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Classification of a new subclass of $\xi^{(as)}$-QSO and its dynamics

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Abstract

A quadratic stochastic operator (QSO) describes the time evolution of different species in biology. The main problem with regard to a nonlinear operator is to study its behavior. This subject has not been studied in depth; even QSOs, which are the simplest nonlinear operators, have not been studied thoroughly. In this paper we introduce a new subclass of $\xi^{(as)}$-QSO defined on 2D simplex. first we classify this subclass into 18 non-conjugate classes. Furthermore, we investigate the behavior of one class.

Keywords: Quadratic stochastic operator, ℓ-Volterra quadratic stochastic operator, $\xi^{(s)}$-quadratic stochastic operator, permuted ℓ-Volterra quadratic stochastic operator, dynamics.
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1. Introduction

The departure point of quadratic stochastic operator can be traced back to Bernstein's work [2]. One of the most important application of QSO is to study the dynamics properties and modeling in several branches of science, for example physics [4], biology [3], mathematics, and economics [6]. One such system related to population genetics is given by a QSO, which is commonly, used to describe the time evolution of species in biology. The QSO in population genetics appears as follows (see [9, 24]). Take a population that consists of m species (traits) $1, 2, \dots, m$. If the probability distribution of initial state denoted by $x^{(0)} = (x_1^{(0)}, \dots, x_m^{(0)})$ and let the heredity coefficient $P_{ij,k}$ stand for probability which individual of the i^{th} and j^{th} species hybridized to give an individual from k^{th} species. Therefore the probability distribution of first generation $x^{(1)} = (x_1^{(1)}, \dots, x_m^{(1)})$ can be found by the formula

$$x_k^{(1)} = \sum_{i,j=1}^{m} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k = \overline{1, m}.$$

This result means that the relation $x^{(0)} \to x^{(1)}$ defines a mapping V called the evolution operator. The population evolves by starting from an arbitrary state $x^{(0)}$ then passing to the state $x^{(1)} = V(x^{(0)})$ then to the state $x^{(2)} = V(x^{(1)}) = V(V(x^{(0)})), \dots$. Hence, the evolution states of the population system are

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described by discrete dynamical system as follows: $x^{(0)}$, $x^{(1)} = V(x^{(0)})$, $x^{(2)} = V(x^{(1)}) = V(V(x^{(0)})), \dots$. In other words, if the probability distribution of intimal state (generation) was given, then a QSO is used to find the next state (generation). In this sense, the quadratic stochastic operator is a primary source for investigations of evolution of population genetics. The detailed exposure of the theory of quadratic stochastic operators is presented in [6, 11].

One of the main topics in the theory of QSO is an investigation of an asymptotic behavior of QSOs. Note that even in the small dimensional simplexes, this is a tricky job [3, 5, 12, 21, 25]. Since there is no general theory for these operators, it is natural to look first at their subclasses. The main problem has been solved for example for the subclass of Volterra operators [1, 6, 9, 14], ℓ- Volter-QSO [12, 19, 20], bistochastic QSO [7, 10, 22], etc.. However, all these classes together would not cover a set of all QSOs. Therefore, many classes of QSO have not been studied. Recently, a new class of QSO called $\xi^{(as)}$ was introduced [12–14, 17]. This class is defined by some partition of the coupled index set $P_m = \{(i, j) : i < j\} \subset I \times I$, where I is a set of integer numbers. In case of two dimensional simplex $(m = 3)$, the coupled index set (the coupled trait set) $\mathbf{P}_3$ has five possible partitions. The dynamics of $\xi^{(as)}$-QSO that correspond to the point partition (the maximal partition) of $\mathbf{P}_3$ have been investigated in [12, 17]. In [13, 16] it has been studied some class of $\xi^{(as)}$-QSO. In this paper, we investigate another new subclass of $\xi^{(as)}$-QSO and classify them into 18 non-conjugate classes. Moreover, we study the dynamic of QSO taken from one class, which is a convex combination of two QSO whose dynamics are studied in detail as well.

2. Preliminaries

Recall that a quadratic stochastic operator (QSO) V is a mapping of the simplex

$$S^{m-1} = \left\{ x = (x_1, \cdots, x_m) \in \mathbb{R}^m : \sum_{i=1}^{m} x_i = 1, \ \ x_i \geqslant 0, \ \ i = \overline{1, m} \right\}$$

into itself, of the form

$$(V(x))_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = \overline{1, m}, \tag{2.1}$$

where $\{P_{ij,k}\}$ are heredity coefficients which satisfy the following conditions

$$P_{ij,k} \geqslant 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{m} P_{ij,k} = 1. \tag{2.2}$$

We denote by $\mathrm{Fix}(V)$ the set of all fixed points of V. By Brouwer's fixed point theorem, one always has $\mathrm{Fix}(V) \neq \emptyset$ for any QSO V.

