y_0)$.

Remark 3.1. It is clear that:

(H_1) (b) ⇒(a) and (b) ⇒(c).

(H_2) If φ is continuous on $y_0 - C$ then (c) ⇒(a).

Example 3.1. In Pareto problems, the function $\varphi(y) = \max_{1\le i\le p}\{vi(yi - zi)\} + \rho\sum_{i=1}^{p} y_i$ where $v \in R_+^p$, $z \in R^p$ and $\rho \in R+$, is monotone if ρ = 0, it is strictly monotone if $v \in \operatorname{int}(R_+^p)$ and ρ = 0, and it is strongly monotone if ρ > 0.

Definition 3.2. Assuming q ∈ int (C) , we define the functional $\varphi_{x_o,\varepsilon}: Y \to R$ by

$$\varphi_{x_o,\varepsilon}(y) = \inf\{s \in R : y \in sq + f(x_o) - \varepsilon q - cl(C)\}, \quad \forall\, y \in Y.$$

Theorem 3.1. (See [2]) Let us consider δ ≥ 0, q ∈C\ {0} and suppose that $x_o \in \operatorname{Inf}(\varphi \circ f, S, \delta)$.

(a) If ε > 0, φ is monotone at $f(x_o) - \varepsilon q$ and $\varphi(f(x_o)) - \varphi(f(x_o) - \varepsilon q) > \delta$, then $x_o \in$ Eff (f, S, C, εq).

(b) If φ is strongly monotone at $f(x_o) - \varepsilon q$ and $\varphi(f(x_o)) - \varphi(f(x_o) - \varepsilon q) \ge \delta$, then $x_o \in$ Eff (f, S, C, εq).

Lemma 3.1. (See [2]) Let us consider q ∈ C\ {0} and φ: Y → R such that

$$\{y \in Y : \varphi(y) < 0\} = f(x_o) - \varepsilon q - \operatorname{int}(C).$$

If $x_o \in$ WEff (f, S, C, εq) , then $x_o \in \operatorname{Inf}(\varphi \circ f, S, \delta)$, $\forall\, \delta \ge \varphi(f(x_o))$.

Lemma 3.2. (See [2]) For all $x_o \in X$, ε ≥ 0, we have:

(a) $\varphi_{x_o,\varepsilon}(\cdot)$ is a continuous, convex and strictly monotone functional satisfying

$$\{y \in Y : \varphi_{x_o,\varepsilon}(y) < 0\} = f(x_o) - \varepsilon q - \text{int}C; \qquad (2)$$

(b) $\varphi_{x_o,\varepsilon}(f(x_o) + \rho q) = \varepsilon + \rho, \forall \rho \in R;$

(c) $\varphi_{x_o,\varepsilon}(y) - \varphi_{x_o,\varepsilon}(y - \rho q) = \rho, \ \forall y \in Y, \forall \rho \in R.$

Theorem 3.2. If $x_o \in \text{WEff}(f, S, C, \varepsilon q)$, then $x_o \in \text{Inf}(\varphi_{x_o,\varepsilon} \circ f, S, \delta), \ \forall \delta \geq \varepsilon.$

Proof: Let $x_o \in \text{WEff}(f, S, C, \varepsilon q)$. By Lemma 3.1 and Lemma 3.2(a) it can be Deduced that $x_o \in \text{Inf}(\varphi_{x_o,\varepsilon} \circ f, S, \delta), \forall \delta \geq \varphi_{x_o,\varepsilon}(f(x_o))$. The theorem follows since $\varphi_{x_o,\varepsilon}(f(x_o)) = \varepsilon$ by Lemma 3.2 (b).

4. Scalarization of variational convergence for vector-valued sequences of functions

Proposition 4.1. (See [4]) Suppose that $f_n, f: X \to Y$, $f_n \xrightarrow{\Gamma_c} f$, and the scalarization functional $g : Y \to [-\infty, +\infty]$ satisfying $g(q) \to 0$ when $q \to 0$. Moreover, assume that g is monotone (i.e.$\forall y_1, y_2 \in Y, y_1 \leq y_2$ implies $g(y_1) \leq g(y_2)$), sub-additive (i.e.$\forall y_1, y_2 \in Y, g(y_1 + y_2) \leq g(y_1) + g(y_2)$). Then var-lim$g \circ f_n = g \circ f$.

