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Variational Homotopy Perturbation Method: An efficient scheme for solving partial differential equations in fluid mechanics

T. Allahviranloo[1], A. Armand[2,*], S. Pirmuhammadi[1]

[1]Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
[2]Young Researchers and Elite Club, Shahr-e-Rey Branch, Islamic Azad University, Tehran, Iran

tofigh@allahviranloo.com, atefeh.armand@ymail.com, s.pirmuhammadi@hotmail.com

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Abstract

In this paper, an application of variationalhomotopy perturbation method is applied to solve Korteweg-de Vries (KdV)and Burgers equations. The study reveals that the method is very effective and simple.

Keywords: Variational homotopy perturbation method, Korteweg-de Vries equation, Burgers equation.

1. Introduction

It is well known that there are many phenomena in engineering, biology, fluid mechanics and other sciences can be modeled as partial differential equations (PDEs), such as KdV equation, Burgers equation, Fisher's equation, wave equation, biharmonic equation and many other important equations. These equations are usually difficult to solve analytically, so these are required to obtain efficient approximately solutions. Therefore, in recent years many authors have studied on solutions of PDEs by some numerical methods. For example, the Adomian decomposition method (ADM) was employed in [8] for solving generalized Boussinesq equation. Application of the variational iteration method (VIM) to the KdV, k(2,2), Burgers and cubic Boussinesq equations are investigated in [22]. He's variational iteration technique is used in [2] for solving Klein-Gordon equation. In [3, 23], the applications of the VIM to burgers, coupled Burgers equation and Blasius equation are provided. The homotopy perturbation method (HPM) is used in [7] for solving KdV type equations. The authors of [20] applied the homotopy analysis method (HAM) to the KdV equation.

In this paper, we extend the application of the variational homotopy perturbation method (VHPM) to find approximate solutions for the KdV and Burgers equations. The VHPM which proposed in [16, 14], is based on the HPM and the VIM. The VIM is a simple and effective method which proposed by Chinese

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* Corresponding author.

mathematician Ji-Huan He [9-12] as modification of a general Lagrange multiplier method [13]. The homotopy perturbation method [4, 15, 17]has been used by many mathematicians and engineers to solve various functional equations. The HPM, using the homotopy technique of topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which considered as a small parameter. This method, which does not require a small parameter, which is the case with other methods, has a significant advantage in that it provides an approximate solution to a wide range of nonlinear problems in applied science [21]. Recently, the application of the variational homotopy perturbation method has been extended to higher dimensional initial boundary problems [16]. This technique is used in [14] for solving the Fisher's equation. Moreover, the VHPM was successfully applied to nonlinear oscillators [6]. For a relatively comprehensive survey of the methods and their applications, the reader is referred to [16].
Consider the KdV equation,

$$u_t - 3(u^2)_x + u_{3x} = 0, \qquad \textbf{(1.1)}$$

First derived by Korteweg de Vries in their study of long water waves in a (relatively shallow) channel in 1895. Also, Eq. (1.1) is the pioneering equation that gives rise to solitary wave solutions. Solutions which are waves with infinite support are generated as a result of the balance between the nonlinear convection $(u^n)x$ and the linear dispersion u_{3x} in these equations. Solutions are localized waves that propagate without change of their shape and velocity properties and stable against mutual collisions [22].

The Burgers equation

$$u_t + \frac{1}{2}(u^2)_x - u_{2x} = 0, \qquad \textbf{(1.2)}$$

appears in fluid mechanics. This equation incorporates both convection and diffusion in fluid dynamics, and is used to describe the structure of shock waves [22]. In the above equations, the unknown function $u = u(x,t)$ is sufficiently often differentiable function and it is usually assumed to be real.

Subscriptsdenote partial differentiations, that is $u_{kx} = \partial^k u / \partial x^k$, $k = 1,2,3$ and $u_t = \partial u / \partial t$ for $x \in \mathbb{R}, t > 0$.

The following structure leads this paper as follows: In section 2, we introduce the variational homotopy perturbation method. In section 3, we try to implement the VHPM on the KdV and Burgers equations. In the last section, we will present the results of this work.