Definition 2.1. a point $\mathbf{x}$ is called fixed point of an operator V if $V(\mathbf{x}) = \mathbf{x}$.

Given a point $x^{(0)} \in S^{m-1}$, a trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ of $V : S^{m-1} \to S^{m-1}$ starting from $x^{(0)}$ is defined by $x^{(n+1)} = V(x^{(n)})$. By $\omega_V\left(x^{(0)}\right)$, we denote a set of limiting points of the trajectory $\{x^{(n)}\}_{n=0}^{\infty}$. Since $\{x^{(n)}\}_{n=0}^{\infty} \subset S^{m-1}$ and S^{m-1} is compact, one has $\omega_V\left(x^{(0)}\right) \neq \emptyset$. Obviously, if $\omega_V(x^{(0)})$ consists of a single point, then the trajectory converges, and a limiting point is a fixed point of V.

In what follows, we denote $I = \{1, 2, \dots, m\}$. A QSO V defined by (2.1), (2.2) is called *Volterra operator* [3] if one has

$$P_{ij,k} = 0 \ \text{ if } \ k \notin \{i, j\}, \ \ \forall i, j, k \in I. \tag{2.3}$$

The biological treatment of condition (2.3) is clear: *the offspring repeats the genotype (trait) of one of its parents.*

One can see that a Volterra QSO has the following form:

$$x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \quad k \in I,$$

where

$$a_{ki} = 2P_{ik,k} - 1 \ \text{ for } i \neq k \text{ and } \ a_{ii} = 0, \ i \in I.$$

Moreover,

$$a_{ki} = -a_{ik} \ \text{ and } \ |a_{ki}| \leqslant 1.$$

This kind of operators are intensively studied in [3–5, 8, 14].

Let $\ell \in I$ be fixed, and assume that the heredity coefficients $\{P_{ij,k}\}$ satisfy

$$P_{ij,k} = 0 \ \text{ if } \ k \notin \{i,j\} \ \text{ for any } \ k \in \{1,\dots,\ell\}, \ \ i,j \in I,$$
$$P_{i_0 j_0,k} > 0 \ \text{ for some } (i_0,j_0), \ i_0 \neq k, \ j_0 \neq k, \ \ k \in \{\ell+1,\dots,m\},$$

then the corresponding QSO defined by (2.1), (2.2), is called *ℓ-Volterra-QSO*.

Remark 2.2. Here, we emphasize the following points [19].

1. An ℓ-Volterra-QSO is a Volterra-QSO if and only if $\ell = m$.
2. No periodic trajectory exists for Volterra-QSO. However, such trajectories may exist for ℓ-Volterra-QSO.

Note that each element $x \in S^{m-1}$ can be considered as a probability distribution on the set $I = \{1,\dots,m\}$. Let $x = (x_1,\cdots,x_m)$ and $y = (y_1,\cdots,y_m)$ be two vectors taken from S^{m-1}. We say that x is *equivalent* to y if $x_k = 0 \Leftrightarrow y_k = 0$. We denote this relation by $x \sim y$.

Let $\operatorname{supp}(x) = \{i : x_i \neq 0\}$ be a support of $x \in S^{m-1}$. We say that x is *singular* to y and denote by $x \perp y$, if $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$. If $x, y \in S^{m-1}$, then $x \perp y$ if and only if $(x,y) = 0$, where $(\cdot,\cdot)$ stands for the standard inner product in $\mathbb{R}^m$.

We denote the set of all coupled indexes by

$$\mathbf{P}_m = \{(i,j) : \ i < j\} \subset I \times I, \quad \Delta_m = \{(i,i) : i \in I\} \subset I \times I.$$

For a given pair $(i,j) \in \mathbf{P}_m \cup \Delta_m$, we set a vector $\mathbb{P}_{ij} = \left(P_{ij,1},\cdots,P_{ij,m}\right)$. It is clear that $\mathbb{P}_{ij} \in S^{m-1}$ (see (2.2)).

Definition 2.3 ([17])**.** A QSO $V : S^{m-1} \to S^{m-1}$ given by (2.1), (2.2), is called a $\xi^{(as)}$-QSO w.r.t. the partitions ξ_1, ξ_2 (where the "as" stands for absolutely continuous-singular) if the following conditions are satisfied:

(i) for each $k \in \{1,\dots,N\}$ and any $(i,j), (u,v) \in A_k$, one has $\mathbb{P}_{ij} \sim \mathbb{P}_{uv}$;

(ii) for any $k \neq \ell$, $k,\ell \in \{1,\dots,N\}$ and any $(i,j) \in A_k$ and $(u,v) \in A_\ell$ one has $\mathbb{P}_{ij} \perp \mathbb{P}_{uv}$;

(iii) for each $d \in \{1,\dots,M\}$ and any $(i,i), (j,j) \in B_d$, one has $\mathbb{P}_{ii} \sim \mathbb{P}_{jj}$;

(iv) for any $s \neq h$, $s,h \in \{1,\dots,M\}$ and any $(u,u) \in B_s$ and $(v,v) \in B_h$ one has $\mathbb{P}_{uu} \perp \mathbb{P}_{vv}$.

