Journal of mathematics and computer science 9 (2014), 426-437

Contents list available at JMCS

Journal of Mathematics and Computer Science

Journal Homepage: www.tjmcs.com

Analytical approach to fractional Fokker–Planck equations by new homotopy perturbation method

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Article history:
Received December 2013
Accepted January 2014
Available online February 2014

Abstract

In this paper, a new form of homotopy perturbation method has been adopted for solving the space-time dependent fractional Fokker-Planck equation. The fractional derivatives are described in the Caputo sense. The method gives an analytic solution in the form of a convergent series with easily computable components, requiring no linearization or small perturbation. The numerical results show that the approaches are easy to implement and accurate when applied to the space-time dependent fractional Fokker-Planck equations. The method introduces a promising tool for solving many space-time fractional partial differential equations.

Keywords: New Homotopy perturbation method, Fokker-Plank equation, functional equation.

1. Introduction

The Fokker–Planck equation (FPE) arises in various fields in natural science, including solid-state physics, quantum optics, chemical physics, theoretical biology and circuit theory. The Fokker–Planck equation was first used by Fokker and Plank to describe the Brownian motion of particles [1]. A FPE describes the change of probability of a random function in space and time. Hence it is naturally used to describe solute transport. The general FPE for the motion of a concentration field $u(x,t)$ of one space variable x at time t has the following form [1,2]

$$\frac{\partial u}{\partial t}=\left[-\frac{\partial}{\partial x}A(x)+\frac{\partial^2}{\partial x^2}B(x)\right]u(x,t), \qquad (1)$$

with initial condition

$$u(x,t)=f(x), \quad x\in R. \qquad (2)$$

Where $B(x) > 0$ is the diffusion coefficient and $A(x) > 0$ is the drift coefficient. The drift and diffusion coefficients may also depend on time. Eq. (1) is a linear second-order partial differential equation of parabolic type.

There is a more general form of FPE which is called nonlinear Fokker–Planck equation. Nonlinear FPE has important applicationsin various areas such as plasma physics, surface physics, population dynamic, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology and marketing [3–5]. In one variable case, the nonlinear FPE is written in the following form

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t,u) + \frac{\partial^2}{\partial x^2} B(x,u,t) \right] u(x,t). \qquad (3)$$

In recent years there has been a great deal of interest in fractional diffusion equations. These equations arise in continuoustime random walks, modeling of anomalous diffusive and subdiffusive systems, unification of diffusion and wave propagation phenomenon, and simplification of the results [6].

Our concern in this work is to consider the numerical solution of the nonlinear FPE with space-time fractional derivatives of the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left[-\frac{\partial^\beta}{\partial x^\beta} A(x,t,u) + \frac{\partial^{2\beta}}{\partial x^{2\beta}} B(x,t,u) \right] u(x,t), \quad t > 0, \quad 0 < \alpha, \beta \le 1, \qquad (4)$$

where α and β are parameters describing the order of the fractional time and space derivatives, respectively. The function $u(x,t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t < 0$ and $x < 0$. The fractional derivatives are considered in the Caputo sense. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses. In the case of $\alpha = 1$ and $\beta = 1$, the fractional equation reduces to the classical nonlinearFPE (3).

In recent years, increasing interest of scientists and engineers has been devoted to analytical asymptotic techniques for solving nonlinear problems, and many new numerical techniques have been widely applied to nonlinear problems. Based on homotopy, which is a basic concept in topology, a general analytical method, namely, the Homotopy Perturbation Method (HPM) was established by He [6-8] in 1998, to obtain a series solution of nonlinear differential equations.

We apply a new version of the HPM that efficiently solves the space and time dependent fractional Fokker–Planck equations. He's HPM has been already used by many mathematicians and engineers to solve various functional equations. In this method, the nonlinear problem is transferred into an infinite number of sub-problems and, the solution is approximated by the sum of the solutions of the first several sub-problems. This simple method has been applied to solve linear and nonlinear equations of heat transfer [9-11], fluid mechanics [12], nonlinear Schrödinger equations [13], boundary value problems [14], fractional KdV-Burgers equation [15] and the nonlinear system of second order boundary value problems [16].