2. The Method

In this section, we will highlight briefly the main point of the variational homotopy perturbation method (VHPM), where details can be found in [6, 14, 16].
To clarify the VHPM, let us assume the following nonlinear differential equation the form

$$L(u(x,t)) + N(u(x,t)) = g(t), \qquad \textbf{(2.1)}$$

where L and N are linear and nonlinear operators, respectively and $g(t)$ is a known analytical function. According to the variational iteration method, we write down a correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(x,s)[Lu_n(x,s) + N\tilde{u}_n(x,s) - g(s)]ds, \qquad \textbf{(2.2)}$$

where λ is a general Lagrangian multiplier, which can be determined optimally via the variational theory, u_n is the n^{th}approximate solution and $\tilde{u}_n$ denotes a restricted variation, i.e. $\delta\tilde{u}_n = 0$.
Now, by homotopy perturbation method [1, 5], we can construct an equation as follows

$$\sum_{i=0}^{\infty} p^i u_i = u_0(x) + p\int_0^x \lambda(x,s)\left[\sum_{i=0}^{\infty} p^i L(u_i(x,s)) + N(\sum_{i=0}^{\infty} p^i \tilde{u}_i(x,s))\right]ds - \int_0^x \lambda(x,s)g(s)ds. \qquad \textbf{(2.3)}$$

By equating the terms (2.3) with identical powers of *p*, and taking the limit as *p* tends to 1, we get

$$u(x,t)=\lim_{p\to 1}\sum_{i=0}^{\infty} p^i u_i(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+\cdots. \quad \textbf{(2.4)}$$

3. The Governing Equations

In what follows, we will apply the variational homotopy perturbation method to the KdV and Burgers equations.

3.1. The KdV equation

First, we consider Eq.(1.1) as

$$u_t-3(u^2)_x+u_{3x}=0, \qquad -\infty<x<+\infty, \quad t>0, \quad \textbf{(2.5)}$$

with initial condition

$$u(0,x)=6x,$$

where subscripts denote differentiations.

By means of the VHPM, we consider

$$L(u)=u_t, \textbf{(3. 4)}$$

and

$$N(u)=-3(u^2)_x+u_{3x}, \textbf{(3. 3)}$$

where L is a linear operator and $N$is a nonlinear operator. In order to construct a correction functional for this system,we can write down the following expression

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(x,s)[(u_n(x,s))_t-3(\tilde{u}_n^2(x,s))_x+(\tilde{u}_n(x,s))_{3x}], \textbf{(3. 4)}$$

where $\tilde{u}_n$ denote a restricted variation, i.e. $\delta(\tilde{u}_n)=0$. To find the optimal value of λ, we make the correction functional(3.4) stationary in the following form

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda(x,s)[(u_n(x,s))_t-3(\tilde{u}_n^2(x,s))_x+(\tilde{u}_n(x,s))_{3x}]ds$$

$$=\delta u_n(x,t)+\lambda(x,s)\delta u_n(x,s)|_{s=t}-\int_0^t \frac{\partial\lambda(x,s)}{\partial s}\delta u_n(x,s)ds$$

$$-\int_0^t \frac{\partial\lambda(x,s)}{\partial s}-3\delta(\tilde{u}_n^2(x,s))_x+\partial(\tilde{u}_n(x,s))_{3x}ds$$

$$=(1+\lambda(x,s))\delta u_n(x,s)|_{s=t}-\int_0^t \frac{\partial\lambda(x,s)}{\partial s}\delta u_n(x,s)ds=0.$$

Hence, we have the following stationary conditions

$$\frac{\partial}{\partial s}\lambda(x,s)|_{s=t}=0,$$

$$1+\lambda(x,s)|_{s=t}=0.$$

This in turn gives $\lambda(x,s)=-1$.By substituting λ into Eq. (3.4), the following formula is obtained

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t\left[(u_n(x,s))_t-3(u_n^2(x,s))_x+(u_n(x,s))_{3x}\right]ds. \quad \textbf{(3. 5)}$$

Then, Eq. (3.5) will enable us to determine the components $u_n\left(x,t\right)$ recursively for $n\geq 0$.

Now, by exerting the VHPM, it is then possible to obtain an equation as follows

$$u_0+pu_1+p^2u_2+\cdots=6x+3p\int_0^t\left[((u_0+pu_1+p^2u_2+\cdots)^2)_x\right]ds$$

$$-p\int_0^t\left[(u_0+pu_1+p^2u_2+\cdots)_{3x}\right]ds$$

$$=6x+3p\int_0^t\left[(u_0^2)_x+p(2u_0u_1)_x+p^2(u_1^2+2u_0u_2)_x+\cdots\right]ds$$

$$-p\int_0^t\left[(u_0)_{3x}+p(u_1)_{3x}+p^2(u_2)_{3x}+\cdots\right]ds.$$

By comparing the terms with identical powers of p, we have the following results

$$p^0:u_0(x,t)=6x,$$

$$p^1:u_1(x,t)=3\int_0^t(u_0^2)_x ds-\int_0^t(u_0)_{3x}ds=216xt,$$

$$p^2:u_2(x,t)=3\int_0^t(2u_0u_1)_x ds-\int_0^t(u_1)_{3x}ds=7776xt^2,$$

$$p^3:u_3(x,t)=3\int_0^t(u_1^2+2u_0u_2)_x ds-\int_0^t(u_2)_{3x}ds=279936xt^3,$$

$$\vdots$$

By calculating seven terms of the series, an approximate solution of the VHPM is obtained

$$u(x,t)\cong\phi_7(x,t)=\lim_{p\to 1}\sum_{i=0}^{6}p^iu_i(x,t)=6x(1+36t+(36t)^2+(36t)^3+(36t)^4+(36t)^5+(36t)^6). \quad \textbf{(3. 6)}$$