Remark 2.4 ([17]). If ξ_2 is the point partition, i.e., $\xi_2 = \{\{(1,1)\},\dots,\{(m,m)\}\}$, then we call the corresponding QSO by $\xi^{(s)}$-QSO (where the "s" stands for singularity), since in this case every two different vectors $\mathbb{P}_{ii}$ and $\mathbb{P}_{jj}$ are singular. If ξ_2 is the trivial, i.e., $\xi_2 = \{\Delta_m\}$, then we call the corresponding QSO by $\xi^{(a)}$-QSO (where the "a" stands for absolute continuous), since in this case every two vectors $\mathbb{P}_{ii}$ and $\mathbb{P}_{jj}$ are equivalent. We note that some classes of $\xi^{(as)}$-QSO have been studied in [13, 16, 17]. In the present paper, we investigate the classification and dynamics of a subclass of $\xi^{(a)}$.

3. Classification of $\xi^{(a)}$-QSO on 2D simplex

In this section, we are going to classify the subclass of $\xi^{(a)}$-QSO in two dimensional simplex, i.e., $m = 3$. In this case, we have the following possible partitions of $\mathbf{P}_3$

$$\xi_1 = \{\{(1,2)\},\{(1,3)\},\{(2,3)\}\},\ |\xi_1| = 3, \qquad \xi_2 = \{\{(2,3)\},\{(1,2),(1,3)\}\},\ |\xi_2| = 2,$$
$$\xi_3 = \{\{(1,3)\},\{(1,2),(2,3)\}\},\ |\xi_3| = 2, \qquad \xi_4 = \{\{(1,2)\},\{(1,3),(2,3)\}\},\ |\xi_4| = 2,$$
$$\xi_5 = \{(1,2),(1,3),(2,3)\},\ |\xi_5| = 1.$$

We note that in [13], it has been investigated $\xi^{(a)}$-QSO related to the partition ξ_1 which is the maximal partition of $\mathbf{P}_3$ and in the paper [17] it has been studied $\xi^{(s)}$-QSO related to the partitions ξ_2, ξ_3, ξ_4.

Let us recall that two operators V_1, V_2 are called *(topologically or linearly) conjugate*, if there is a permutation matrix π such that $\pi^{-1}V_1\pi = V_2$. We say that two classes C_1 and C_2 of operators are conjugate if every operator taken from C_1 is conjugate to some operator taken from C_2. Using the same argument of [17, Proposition 5] one can prove the following.

Proposition 3.1. *A class of all $\xi^{(a)}$-QSO corresponding to the partition ξ_3 (or ξ_4) is conjugate to a class of all $\xi^{(a)}$-QSO corresponding to the partition ξ_2.*

Therefore, it is enough to study a class of all $\xi^{(a)}$-QSO corresponding to the partition ξ_2. Now, we shall consider some sub-class of a class of all $\xi^{(a)}$-QSO corresponding to the partition ξ_2 by choosing coefficients $(P_{ij,k})_{i,j,k=1}^3$ in special forms as in Tables 1 and 2, where $a, b, c \in [0,1]$, and $a + b + c = 1$.

Table 1: Construction of subclass of $\xi^{(a)}$-QSO.

Case	$\mathbb{P}_{12}$	$\mathbb{P}_{13}$	$\mathbb{P}_{23}$
$\mathbf{I}_1$	$(1,0,0)$	$(1,0,0)$	$(0,0,1)$
$\mathbf{I}_2$	$(0,1,0)$	$(0,1,0)$	$(1,0,0)$
$\mathbf{I}_3$	$(1,0,0)$	$(1,0,0)$	$(0,1,0)$
$\mathbf{I}_4$	$(0,0,1)$	$(0,0,1)$	$(1,0,0)$
$\mathbf{I}_5$	$(1,0,0)$	$(1,0,0)$	$(0,1,0)$
$\mathbf{I}_6$	$(0,1,0)$	$(0,1,0)$	$(1,0,0)$

Table 2: Construction of subclass of $\xi^{(a)}$-QSO.

Case	$\mathbb{P}_{11}$	$\mathbb{P}_{22}$	$\mathbb{P}_{33}$
$\mathbf{II}_1$	(a,b,c)	(a,b,c)	(a,b,c)
$\mathbf{II}_2$	(a,c,b)	(a,c,b)	(a,c,b)
$\mathbf{II}_3$	(b,a,c)	(b,a,c)	(b,a,c)
$\mathbf{II}_4$	(b,c,a)	(b,c,a)	(b,c,a)
$\mathbf{II}_5$	(c,a,b)	(c,a,b)	(c,a,b)
$\mathbf{II}_6$	(c,b,a)	(c,b,a)	(c,b,a)

Remark 3.2. We notice that in [13] it has been studied $\xi^{a)}$-QSO related to the partition ξ_1. In this paper we will study the subclass of $\xi^{(a)}$-QSO related to the partitions ξ_2, ξ_3, ξ_4.