Theorem 4.1.(see [4]) Assume that $f_n, f : X \to Y, x_n \to \bar{x}, f_n \xrightarrow{\Gamma_c} f$ and f is strongly upper C-semicontinuous. Then

(a) $\forall \varepsilon \geq 0,$ var-lim $\varphi_{x_n,\varepsilon} \circ f_n = \varphi_{\bar{x},\varepsilon} \circ f$.

(b) $\forall \varepsilon_n \geq 0, \varepsilon_n \to 0,$ var-lim $\varphi_{x_n,\varepsilon_n} \circ f_n = \varphi_{\bar{x},0} \circ f$.

5. Hadamard well-posedness properties of vector optimization problems

In this section, we extend some basic results of Hadamard well-posedness of scalar optimi- zation problems to the cases of vector-valued optimization problems and then get sufficient conditions for Hadamard well-posedness of vector-valued optimization problems.

From Theorem 5 in Chapter 4 of [1], we have the following lemma.

Lemma 5.1. (See [4]) Assume that var-lim $I_n = I$. Then

(a) lim sup val $(S, I_n) \leq$ val (S, I) ;

(b) $\text{Lim sup}_n [\text{Inf}(I_n, S, \varepsilon)] \subset \text{Inf}(I, S, \varepsilon)$ for all sufficiently small $\varepsilon \geq 0$;

If $\varepsilon_n \geq 0, \varepsilon_n \to 0$, then $\text{Lim sup}_n [\text{Inf}(I_n, S, \varepsilon_n)] \subset \text{Inf}(I, S)$

Lemma 5.2. (See [2, Theorem 5.2]) Assume that $f: S \to Y$ and $\varepsilon \geq 0$. Then

$$x_0 \in \mathrm{WEff}\,(f, S, C, \varepsilon q) \quad \Leftrightarrow \quad x_0 \in \mathrm{Inf}\,(\varphi_{x_0,\varepsilon}\, f, S, \varepsilon)\,.$$

Theorem 5.1. Assume that f_n , $f : S \to Y$, $f_n \overset{\Gamma_c}{\to} f$ and f is strongly upper C-semicontinuous . Then

(a) $\forall x_n \to \bar{x}$, $\forall \varepsilon_n \geq 0$, $\varepsilon_n \to 0$, $\mathrm{Lim\,sup}_n\, \mathrm{val}(S, \varphi_{x_n,\varepsilon_n}\ 0\, f_n) \leq \mathrm{val}(S, \varphi_{\bar{x},0}\ o\ f)$, and for arbitrarily chosen $\varepsilon \geq 0$, $\mathrm{Lim\,sup}_n\, \mathrm{val}(S, \varphi_{x_n,\varepsilon}\ o\ f_n) \leq \mathrm{val}(S, \varphi_{\bar{x},\varepsilon}\ o\ f)$;

(b) (S, f) is extended Hadamard well-posed with respect to $\{ f_n \}$.

Proof . The proof of (a) is clear. We only need to prove (b).

Let $\bar{x} \in \mathrm{Lim\,sup}_n$ [WEff $(f_n, S, C, \varepsilon q)$], i.e. $\exists \{n_k\} \subset N$, $x_{n_k} \in \mathrm{WEff}\,(f_{n_k}, S, C, \varepsilon q)$ such that $x_{n_k} \to \bar{x}$. From Lemma 5.2 , $x_{n_k} \in \mathrm{Inf}(\varphi_{x_{n_k},\varepsilon} o f_{n_k}, S, \varepsilon)$. Therefore, $\bar{x} \in \mathrm{Lim\,sup}_{n_k}$ [Inf$(\varphi_{x_{n_k},\varepsilon}\ o f_{n_k}$, S, ε)]. By Theorem 4.1(a), we have var-lim$\varphi_{x_{n_k},\varepsilon} o f_{n_k} = \varphi_{\bar{x},\varepsilon} o f$ From Lemma 5.1, it can be deduced that there exists $\varepsilon_0 > 0$ such that

$$\mathrm{Lim\,sup}_{n_k}\, [\mathrm{Inf}(\varphi_{x_{n_k},\varepsilon} o f_{n_k}, S, \varepsilon)] \subset \mathrm{Inf}\,(\varphi_{\bar{x},\varepsilon} o f, S, \varepsilon), \quad \forall 0 \leq \varepsilon \leq \varepsilon_0\,.$$

