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A Modified Explicit Method for the Black-Scholes Equation with Positivity Preserving Property

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Abstract

In this paper, we show that the standard finite difference scheme can generate numerical drawbacks such as spurious oscillations in the solution of the famous Black-Scholes partial differential equation, in the presence of discontinuities. We propose a modification of this scheme based on a nonstandard discretization. The proposed scheme is free of spurious oscillations and satisfies the positivity requirement, as is demanded for the financial solution of the Black-Scholes equation.

Keywords: Black-Scholes equation, Nonstandard finite differences, Positivity preserving, Stability.

1. Introduction

The famous Black-Scholes equation is an effective model for option pricing, i.e. to compute a fair value for the double barrier knock-out call option. A modified version of this model for the European option pricing in the form of initial value problem can be written [8, 9, 10, 11] as:

$$-\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}-rV=0, \tag{1}$$

where $V(S, \mathrm{t})$ is the price of the option and endowed with initial and boundary conditions:

$$V(S,0)=\max(\mathrm{S}-\mathrm{K},0)\,\mathrm{l}_{[\mathrm{L},\mathrm{U}]}(\mathrm{S}),$$
$$V(S,t)\to 0 \quad \text{as} \quad \mathrm{S}\to 0 \quad \text{or} \quad \mathrm{S}\to\infty,$$

with updating of the initial condition at the monitoring dates t_i ,i=1, …, F:

$$V(S,t_i)=V(\mathrm{S}-t_i^-)\,\mathrm{l}_{[\mathrm{L},\mathrm{U}]}(\mathrm{S}),\qquad 0=t_0<t_1<\cdots<t_F=T,$$

where $\mathrm{l}_{[\mathrm{L},\mathrm{U}]}(\mathrm{S})$ is the indicator function, i.e.,

$$1_{[L,U]}(S)=\begin{cases}1 & if \quad S\in[L,U]\\ 0 & if \quad S\notin[L,U]\end{cases}, \tag{2}$$

where the parameter $r>0$ is the interest rate and the reference volatility is $\sigma>0$.

To obtain the finite difference approximation for equation (1), let the computational domain $[0,S_{\max}]\times[0,T]$ is discretized by a uniform mesh with steps ΔS, Δt in order to obtain grid points $(j\Delta S, n\Delta t)$, $j=1,\cdots,M$ and $n=0,1,\cdots,N$ so that $S_M=S_{\max}=M\Delta S$ and $T=N\Delta t$. By the forward difference for $\frac{\partial V}{\partial t}$ and centered difference for discretization of $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$ and approximations V_j^n of V at the grid points, we have the following explicit finite difference method:

$$-\frac{V_j^{n+1}-V_j^n}{\Delta t}+rS_j\frac{V_{j+1}^n-V_{j-1}^n}{2\Delta S}+\frac{1}{2}(\sigma S_j)^2\frac{V_{j-1}^n-2V_j^n+V_{j+1}^n}{\Delta S^2}-rV_j^n=0. \tag{3}$$

This method has low accuracy and often generates numerical drawbacks such as spurious oscillations and negative values in the solution whenever the financial parameters of the Black-Scholes model σ and r satisfy the relationship $\sigma^2 \ll r$, see Figure 1. In the case of larger time step, we see the same behavior, see Figure 2. The values of the parameters used in our simulation are taken from [8].