The choices of the cases $(\mathbf{I}_i, \mathbf{II}_j)$, where $i, j = \overline{1,6}$, will give 36 operators from the class of $\xi^{(s)}$-QSO corresponding to the partition ξ_2. Finally, we obtain 36 parametric operators which are defined as follows:

$$V_1 : \begin{cases} x' = a(x^2+y^2+z^2) + 2x(1-x), \\ y' = b(x^2+y^2+z^2) + 2yz, \\ z' = c(x^2+y^2+z^2), \end{cases} \qquad V_2 : \begin{cases} x' = b(x^2+y^2+z^2) + 2x(1-x), \\ y' = a(x^2+y^2+z^2) + 2yz, \\ z' = c(x^2+y^2+z^2), \end{cases}$$

$$V_3: \begin{cases} x' = c(x^2+y^2+z^2)+2x(1-x), \\ y' = a(x^2+y^2+z^2)+2yz, \\ z' = b(x^2+y^2+z^2), \end{cases} \qquad V_4: \begin{cases} x' = c(x^2+y^2+z^2)+2x(1-x), \\ y' = b(x^2+y^2+z^2)+2yz, \\ z' = a(x^2+y^2+z^2), \end{cases}$$

$$V_5: \begin{cases} x' = a(x^2+y^2+z^2)+2x(1-x), \\ y' = c(x^2+y^2+z^2)+2yz, \\ z' = b(x^2+y^2+z^2), \end{cases} \qquad V_6: \begin{cases} x' = b(x^2+y^2+z^2)+2x(1-x), \\ y' = c(x^2+y^2+z^2)+2yz, \\ z' = a(x^2+y^2+z^2), \end{cases}$$

$$V_7: \begin{cases} x' = a(x^2+y^2+z^2)+2x(1-x), \\ y' = b(x^2+y^2+z^2), \\ z' = c(x^2+y^2+z^2)+2yz, \end{cases} \qquad V_8: \begin{cases} x' = b(x^2+y^2+z^2)+2x(1-x), \\ y' = a(x^2+y^2+z^2), \\ z' = c(x^2+y^2+z^2)+2yz, \end{cases}$$

$$V_9: \begin{cases} x' = c(x^2+y^2+z^2)+2x(1-x), \\ y' = a(x^2+y^2+z^2), \\ z' = b(x^2+y^2+z^2)+2yz, \end{cases} \qquad V_{10}: \begin{cases} x' = c(x^2+y^2+z^2)+2x(1-x), \\ y' = b(x^2+y^2+z^2), \\ z' = a(x^2+y^2+z^2)+2yz, \end{cases}$$

$$V_{11}: \begin{cases} x' = a(x^2+y^2+z^2)+2x(1-x), \\ y' = c(x^2+y^2+z^2), \\ z' = b(x^2+y^2+z^2)+2yz, \end{cases} \qquad V_{12}: \begin{cases} x' = b(x^2+y^2+z^2)+2x(1-x), \\ y' = c(x^2+y^2+z^2), \\ z' = a(x^2+y^2+z^2)+2yz, \end{cases}$$

$$V_{13}: \begin{cases} x' = a(x^2+y^2+z^2)+2yz, \\ y' = b(x^2+y^2+z^2)+2x(1-x), \\ z' = c(x^2+y^2+z^2), \end{cases} \qquad V_{14}: \begin{cases} x' = b(x^2+y^2+z^2)+2yz, \\ y' = a(x^2+y^2+z^2)+2x(1-x), \\ z' = c(x^2+y^2+z^2), \end{cases}$$

$$V_{15}: \begin{cases} x' = c(x^2+y^2+z^2)+2yz, \\ y' = a(x^2+y^2+z^2)+2x(1-x), \\ z' = b(x^2+y^2+z^2), \end{cases} \qquad V_{16}: \begin{cases} x' = c(x^2+y^2+z^2)+2yz, \\ y' = b(x^2+y^2+z^2)+2x(1-x), \\ z' = a(x^2+y^2+z^2), \end{cases}$$

$$V_{17}: \begin{cases} x' = a(x^2+y^2+z^2)+2yz, \\ y' = c(x^2+y^2+z^2)+2x(1-x), \\ z' = b(x^2+y^2+z^2), \end{cases} \qquad V_{18}: \begin{cases} x' = b(x^2+y^2+z^2)+2yz, \\ y' = c(x^2+y^2+z^2)+2x(1-x), \\ z' = a(x^2+y^2+z^2), \end{cases}$$