In this letter, we use from Riemann–Liouville and Caputo fractional calculus theory [17-20].

2. Basic ideas of the NHPM

To illustrate the basic ideas of this method, let us consider the following nonlinear differential equation [21-22],

$$A\left(u\left(X,t\right)\right)-f\left(r\right)=0,\qquad r\in\Omega,\tag{5}$$

with the following boundary conditions

$$B\left(u\left(X,t\right),\frac{\partial u}{\partial n}\right)=0,\qquad r\in\Gamma,\tag{6}$$

where A is a general differential operator, B is a boundary operator, $f\left(r\right)$ is a known analytical function, Γ is the boundary of the domain Ω, and $X=\left(x_1,x_2,\ldots,x_n\right)$. The operator A can be divided into two parts, L and N, where L is a linear and N is a nonlinear operator.

Therefore Eq. (5) can be rewritten as

$$L\left(u\right)+N\left(u\right)-f\left(r\right)=0.\tag{7}$$

By the homotopy technique, we construct a homotopy $U\left(r,p\right):\Omega\times\left[0,1\right]\rightarrow\mathbb{R}$, which satisfies

$$H\left(U,p\right)=\left(1-p\right)\left(L\left(U\right)-u_0\right)+p\left(A\left(u\right)-f\left(r\right)\right)=0,\quad p\in\left[0,1\right],\quad r\left(x\right)\in\Omega,\tag{8}$$

or equivalently,

$$H\left(U,p\right)=L\left(U\right)-u_0+pu_0+p\left(N\left(U\right)-f\left(r\right)\right)=0,\tag{9}$$

where $p\in\left[0,1\right]$, is an embedding parameter, u_0 is an initial approximation of the solution of Eq. (5). Clearly, we have from Eqs. (8) and (9)

$$H\left(U,0\right)=L\left(U\right)-L\left(u_0\right)=0,\tag{10}$$

$$H\left(U,1\right)=A\left(U\right)-f\left(r\right)=0.\tag{11}$$

According to the HPM, we can first use the embedding parameter p, as a small parameter, and assume that the solutions of Eqs. (8) and (9) can be represented as a power series in p as

$$U\left(X,t\right)=\sum_{n=0}^{\infty}p^nU_n\left(X,t\right).\tag{12}$$

Now let us write the Eq. (9) in the following form

$$L\left(U\left(X,t\right)\right)=u_0\left(X,t\right)+p\left[f\left(r\right)-u_0\left(X,t\right)-N\left(U\left(X,t\right)\right)\right].\tag{13}$$

By applying the inverse operator, L^{-1}, to both sides of Eq. (13), we derive

$$U\left(X,t\right)=L^{-1}\left(u_0\left(X,t\right)\right)+p\left[L^{-1}\left(f\left(r\right)\right)-L^{-1}\left(u_0\left(X,t\right)\right)-L^{-1}N\left(U\left(X,t\right)\right)\right].\tag{14}$$

Suppose that the initial approximation of Eq. (5) has the following form

$$u_0(X,t)=\sum_{n=0}^{\infty}a_n(X)p_n(t), \quad (15)$$

where $a_0(X), a_1(X), a_2(X), \ldots,$ are unknown coefficients and $p_0(t), p_1(t), p_2(t), \ldots,$ are specific functions depending on the problem. Now by substituting (12) and (15) into the Eq. (14), we get

$$\sum_{n=0}^{\infty}p^n U(X,t)=L^{-1}\left(\sum_{n=0}^{\infty}a_n(X)p_n(t)\right) + p\left[L^{-1}(f(r))-L^{-1}\left(\sum_{n=0}^{\infty}a_n(X)p_n(t)\right)-L^{-1}N\left(\sum_{n=0}^{\infty}p^n U(X,t)\right)\right]. \quad (16)$$