So the solution in a closed form is:

$$u(x,t)=\frac{6x}{1-36t}.$$

If we set *t=0.005* and *t=0.01*, comparing the VHPM and the VIM with exact solution yields our method is more accurate. Results are shown inTable(3.1)and Fig. (3.1).

Table (3. 1). Absolute error of the KdV equation

x	**VIM** in *t=0.005*	**VHPM** in *t=0.005*	**VIM** in *t=0.01*	**VHPM** in *t=0.01*
0.1	3.09037*E*−03	4.47966*E*−06	3.39311*E*−02	7.34664*E*−04
0.2	6.18074*E*−03	8.95932*E*−06	6.78622*E*−02	146933*E*−03
0.3	9.27112*E*−03	1.34390*E*−05	1.01793*E*−01	220399*E*−03
0.4	1.23615*E*−02	1.79186*E*−05	1.35724*E*−01	293866*E*−03
0.5	1.54519*E*−02	2.23983*E*−05	1.69655*E*−01	367332*E*−03
0.6	1.85422*E*−02	2.68780*E*−05	2.03587*E*−01	440798*E*−03
0.7	2.16326*E*−02	3.13576*E*−05	2.37518*E*−01	514265*E*−03
0.8	2.47230*E*−02	3.58373*E*−05	2.71449*E*−01	587731*E*−03
0.9	2.78134*E*−02	4.03169*E*−05	3.05380*E*−01	661198*E*−03
1.0	3.09037*E*−02	4.47966*E*−05	3.39311*E*−01	734664*E*−03

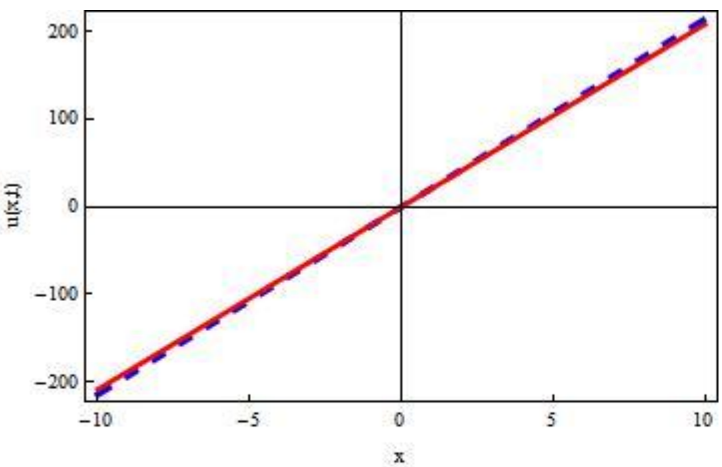

Fig. (3. 1). Comparison of the exact solution with approximate solution (3. 6) of the KdV equation for $t=1$ and $-10\le x\le 10$ .Symbols: solid line: exact solution; dash line: VHPM

3.2. The Burgers equation

Our last consideration is to takeplacein Burgers equation as

$$u_t+\frac{1}{2}(u^2)_x-u_{2x}=0,\qquad t>0,\quad x\in\mathbb{R}\ ,\textbf{(3. 7)}$$

with the following initial condition

$$u(x,0)=x.$$

By means of the first step of the VHPM, we assume that

$$L(u)=u_t\ ,\textbf{(3. 8)}$$

and

$$N(u)=\frac{1}{2}(u^2)_x-u_{2x}. \quad \textbf{(3. 9)}$$

Now, like before we try to construct the correction functional. So, take the form of

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda(x,s)[(u_n(x,s))_t+\frac{1}{2}(\tilde{u}_n^2(x,s))_x-(\tilde{u}_n(x,s))_{2x}]ds\,, \quad \textbf{(3. 10)}$$

again we obtain $\lambda(x,s)=-1$. Therefore, Eq. (3.10) changes to

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^t[(u_n(x,s))_t+\frac{1}{2}(u_n^2(x,s))_x-(u_n(x,s))_{2x}]ds. \qquad \textbf{(3. 11)}$$