$$V_{19}: \begin{cases} x' = a(x^2+y^2+z^2), \\ y' = b(x^2+y^2+z^2)+2x(1-x), \\ z' = c(x^2+y^2+z^2)+2yz, \end{cases} \qquad V_{20}: \begin{cases} x' = b(x^2+y^2+z^2), \\ y' = a(x^2+y^2+z^2)+2x(1-x), \\ z' = c(x^2+y^2+z^2)+2yz, \end{cases}$$

$$V_{21}: \begin{cases} x' = c(x^2+y^2+z^2), \\ y' = a(x^2+y^2+z^2)+2x(1-x), \\ z' = b(x^2+y^2+z^2)+2yz, \end{cases} \qquad V_{22}: \begin{cases} x' = c(x^2+y^2+z^2), \\ y' = b(x^2+y^2+z^2)+2x(1-x), \\ z' = a(x^2+y^2+z^2)+2yz, \end{cases}$$

$$V_{23}: \begin{cases} x' = a(x^2+y^2+z^2), \\ y' = c(x^2+y^2+z^2)+2x(1-x), \\ z' = b(x^2+y^2+z^2)+2yz, \end{cases} \qquad V_{24}: \begin{cases} x' = b(x^2+y^2+z^2), \\ y' = c(x^2+y^2+z^2)+2x(1-x), \\ z' = a(x^2+y^2+z^2)+2yz, \end{cases}$$

$$V_{25}: \begin{cases} x' = a(x^2+y^2+z^2)+2yz, \\ y' = b(x^2+y^2+z^2), \\ z' = c(x^2+y^2+z^2)+2x(1-x), \end{cases} \qquad V_{26}: \begin{cases} x' = b(x^2+y^2+z^2)+2yz, \\ y' = a(x^2+y^2+z^2), \\ z' = c(x^2+y^2+z^2)+2x(1-x), \end{cases}$$

$$V_{27}: \begin{cases} x' = c(x^2+y^2+z^2)+2yz, \\ y' = a(x^2+y^2+z^2), \\ z' = b(x^2+y^2+z^2)+2x(1-x), \end{cases} \qquad V_{28}: \begin{cases} x' = c(x^2+y^2+z^2)+2yz, \\ y' = b(x^2+y^2+z^2), \\ z' = a(x^2+y^2+z^2)+2x(1-x), \end{cases}$$

$$V_{29}: \begin{cases} x' = a(x^2+y^2+z^2)+2yz, \\ y' = c(x^2+y^2+z^2), \\ z' = b(x^2+y^2+z^2)+2x(1-x), \end{cases} \qquad V_{30}: \begin{cases} x' = b(x^2+y^2+z^2)+2yz, \\ y' = c(x^2+y^2+z^2), \\ z' = a(x^2+y^2+z^2)+2x(1-x), \end{cases}$$

$$V_{31}: \begin{cases} x' = a(x^2+y^2+z^2), \\ y' = b(x^2+y^2+z^2)+2yz, \\ z' = c(x^2+y^2+z^2)+2x(1-x), \end{cases} \qquad V_{32}: \begin{cases} x' = b(x^2+y^2+z^2), \\ y' = a(x^2+y^2+z^2)+2yz, \\ z' = c(x^2+y^2+z^2)+2x(1-x), \end{cases}$$

$$V_{33}: \begin{cases} x' = c(x^2+y^2+z^2), \\ y' = a(x^2+y^2+z^2)+2yz, \\ z' = b(x^2+y^2+z^2)+2x(1-x), \end{cases} \qquad V_{34}: \begin{cases} x' = c(x^2+y^2+z^2), \\ y' = b(x^2+y^2+z^2)+2yz, \\ z' = a(x^2+y^2+z^2)+2x(1-x), \end{cases}$$

$$V_{35}: \begin{cases} x' = a(x^2+y^2+z^2), \\ y' = c(x^2+y^2+z^2)+2yz, \\ z' = b(x^2+y^2+z^2)+2x(1-x), \end{cases} \qquad V_{36}: \begin{cases} x' = b(x^2+y^2+z^2), \\ y' = c(x^2+y^2+z^2)+2yz, \\ z' = a(x^2+y^2+z^2)+2x(1-x). \end{cases}$$

Theorem 3.3. *The above obtained 36 operators from the class of $\xi^{(a)}$-QSO, corresponding to the partition ξ_2, are classified into 18 non-conjugate classes:*

$$\begin{array}{cccc} C_1 = \{V_1, V_{11}\}, & C_2 = \{V_2, V_{12}\}, & C_3 = \{V_3, V_{10}\}, & C_4 = \{V_4, V_9\}, \\ C_5 = \{V_5, V_7\}, & C_6 = \{V_6, V_8\}, & C_7 = \{V_{13}, V_{29}\}, & C_8 = \{V_{14}, V_{30}\}, \\ C_9 = \{V_{15}, V_{28}\}, & C_{10} = \{V_{12}, V_{27}\}, & C_{11} = \{V_{17}, V_{25}\}, & C_{12} = \{V_{18}, V_{26}\}, \\ C_{13} = \{V_{19}, V_{35}\}, & C_{14} = \{V_{20}, V_{36}\}, & C_{15} = \{V_{21}, V_{34}\}, & C_{16} = \{V_{22}, V_{33}\}, \\ C_{17} = \{V_{23}, V_{32}\}, & C_{18} = \{V_{24}, V_{32}\}. & & \end{array}$$