Comparing coefficients of terms with identical powers of p, leads to

$$\begin{aligned}
&p^0: U_0(X,t)=L^{-1}\left(\sum_{n=0}^{\infty}a_n(X)p_n(t)\right),\\
&p^1: U_1(X,t)=L^{-1}(f(r))-L^{-1}\left(\sum_{n=0}^{\infty}a_n(X)p_n(t)\right)-L^{-1}N\left(U_0(X,t)\right),\\
&p^2: U_2(X,t)=-L^{-1}N\left(U_0(X,t),U_1(X,t)\right),\\
&\vdots\\
&p^j: U_j(X,t)=-L^{-1}N\left(U_0(X,t),U_1(X,t),\ldots,U_{j-1}(X,t)\right),\\
&\vdots
\end{aligned} \quad (17)$$

Now if we solve these equations in such a way that $U_1(X,t)=0$, then Eq. (17) results in $U_2(X,t)=U_3(X,t)=\cdots=0$. Therefore the exact solution may be obtained as the following

$$u(X,t)=U_0(X,t)=L^{-1}\left(\sum_{n=0}^{\infty}a_n(X)p_n(t)\right).$$

It is worthwhile to mention that if $f(r)$ and u_0 are analytic at $x=x_0$, then their Taylor series defined as

$$u_0(X,t)=\sum_{n=0}^{\infty}a_n(X)(t-t_0)^n, \qquad f(r)=\sum_{n=0}^{\infty}a_n^*(X)(t-t_0)^n,$$

can be used in Eq. (16), where $a_0^*(X), a_1^*(X), a_2^*(X), \ldots,$ are known coefficients and $a_0(X), a_1(X), a_2(X), \ldots,$ are unknown ones, which must be computed.

4. NHPM applied to the fractional Fokker–Planck equation

To solve the fractional Fokker–Planck equation by means of NHPM, rewrite Eq. (1) in the form

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}}=-\frac{\partial^{\beta}}{\partial x^{\beta}}A^{*}(x,t,u)+\frac{\partial^{2\beta}}{\partial x^{2\beta}}B^{*}(x,t,u)=0, \quad t>0, \quad x>0, \qquad (18)$$

where $0<\alpha\le 1,\ \beta\le 1,\ A^{*}(x,t,u)=A(x,t,u)u(x,t)$ and $B^{*}(x,t,u)=B(x,t,u)u(x,t)$. with the initial condition as follows

$$u(x,t)=f(x). \qquad (19)$$

To solve Eq. (18) by homotopy perturbation method, we construct the following homotopy:

$$(1-p)\left(\frac{\partial^{\alpha} U}{\partial t^{\alpha}}-u_0\right)+p\left(\frac{\partial^{\alpha} U}{\partial t^{\alpha}}+\frac{\partial^{\beta}}{\partial x^{\beta}}A^{*}(x,t,U)-\frac{\partial^{2\beta}}{\partial x^{2\beta}}B^{*}(x,t,U)\right)=0. \qquad (20)$$

Applying the operator J^{α}, the inverse of the operator $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$, to both sides of Eq. (20), yields to

$$U(x,t)=U(x,0)+J^{\alpha}u_0-pJ^{\alpha}\left(u_0+\frac{\partial^{\beta}}{\partial x^{\beta}}A^{*}(x,t,U)-\frac{\partial^{2\beta}}{\partial x^{2\beta}}B*(x,t,\mathrm{U})\right). \qquad (21)$$

Substituting Eq. (12) into Eq. (21), collecting the terms with the same powers of p, and equating each coefficient of p to zero, results in

$$\begin{aligned}
&p^{0}:U_0(x,t)=U(x,0)+J^{\alpha}u_0,\\
&p^{1}:U_1(x,t)=J^{\alpha}\left(-u_0-\frac{\partial^{\beta}}{\partial x^{\beta}}A^{*}(x,t,U_0)+\frac{\partial^{2\beta}}{\partial x^{2\beta}}B^{*}(x,t,U_0)\right),\\
&p^{2}:U_2(x,t)=J^{\alpha}\left(-\frac{\partial^{\beta}}{\partial x^{\beta}}A^{*}(x,t,U_0,U_1)+\frac{\partial^{2\beta}}{\partial x^{2\beta}}B^{*}(x,t,U_0,U_1)\right),\\
&\vdots\\
&p^{j}:U_j(x,t)=J^{\alpha}\left(-\frac{\partial^{\beta}}{\partial x^{\beta}}A^{*}(x,t,U_0,U_1,\ldots,U_{j-1})+\frac{\partial^{2\beta}}{\partial x^{2\beta}}B^{*}(x,t,U_0,U_1,\ldots,U_{j-1})\right),\\
&\vdots
\end{aligned} \qquad (22)$$