It follows that $\bar{x} \in \mathrm{Inf}(\varphi_{\bar{x},\varepsilon} o f, S, \varepsilon)$. By Lemma 5.2, $\bar{x} \in \mathrm{WEff}\,(f, S, C, \varepsilon q)$. Therefore, there exists $\varepsilon_0 > 0$ such that $\forall 0 \leq \varepsilon \leq \varepsilon_0$,

$$\mathrm{Lim\,sup}_n\, [\mathrm{WEff}\,(f_n, S, C, \varepsilon q)] \subset \mathrm{WEff}\,(f, S, C, \varepsilon q).$$

We conclude that (S, f) is extended Hadamard well-posed with respect to $\{f_n\}$.

Remark 5.1. From Theorem 5.1(b) and Proposition 2.2, if the conditions of Theorem 5.1 hold, the problem (S, f) is generalized well-posedness with respect to $\{f_n\}$.

Example 5.1. The following example shows that without the assumption of strongly upper C-semicontinuous of f , conclusions of Theorem 5.1 may not hold. Assume that f_n, f: $R \to R^2$ defined as

$f_n = \left(x\,, nxe^{-2n^2x^2}\right)$, for any $n \in N$, and

$$f(x) = \begin{cases} (x, 0\,) & \text{if } x \neq 0 \\ \left(0\,, \dfrac{1}{2} e^{-\frac{1}{2}}\right), & \text{if } x = 0 \end{cases}$$

respectively. Now we show that $f_n \overset{\Gamma_c}{\to} f$,

In fact, if $x \neq 0$, we notice that $nx_n e^{-2n^2x_n^2} \longrightarrow 0$, when $x_n \to x$. Then , we have that

$$\forall x_n \longrightarrow x\ \ , \forall\, U \in u(x), \forall\, q_0 \in \mathrm{int}C\ \ , \exists\, n_{q_0,U} \in N \quad \text{s.t} \quad \forall\, n \geq n_{q_0,U}$$

$$\left(x_n\,, nx_n e^{-2n^2x_n^2}\right) \leq (\,x, 0\,) + q_0 \qquad (3)$$

Moreover,

$\forall q_0 \in \mathrm{int}C,\quad \exists\, U_{q_0} \in u(x),\quad k_{q_0} \in N \quad \text{s.t} \quad \forall\, \acute{x} \in U_{q_0},$

$\forall\, n \geq k_{q_0}\ \left(\acute{x}, n\acute{x}e^{-2n^2\acute{x}^2}\right) \geq (x, 0) - q_0$ (4)

If $x = 0$, by taking $x_n = -\frac{1}{2n}$, we have that $\forall U \in u(x)$, $\forall\, q_0 \in \mathrm{int}C$, $\exists\, \acute{n}_{q_{0.U}} \in N$ s.t $\forall\, n \geq \acute{n}_{q_{0.U}}$

$\left(x_n, nx_ne^{-2n^2x_n^2}\right) = \left(\frac{-1}{2n}, \frac{-1}{2}e^{-\frac{1}{2}}\right) \leq \left(0, -\frac{1}{2}e^{-\frac{1}{2}}\right) + q_0$ (5)

And since $nxe^{-2n^2x_n^2} \geq -\frac{1}{2}e^{-\frac{1}{2}}$ for all $x \in R$,

We have that $\forall q_0 \in \mathrm{int}C$, $\exists\, \acute{U}_{q_0} \in u(0)$, $\acute{k}_{q_0} \in N$ s.t $\forall\, \acute{x} \in \acute{U}_{q_0}$, $\forall\, n \geq \acute{k}_{q_0}$

$$\left(\acute{x}, n\acute{x}e^{-2n^2\acute{x}^2}\right) \geq \left(0, -\frac{1}{2}e^{-\frac{1}{2}}\right) - q_0 \qquad (6)$$