Proof. The proof is straightforward. One can see the partition

$$\xi_2 = \{\{(2,3)\}, \{(1,2),(1,3)\}\}$$

is invariant under only one permutation $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$. Therefore, we just perform π_1. □

4. Dynamics of operator taken from class C_1

In this section, we are going to study the dynamics of the operator V_1 taken from the class C_1 when $b = 0$. In this case, we denote the corresponding operator by V_a (since $c = 1-a$). Then one can see that $V_a = aW_1 + (1-a)W_2$, where

$$W_1: \begin{cases} x' = x^2+y^2+z^2+2xy+2xz, \\ y' = 2yz, \\ z' = 0, \end{cases} \qquad W_2: \begin{cases} x' = 2xy+2xz, \\ y' = 2yz, \\ z' = x^2+y^2+z^2. \end{cases}$$

One can see immediately that the operators W_1 and W_2 are two ℓ-Volterra QSOs on S^2.

Now we are aiming to study the dynamics of both operators.

Lemma 4.1. *For the operator W_1 one has $\mathrm{Fix}(W_1) = e_1$. Moreover, $\omega_{W_1}(x) = e_1$, for any $x \in S^2$, where $e_1 = (1,0,0)$.*

Proof. A direct substitution gives us that $e_1 = (1,0,0)$ is a fixed point of W_1. Moreover, $W_1^n(x^{(0)})$ goes to e_1. □

Now let us define the following regions:

$$B_1 := \{(x,y,z):\ x \geqslant \frac{1}{2},\ \ 0 \leqslant z \leqslant \frac{1}{3}\},$$
$$B_2 := \{(x,y,z):\ x \leqslant \frac{1}{2},\ \ 0 \leqslant z \leqslant \frac{1}{3}\},$$
$$B_3 := \{(x,y,z):\ 0 \leqslant x \leqslant \frac{1}{3},\ \ \frac{1}{2} \leqslant z \leqslant 1,\ 0 \leqslant y \leqslant \frac{1}{3}\},$$
$$B_4 := \{(x,y,z):\ 0 \leqslant x \leqslant \frac{1}{2},\ \ \frac{1}{3} \leqslant z \leqslant \frac{1}{2},\ 0 \leqslant y \leqslant \frac{1}{2}\}.$$

One can see that $\bigcup_{i=1}^{4} B_i = S^2$.

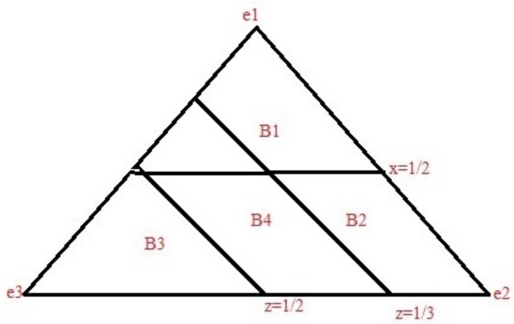

Figure 1: Sub-regions of simplex.

Denote

$$\Gamma_1 := \{(x,y,z):\ x = 0\},\quad \Gamma_2 := \{(x,y,z):\ y = 0\},\quad \Gamma_3 := \{(x,y,z):\ z = 0\}.$$

Proposition 4.2. *Let $f(x) = x - 2x^2$ and $g(x,z) = x^2 + (1-x-z)^2 + z^2$. Then the following statements hold.*

(i) *The function f is decreasing when $x \in (\frac{1}{2}, 1]$ and increasing when $x \in [0, \frac{1}{2})$.*

(ii) *If $x^{(0)} \in B_2$, then the minimum value of the function $g(x,z)$ is $\frac{1}{3}$.*

(iii) *One has $\frac{1}{3} \leqslant g(x,z) \leqslant \frac{1}{2}$, one B_4.*

Proof. The proof is obvious. □

We are now in the position to study the dynamic of W_2.

Theorem 4.3. *The following statements hold.*

(i) $\mathrm{Fix}(W_2) = \{e_3, (0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2})\}$*, where* $e_3 = (0,0,1)$.

(ii) *Γ_1 and Γ_2 are invariant.*

(iii) *B_4 is invariant region.*

(iv) *If $x \in S^2 \setminus B_4$ then there exist $n_k \in \mathbb{N}$ such that $W_2^{n_k}(x) \in B_4$.*

Proof.