Assuming $u_0(x,t)=\sum_{n=0}^{\infty} a_n(x,t)t^n,\ U(x,0)=u(x,0)=f(x)$. By solving equation $U_1(x,\mathrm{t})=0,$ and to obtain the unknown coefficients $a_i,\ i=0,1,2,\ldots,$ the exact solution can be obtained as follows:

$$u(x,t)=U_0(x,t)=f(x)+J^{\alpha}\left(\sum_{n=0}^{\infty} a_n t^n\right). (23)$$

5. Numerical Examples

Example 1. Consider the linear space fractional FPE [23]

$$\frac{\partial u}{\partial t}=\left[-\frac{\partial^{\beta}}{\partial x^{\beta}}.\mathrm{x}+\frac{\partial^{2\beta}}{\partial x^{2\beta}}.\frac{x^2}{2}\right]u(x,t),\quad t>0,\quad x>0, \tag{24}$$

where $0<\beta\le 1,$ $\mathrm{A}^*(x,y,t)=x.u(x,t),$ and $B^*(x,y,t)=\frac{x^2}{2}.u(x,t)$. Subject to the initial condition

$$u(x,0)=x. \qquad (25)$$

To solve Eq. (24) by NHPM, we construct the following homotopy:

$$(1-p)\left(\frac{\partial U}{\partial t}-u_0\right)+p\left(\frac{\partial U}{\partial t}+\frac{\partial^{\beta}}{\partial x^{\beta}}(xU)-\frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\frac{x^2U}{2}\right)\right)=0. \tag{26}$$

According to the Eq. (22), for Eq. (24) we derive

$$p^0: U_0(x,t)=U(x,0)+\int_0^t u_0(x,\xi)d\xi,$$

$$p^1: U_1(x,t)=\int_0^t\left(-u_0(x,\xi)-\frac{\partial^{\beta}}{\partial x^{\beta}}(xU_0(x,\xi))+\frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\frac{x^2U_0(x,\xi)}{2}\right)\right)d\xi,$$

$$p^2: U_2(x,t)=\int_0^t\left(-\frac{\partial^{\beta}}{\partial x^{\beta}}(xU_1(x,\xi))+\frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\frac{x^2U_1(x,\xi)}{2}\right)\right)d\xi,$$

$$\vdots$$

$$p^j: U_j(x,t)=\int_0^t\left(-\frac{\partial^{\beta}}{\partial x^{\beta}}(xU_{j-1}(x,\xi))+\frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\frac{x^2U_{j-1}(x,\xi)}{2}\right)\right)d\xi,$$

$$\vdots$$

Now assume that $u_0(x,t)=\sum_{n=0}^{\infty} a_n(x,t)t^n$, $U(x,0)=u(x,0)=x$, and $U_1(x,t)=0$. Then we have

$$U_1(x,t)=\left(-a_0-\frac{\partial^{\beta}}{\partial x^{\beta}}(x^2)+\frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\frac{x^3}{2}\right)\right)t+\left(-a_1-\frac{\partial^{\beta}}{\partial x^{\beta}}(x\,a_0)+\frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\frac{x^2a_0}{2}\right)\right)\frac{t^2}{2}$$
$$+\left(-a_2-\frac{\partial^{\beta}}{\partial x^{\beta}}(x\,a_1)+\frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\frac{x^2a_1}{2}\right)\right)\frac{t^3}{3}+\cdots=0.$$