Now, by preforming the VHPM, it is then possible to obtain the following relation

$$u_0+pu_1+p^2u_2+\cdots=x-\frac{1}{2}p\int_0^t\left[((u_0+pu_1+p^2u_2+\cdots)^2)_x\right]ds$$

$$+p\int_0^t\left[(u_0+pu_1+p^2u_2+\cdots)_{2x}\right]ds$$

$$=x-\frac{1}{2}p\int_0^t\left[(u_0^2)_x+p(2u_0u_1)_x+p^2(u_1^2+2u_0u_2)_x+\cdots\right]ds$$

$$+p\int_0^t\left[(u_0)_{2x}+p(u_1)_{2x}+p^2(u_2)_{2x}+\cdots\right]ds.$$

By equating the coefficients of p with the same power, one gets

$$p^0: u_0(x,t)=x,$$

$$p^1: u_1(x,t)=-\frac{1}{2}\int_0^t(u_0^2)_x\,ds+\int_0^t(u_0)_{2x}\,ds=-xt,$$

$$p^2: u_2(x,t)=-\frac{1}{2}\int_0^t(2u_0u_1)_x\,ds+\int_0^t(u_1)_{2x}\,ds=xt^2,$$

$$p^3: u_3(x,t)=-\frac{1}{2}\int_0^t(u_1^2+2u_0u_2)_x\,ds+\int_0^t(u_2)_{2x}\,ds=-xt^3,$$

$$\vdots$$

By calculating seven terms of the series, an approximate solution of the VHPM is obtained

$$u(x,t)\cong\phi_7(x,t)=\lim_{p\to1}\sum_{i=0}^{6}p^iu_i(x,t)=x(1-t+(t)^2-(t)^3+(t)^4-(t)^5+(t)^6)\,, \quad \textbf{(3.12)}$$

with the following closed form

$$u(x,t)=\frac{x}{1+t}\,.$$

The results corresponding absolute errors are presented in Table (3.2)and Fig. (3.2).

Table (3.2). Absolute error of the Burgers equation

x \ *t*	***0.005***	***0.01***	***0.1***
0.1	1.38777*E*−17	9.85323*E*−16	9.09091*E*−09
0.2	2.77555*E*−17	1.97065*E*−15	1.81818*E*−08
0.3	5.55111*E*−17	2.94209*E*−15	2.72727*E*−08
0.4	5.55111*E*−17	3.94129*E*−15	3.63636*E*−08
0.5	5.55111*E*−17	4.94049*E*−15	4.54545*E*−08
0.6	1.11022*E*−16	5.88418*E*−15	5.45455*E*−08
0.7	1.11022*E*−16	6.88338*E*–15	6.36364*E*−08
0.8	1.11022*E*−16	7.88258*E*−15	7.27273*E*−08
0.9	1.11022*E*−16	8.88178*E*−15	8.18182*E*−08
1.0	1.11022*E*−16	9.88098*E*−15	9.09091*E*−08

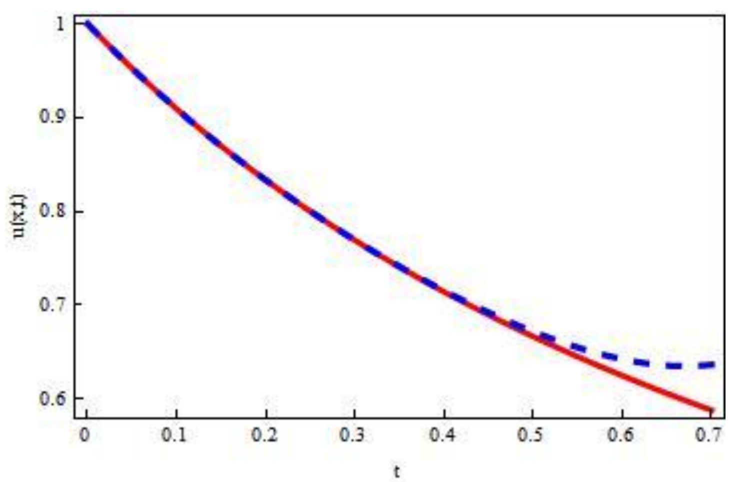

Fig. (3.2). Comparison of the exact solution with approximate solution (3.12) of the Burgers equation for $x=1$ and $0 \le t \le 0.7$ .Symbols: solid line: exact solution; dash line: VHPM

4. Conclusion

In this paper, the variational homotopy perturbation method was successfully employed for solving the KdV and Burgers equations. This method is based onthe homotopy perturbation method and variational iteration method. For our equations, the results of this method are exactly the same as those obtained bythe homotopy perturbation method. As an advantage of this method over the homotopy perturbation method, we do not need to solve a differential equation in eachiteration. It is important to note that this technique unlike most numerical method provides a closed form of the solution. The computations in this paper were performed by using Mathematica 7.

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