(i). To find fixed point of W_2 we have to solve the following system

$$x = 2x - 2x^2,\quad y = 2yz,\quad z = x^2 + y^2 + z^2.$$

It is easy to see that $x \in \{0, \frac{1}{2}\}$ and from the second equation of above system, we obtain $y = 0$ or $z = \frac{1}{2}$. If $y = 0$ we have the fixed points $\{e_3, (\frac{1}{2}, 0, \frac{1}{2})\}$. If $z = \frac{1}{2}$ we have the fixed point $(0, \frac{1}{2}, \frac{1}{2})$.

(ii). This statement is obvious.

(iii). We are going to prove if $x^{(0)} \in B_4$ then $W_2(x^{(0)}) \in B_4$. To end this job, let us consider the functions $f(x) = 2x - 2x^2$, and $g(x,z) = x^2 + (1-x-z)^2 + z^2$. Here we have to find the minimum and maximum values of f and g over the region B_4. After long, but simple calculations, we obtain that $0 \leqslant f(x) \leqslant \frac{1}{2}$, and $\frac{1}{3} \leqslant g \leqslant \frac{1}{2}$ which imply that B_4 is invariant.

(iv). Let us first consider $x^{(0)} \in B_1$. Then by (i) of Proposition 4.2 we have $W_2(x^{(0)})$ in $B_2 \cup B_3 \cup B_4$. Now let $x^{(0)} \in B_2$. Then by (ii) of Proposition 4.2 one has the minimum value of next iteration is $\frac{1}{3}$ it follows that the next iteration will be in $B_3 \cup B_4$.

Due to (iii) it is enough to consider $x \in B_3$. To prove the statement, we suppose the region B_3 is invariant, i.e. for any $x \in B_3$ one has $W_2^n(x) \in B_3$, for all $n \in \mathbb{N}$. It is clear that the sequence $\{x^n\}$ is increasing and converges to $1/2$. Furthermore, the sequence $\{y^n\}$ is bounded and increasing, since B_3 is invariant and $z > \frac{1}{2}$. Therefore, it is convergent to a fixed point. The only possible (belonging to B_3) fixed point is $\frac{1}{2}$. It follows that $\{z^n\}$ goes to zero which is impossible because the point $(\frac{1}{2}, \frac{1}{2}, 0)$ is not a fixed point. Therefore, there is $n_k \in N$ such that $W_2^{n_k} \in B_4$. It follows that for any initial point $x \notin \text{Fix}(W_2)$ the trajectory $W_2^n(x)$ will be in B_4. □

Theorem 4.4. *Let $W_2 : S^2 \to S^2$ be a QSO defined above, and let $x \notin \text{Fix}(W_2)$ be an initial point, then*

$$\omega_{W_2}(x^{(0)}) = \begin{cases} (\frac{1}{2}, 0, \frac{1}{2}) : & x \in S^2 \setminus \Gamma_1, \\ (0, \frac{1}{2}, \frac{1}{2}) : & x \in \Gamma_1. \end{cases}$$

Proof. Let $x \in S^2 \setminus \Gamma_1$. Then by Theorem 4.3, it is enough to study the behavior of W_2 on B_4. Let us consider the function $f(x) = x - 2x^2$. By Proposition 4.2 we have that $f^n(x)$ converges to $1/2$. In addition, the sequence $\{y^n\}$ is decreasing and bounded, which means that it is convergent to fixed point $y = 0$. It follows that the trajectory $W_2^n(x)$ tends to $(\frac{1}{2}, 0, \frac{1}{2})$.

Let us now consider $x \in \Gamma_1$. Then $y^{(n)}$ converges to $1/2$. Finally we get the trajectory $W_2^n(x)$ goes to $(0, \frac{1}{2}, \frac{1}{2})$. □

Now, we consider the following l-Volterra QSO on the two-dimensional simplex

$$V_a : \begin{cases} x' = a(x^2 + y^2 + z^2) + 2xy + 2xz, \\ y' = 2yz, \\ z' = (1 - a)(x^2 + y^2 + z^2). \end{cases} \tag{4.1}$$

In [19, 20] ℓ-Volterra QSOs have been introduced and some of them are also studied. But, the QSO given by (4.1) is not studied yet. In general, still it is an open problem to investigate whole trajectories of ℓ-Volterra QSOs.

Remark 4.5. As we pointed out that V_a is a convex combination of W_1 and W_2, and both of them are regular, therefore, it is expected that V_a is also regular. In general, it is an open problem to study the behavior of convex combination of two regular QSOs. In this paper, we are partially going to study the mentioned problem.