It can be easily shown that

$$a_0=-\frac{2x^{2-\beta}}{\Gamma(3-\beta)}+\frac{3x^{3-2\beta}}{\Gamma(4-2\beta)},$$
$$a_1=\frac{2\Gamma(4-\beta)x^{3-2\beta}}{\Gamma(4-2\beta)\Gamma(3-\beta)}-\frac{3\Gamma(5-2\beta)x^{4-3\beta}}{\Gamma(4-2\beta)\Gamma(5-3\beta)}-\frac{\Gamma(5-\beta)x^{4-3\beta}}{\Gamma(3-\beta)\Gamma(5-3\beta)}+\frac{3\Gamma(6-2\beta)x^{5-4\beta}}{2\Gamma(4-2\beta)\Gamma(6-4\beta)},$$
$$a_2=\frac{3\Gamma(6-2\beta)\Gamma(8-4\beta)x^{7-6\beta}}{8\Gamma(4-2\beta)\Gamma(6-4\beta)\Gamma(8-6\beta)}-\left(\frac{3\Gamma(6-2\beta)\Gamma(7-4\beta)}{4\Gamma(4-2\beta)\Gamma(6-4\beta)}+\frac{3\Gamma(5-2\beta)\Gamma(7-3\beta)}{4\Gamma(4-2\beta)\Gamma(5-3\beta)}\right.$$
$$\left.+\frac{\Gamma(5-\beta)\Gamma(7-3\beta)}{4\Gamma(3-\beta)\Gamma(5-3\beta)}\right)\frac{x^{6-5\beta}}{\Gamma(7-5\beta)}+\left(\frac{\Gamma(4-\beta)\Gamma(6-2\beta)}{2\Gamma(3-\beta)\Gamma(4-2\beta)}+\frac{3\Gamma(5-2\beta)\Gamma(6-3\beta)}{2\Gamma(4-2\beta)\Gamma(5-3\beta)}\right.$$
$$\left.+\frac{\Gamma(5-\beta)\Gamma(6-3\beta)}{2\Gamma(3-\beta)\Gamma(6-4\beta)}\right)\frac{x^{5-4\beta}}{\Gamma(6-4\beta)}-\frac{\Gamma(4-\beta)\Gamma(5-2\beta)x^{4-3\beta}}{\Gamma(3-\beta)\Gamma(4-2\beta)\Gamma(5-3\beta)},$$
$$\vdots$$

This implies that

$$u(x,t)=U_0(x,t)=x+a_0t+\frac{a_1}{2}t^2+\frac{a_2}{3}t^3+\cdots$$
$$=x+\left[-\frac{2x^{2-\beta}}{\Gamma(3-\beta)}+\frac{3x^{3-2\beta}}{\Gamma(4-2\beta)}\right]t+\left[\frac{2\Gamma(4-\beta)x^{3-2\beta}}{\Gamma(4-2\beta)\Gamma(3-\beta)}\right.$$
$$\left.-\frac{3\Gamma(5-2\beta)x^{4-3\beta}}{\Gamma(4-2\beta)\Gamma(5-3\beta)}-\frac{\Gamma(5-\beta)x^{4-3\beta}}{\Gamma(3-\beta)\Gamma(5-3\beta)}+\frac{3\Gamma(6-2\beta)x^{5-4\beta}}{2\Gamma(4-2\beta)\Gamma(6-4\beta)}\right]\frac{t^2}{2}+\cdots. \tag{27}$$

Setting $\beta=1$, in (27), we reproduce the solution of problem as follows

$$u(x,t)=x\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right), \tag{28}$$

where this is equivalent to the exact solution in closed form

$$u(x,t)=xe^{t}.$$

It is clear that no linearization or perturbation was used and a closed form solution is obtainable by adding more terms to the homotopy perturbation series.

Example 2. Consider the nonlinear time-fractional FPE

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\left[-\frac{\partial}{\partial x}.\left(\frac{4u}{x}-\frac{x}{3}\right)+\frac{\partial^{2}}{\partial x^{2}}u\right]u(x,t),\qquad t>0,\quad x>0, \tag{29}$$

where $0<\alpha\le 1$, $A^{*}(x,y,t)=\left(\frac{4u^{2}}{x}-\frac{xu}{3}\right)$, and $B^{*}(x,y,t)=u^{2}$. Subject to the initial condition

$$u(x,t)=x^{2}.$$

To solve Eq. (29) by NHPM, we construct the following homotopy:

$$(1-p)\left(\frac{\partial^{\alpha}U}{\partial t^{\alpha}}-u_{0}\right)+p\left(\frac{\partial^{\alpha}U}{\partial t^{\alpha}}+\frac{\partial}{\partial x}.\left(\frac{4U^{2}}{x}-\frac{xU}{3}\right)-\frac{\partial^{2}}{\partial x^{2}}.U^{2}\right)=0. \tag{30}$$