Therefore, it follows from (3), (4), (5), (6) and Definition 2.2 that $f_n \xrightarrow{\Gamma_c} f$. However, because

$$\mathrm{WEff}(f, S, C) = \left\{\left(0, -\frac{1}{2}e^{-\frac{1}{2}}\right)\right\}$$

and

$$\mathrm{WEff}(f_n, S, c) = \left\{(x, nxe^{-2n^2x_n^2}) \middle| x \leq -\frac{1}{2n}\right\}$$

We have $\mathrm{Lim\,sup}_n\,[\mathrm{WEff}(f_n, S, c)] \not\subset \mathrm{WEff}(f, S, c)$. It is said that (S, f) is not extended well-posed with respect to $\{f_n\}$.

Lemma 5.3. (See [2, Theorem 5.1]) Assume that $f: S \to Y$ and $\varepsilon \geq 0$.

(a) $x_0 \in \mathrm{Eff}(f, S, C, \varepsilon q) \Rightarrow x_0 \in \mathrm{Inf}\left(\varphi_{x_0,\varepsilon} of, S, \varepsilon\right)$

(b) $x_0 \in \mathrm{Inf}\left(\varphi_{x_0,\varepsilon} of, S, \varepsilon\right) \Rightarrow x_0 \in \mathrm{Eff}(f, S, C, vq),\ \forall\, v > \varepsilon$

Theorem 5.2. Assume that f_n, $f: S \to Y$, $f_n \xrightarrow{\Gamma_c} f$, and f is strongly upper C-semicontinuous. Then $\forall \varepsilon_n \geq 0$, $\varepsilon_n \to 0$ $\mathrm{Limsup}_n[\mathrm{Eff}(f_n, S, C, \varepsilon_n q] \subset \mathrm{Eff}(f, S, C, vq)$ $\forall\, v > 0$.

Proof. $\forall\ \varepsilon_n \to 0$, $\forall\, \varepsilon_n \geq 0$, let $\bar{x} \in \mathrm{Limsup}_n[\mathrm{Eff}(f_n, S, C, \varepsilon_n q)]]$, i.e. $\exists\, \{n_k\} \subset N$ $x_{n_k} \in \mathrm{Eff}\left(f_n, S, C, \varepsilon_{n_k} q\right)$, such that $x_{n_k} \to \bar{x}$. From Lemma 5.3(a), $x_{n_k} \in \mathrm{Inf}\left(\varphi_{x_{n_k}.\varepsilon_{n_k}} of_{n_k}, S, \varepsilon_{n_k}\right)$.

Therefore $\bar{x} \in \mathrm{Limsup}_{n_k}\left[\mathrm{Inf}\left(\varphi_{x_{n_k}.\varepsilon_{n_k}} of_{n_k}, S, \varepsilon_{n_k}\right)\right]$. By Theorem 4.1(b), we Have $\mathrm{var}-\lim \varphi_{x_{n_k}.\varepsilon_{n_k}} of_{n_k} = \varphi_{\bar{x},0}\, of$. It can be deduced that

$$\mathrm{Limsup}_{nk}\left[\mathrm{Inf}\left(\varphi_{x_{n_k},\varepsilon_{n_k}} \mathrm{o} f_{n_k}, S, \varepsilon_{n_k}\right)\right] \subset \mathrm{Inf}\left(\varphi_{\bar{x},0} \mathrm{o} f, S\right)$$

Thus $\bar{x} \in \mathrm{Inf}\left(\varphi_{\bar{x},0} \mathrm{o} f, S\right)$. By Lemma 5.3 (b), we have $\bar{x} \in \mathrm{Eff}(f, S, C, vq), \quad \forall\, v > 0$. Hence $\forall \varepsilon_n \geq 0$, $\varepsilon_n \to 0$ $\quad \mathrm{Limsup}_n[\mathrm{Eff}\,(f_n, S, C, \varepsilon_n q)] \subset \mathrm{Eff}(f, S, C, vq), \quad \forall\, v > 0.$

11. References

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