Theorem 4.6. *Let $V_a : S^2 \to S^2$ be a QSO given by (4.1). Then the following statements hold:*

(i)

$$\text{Fix}(V_a) = \begin{cases} \{(\frac{2a-1-\sqrt{-4a^2+4a+1}}{4a-4}, 0, \frac{2a-3+\sqrt{-4a^2+4a+1}}{4a-4}), e_3\} : & a \neq 1, \\ (1, 0, 0) : & a = 1. \end{cases}$$

(ii) $\text{Eigenvalues}(V_a) = \{2, l_1, l_2\}$, $a \neq 1$, *where* $l_1 = 2 - 2x^*$, $l_2 = 2(1-a)(1-2x^*)$, *here*

$$x^* = \frac{2a - 1 - \sqrt{-4a^2 + 4a + 1}}{4a - 4}.$$

(iii) *Let $a \in (0, 1)$. Then the fixed point is attracting, i.e.* $|\lambda_1| < 1$, $|\lambda_2| < 1$.

Proof.

(i). To find the fixed points of V_a we have to solve the following system of equations. Namely,

$$x = a(x^2 + y^2 + z^2) + 2x(1 - x), \quad y = 2yz, \quad z = (1 - a)(x^2 + y^2 + z^2).$$

First of all let $a \neq 1$, then from the second equation of the above system we obtain $y = 0$ or $z = \frac{1}{2}$. If

$y = 0$, then the first equation of system becomes $(2a-2)x^2 + (-2a+1)x + a = 0$. It is not difficult to see the solutions of last quadratic equation are $x_1 = \frac{2a-1-\sqrt{-4a^2+4a+1}}{4a-4}, x_2 = \frac{2a-1+\sqrt{-4a^2+4a+1}}{4a-4}$. If $a = 0$ then we have the fixed point e_3 and $(\frac{1}{2}, 0, \frac{1}{2})$. It is clear that $x_1 \in [0,1]$ but $x_2 \notin [0,1]$. Therefore, we have the fixed point $(\frac{2a-1-\sqrt{-4a^2+4a+1}}{4a-4}, 0, \frac{2a-3+\sqrt{-4a^2+4a+1}}{4a-4})$. Now if $z = \frac{1}{2}$, then the first equation of above system takes the following form $(2a-2)x^2 + (-a+1)x + 1/2\,a$. It is easy to find the solutions of last quadratic equation are $x_1 = \frac{a-1+\sqrt{-3a^2+2a+1}}{4a-4}, x_2 = \frac{a-1-\sqrt{-3a^2+2a+1}}{4a-4}$. One can have $x_1 \notin [0,1]$ and the minimum value of x_2 is $\frac{1}{2}$, this means that $y \leqslant 0$, which is impossible. Hence if $a \neq 1$, we have the only fixed point $(\frac{2a-1-\sqrt{-4a^2+4a+1}}{4a-4}, 0, \frac{2a-3+\sqrt{-4a^2+4a+1}}{4a-4})$. If $a = 1$, then we immediately have the fixed point $e_1 = (1,0,0)$.

(ii). The Jacobian of the fixed point is

$$J(V_a) = \begin{pmatrix} 2ax + 2y + 2z & 2ay + 2x & 2az + 2x \\ 0 & 2z & 2y \\ 2(1-a)x & 2(1-a)y & 2(1-a)z \end{pmatrix}.$$

Then its eigenvalues are solutions of the equation

$$\lambda^3 - A\lambda^2 + Bl + C = 0,$$

where

$$\begin{aligned} A &= 6 - 6x^* - 2a + 4ax^*, \\ B &= -20ax^* + 8ax^{*2} - 8x^{*2} - 12 + 24x^* + 8a, \\ C &= -8 + 8a + 24x^* - 24ax^* + 16ax^{*2} - 16x^{*2}. \end{aligned}$$

Then one can calculate that the eigenvalues are $\lambda_0 = 2, \lambda_1 = 2 - 2x^*$, and $\lambda_2 = 2(1-a)(1-2x^*)$, where $x^* = \frac{2a-1-\sqrt{-4a^2+4a+1}}{4a-4}$.

(iii). This statement is clear. □

Remark 4.7. If $a = 1$, then the eigenvalues of the fixed point e_1 are $\lambda_0 = 2$, $\lambda_1 = \lambda_2 = 0$. If $a = 0$ then the fixed point e_3 is repelling, since its eigenvalues are $\lambda_0 = \lambda_1 = \lambda_2 = 2$.

Note: The trajectory of V_a where $a \in (0,1)$ goes to the fixed point $(x^*, 0, 1-x^*)$ as shown in Figure 2.

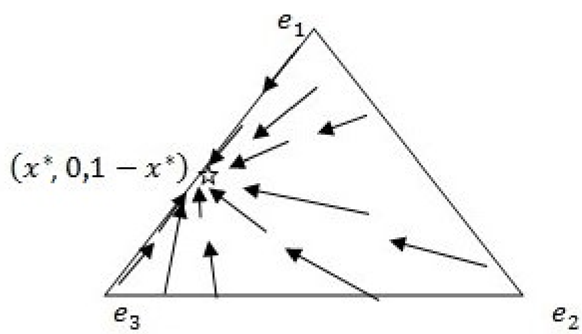

Figure 2: The trajectory of V_a, where $a \in (0,1)$.

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