According to the Eq. (22), the following results will be obtained

$$p^{0}:U_{0}(x,t)=U(x,0)+J^{\alpha}u_{0},$$

$$p^{1}:U_{1}(x,t)=J^{\alpha}\left(-u_{0}-\frac{\partial}{\partial x}.\left(\frac{4U_{0}^{2}}{x}-\frac{x}{3}U_{0}\right)+\frac{\partial^{2}}{\partial x^{2}}.U_{0}^{2}\right),$$

$$p^{2}:U_{2}(x,t)=J^{\alpha}\left(-\frac{\partial}{\partial x}.\left(\frac{8U_{0}U_{1}}{x}-\frac{x}{3}U_{1}\right)+\frac{\partial^{2}}{\partial x^{2}}.(2U_{0}U_{1})\right),$$

$$\vdots$$

$$p^{j}:U_{j}(x,t)=J^{\alpha}\left(-\frac{\partial}{\partial x}.\left(\frac{4\left(\sum_{k=0}^{j-1}U_{k}U_{j-1-k}\right)}{x}-\frac{x}{3}U_{j-1}\right)+\frac{\partial^{2}}{\partial x^{2}}.\left(\left(\sum_{k=0}^{j-1}U_{k}U_{j-1-k}\right)\right)\right),$$

$$\vdots$$

Assuming $u_{0}(x,t)=\sum_{n=0}^{\infty}a_{n}(x,t)t^{n}$, $U(x,0)=u(x,0)=x^{2}$. By solving equation $U_{1}(x,\mathrm{t})=0,$ unknown coefficients a_{i}, $i=0,1,2,\ldots,$ are obtained as follows,

$$a_0 = x^2, \quad a_1 = \frac{x^2\Gamma(\alpha+2)}{\Gamma(2\alpha+1)}t^{\alpha-1}, \quad a_2 = \frac{x^2\Gamma(\alpha+3)}{2\Gamma(3\alpha+1)}t^{2\alpha-2}, \ \cdots. (31)$$

Therefore the following solution will be derived

$$u(x,t) = U_0(x,t) = x^2 + a_0\frac{t^{\alpha}}{\Gamma(\alpha+1)} + a_1\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + a_2\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} + \cdots$$
$$= x^2 + \frac{x^2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^2t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots = x^2\left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right). \quad (32)$$

Setting $\alpha = 1$, in (32), the solution of problem is obtained as follows

$$u(x,t) = x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right).$$

This solution is equivalent to the exact solution in closed form

$$u(x,t) = x^2e^t.$$

Example 3. Consider the linear space- time dependent fractional FPE

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \left[-\frac{\partial^{\beta}}{\partial x^{\beta}}.\left(\frac{\mathrm{x}}{6}\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}.\left(\frac{x^2}{12}\right)\right]u(x,t), \quad t > 0, \quad x > 0, \quad (33)$$

where $0 < \alpha, \beta \le 1$, $\mathrm{A}^*(x,t,u) = \frac{xu}{6}$, and $\mathrm{A}^*(x,t,u) = \frac{x^2u}{12}$. Subject to the initial condition

$$u(x,t) = x^2. \quad (34)$$

To solve Eq. (33) by NHPM, construct the following homotopy:

$$(1-p)\left(\frac{\partial^{\alpha}U}{\partial t^{\alpha}} - u_0\right) + p\left(\frac{\partial^{\alpha}U}{\partial t^{\alpha}} + \frac{\partial^{\beta}}{\partial x^{\beta}}.\left(\frac{\mathrm{xU}}{6}\right) - \frac{\partial^{2\beta}}{\partial x^{2\beta}}.\left(\frac{x^2U}{12}\right)\right) = 0. \quad (35)$$

According to the Eq. (22), we have

$$
\begin{aligned}
&p^0: U_0(x,t) = U(x,0) + J^{\alpha} u_0,\\
&p^1: U_1(x,t) = J^{\alpha}\left(-u_0(x,\xi) - \frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{xU_0}{6}\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\frac{x^2 U_0}{12}\right)\right),\\
&p^2: U_2(x,t) = J^{\alpha}\left(-\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{xU_1}{6}\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\frac{x^2 U_1}{12}\right)\right),\\
&\vdots\\
&p^j: U_j(x,t) = J^{\alpha}\left(-\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{xU_{j-1}}{6}\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\frac{x^2 U_{j-1}}{12}\right)\right),\\
&\vdots
\end{aligned}
\tag{36}
$$

Let's consider $u_0(x,t) = \sum_{n=0}^{\infty} a_n(x,t)t^n, U(x,0) = u(x,0) = x^2,$ by solving equation $U_1(x,\mathrm{t}) = 0,$ unknown coefficients a_i, $i = 0,1,2,\ldots,$ are obtained as follows

$$
\begin{aligned}
&a_0 = \frac{2x^{4-2\beta}}{\Gamma(5-2\beta)} - \frac{x^{3-\beta}}{\Gamma(4-\beta)},\\
&a_1 = \left[\frac{\Gamma(5-\beta)x^{4-2\beta}}{6\Gamma(4-\beta)\Gamma(5-2\beta)} - \left(\frac{\Gamma(6-2\beta)}{3\Gamma(5-2\beta)} + \frac{\Gamma(6-\beta)}{12\Gamma(4-\beta)}\right)\frac{x^{5-3\beta}}{\Gamma(6-3\beta)} + \frac{\Gamma(7-2\beta)x^{6-4\beta}}{6\Gamma(5-2\beta)\Gamma(7-4\beta)}\right]\frac{\Gamma(\alpha+2)t^{\alpha-1}}{\Gamma(2\alpha+1)},\\
&\vdots
\end{aligned}
$$

Therefore, we gain the following solution of Eq. (33)

$$
\begin{aligned}
u(x,t) = U_0(x,t) &= x^2 + a_0\frac{t^{\alpha}}{\Gamma(\alpha+1)} + a_1\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + a_2\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} + \cdots\\
&= x^2 + \left[\frac{2x^{4-2\beta}}{\Gamma(5-2\beta)} - \frac{x^{3-\beta}}{\Gamma(4-\beta)}\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left[\frac{\Gamma(5-\beta)x^{4-2\beta}}{6\Gamma(4-\beta)\Gamma(5-2\beta)} - \left(\frac{\Gamma(6-2\beta)}{3\Gamma(5-2\beta)} + \frac{\Gamma(6-\beta)}{12\Gamma(4-\beta)}\right)\frac{x^{5-3\beta}}{\Gamma(6-3\beta)}\right.\\
&\quad \left. + \frac{\Gamma(7-2\beta)x^{6-4\beta}}{6\Gamma(5-2\beta)\Gamma(7-4\beta)}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots.
\end{aligned}
\tag{37}
$$

Setting $\alpha = 1,$ and $\beta = 1,$ in (37), we reproduce the solution of problem as follows

$$
u(x,t) = x^2\left(1 + \left(\frac{t}{2}\right) + \frac{\left(\frac{t}{2}\right)^2}{2!} + \frac{\left(\frac{t}{2}\right)^3}{3!} + \cdots\right),
$$

which is equivalent to the following closed form

$$u(x,t) = \mathrm{x}^2 e^{t/2}.$$

6. Conclusion

In this paper, the NewHomotopy perturbation method is implemented to solve the space- time dependent fractional Fokker-Planck equation. It may be concluded that the method is verypowerful and efficient in finding analytical as well as numericalsolutions for wide classes of space-time fractional partial differentialequations.The study shows that the technique requires less computational work than existing approaches while supplying quantitatively reliable results. Finally, the NewHomotopy perturbation method is more effective and overcome the difficulty of traditional methods.

7. Reference

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