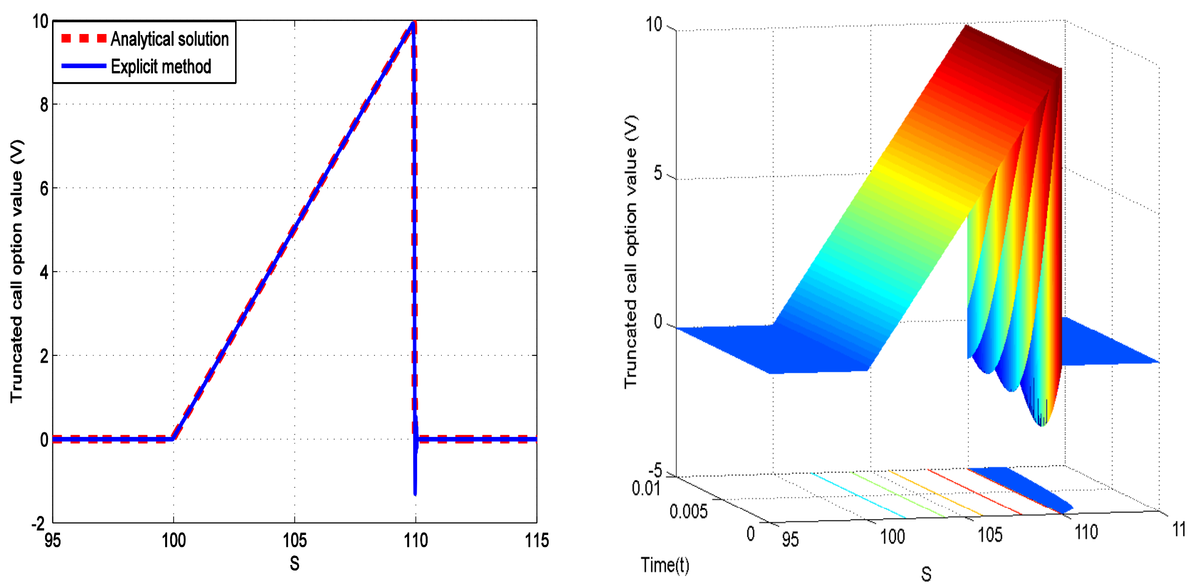

Figure 1. Truncated call option value for explicit method with $\Delta S=0.01$, $\Delta t=10^{-6}$. parameters: $L=90$, $K=100$, $U=110$, $r=0.05$, $\sigma=0.001$, $T=0.01$, $S_{\max}=120$.

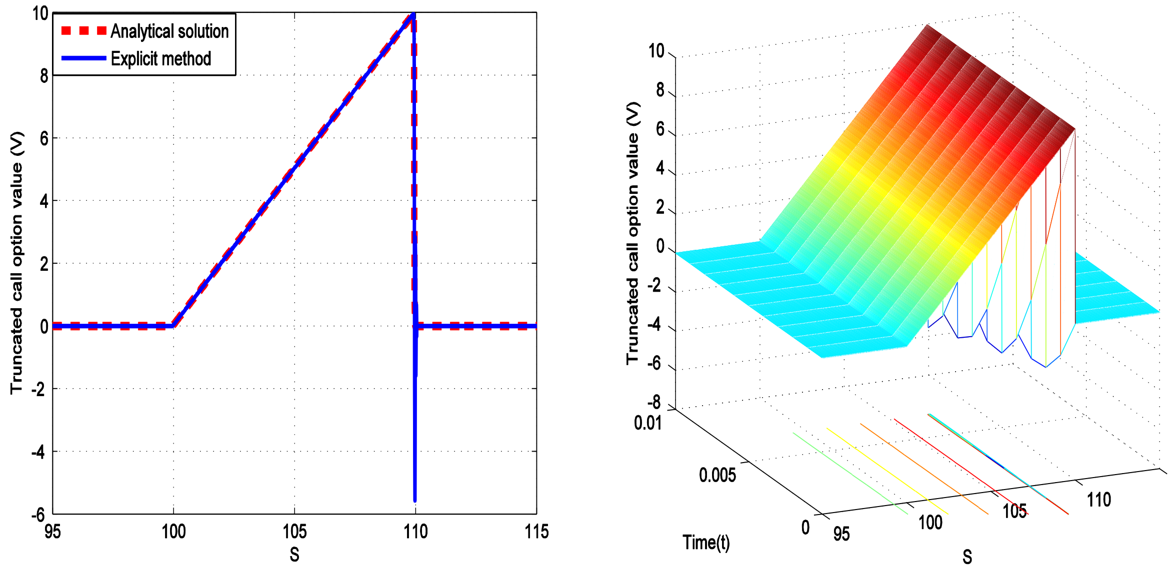

Figure 2. Truncated call option value for explicit method with $\Delta S = 0.01$, $\Delta t = 10^{-3}$. parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 0.01$, $S_{\max} = 120$.

2. Scheme construction

To overcome the drawbacks mentioned above, we develop an explicit nonstandard finite difference method [2,3,4,5,6,7] within the strategy suggested by Milev in [8]. We propose our nonstandard finite difference scheme as:

$$-\frac{V_j^{n+1} - V_j^n}{\Delta t} + rS_j \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S} + \frac{1}{2}(\sigma S_j)^2 \frac{V_{j-1}^n - 2V_j^n + V_{j+1}^n}{\Delta S^2} - ra(V_{j+1}^n + V_{j-1}^n) - r(1-2a)V_j^n = 0, \tag{4}$$

Where it can be written in the following form:

$$\frac{1}{\Delta t} V_j^{n+1} = \left(-\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - ra\right)V_{j-1}^n + \left(\frac{1}{\Delta t} - \left(\frac{\sigma S_j}{\Delta S}\right)^2 - r(1-2a)\right)V_j^n + \left(\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - ra\right)V_{j+1}^n. \tag{5}$$

Theorem 1. Sufficient for scheme (5) to be positive is

$$a \le -\frac{r}{8\sigma^2} \quad , \quad \Delta t < \frac{1}{(\sigma j)^2 + r(1-2a)}. \tag{6}$$

Proof. From (5) it is enough to show that

$$-\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - ra \ge 0, \tag{7}$$

$$\frac{rS_j}{2\Delta S} + \frac{1}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - ra \ge 0, \tag{8}$$

$$\frac{1}{\Delta t}-(\frac{\sigma S_j}{\Delta S})^2-r(1-2a)\geq 0. \qquad (9)$$

From (7) we can write

$$ra \leq \frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2-\frac{rS_j}{2\Delta S},$$

$$\Leftrightarrow a \leq \frac{1}{r}\left[\frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2-\frac{rS_j}{2\Delta S}\right],$$

$$\Leftrightarrow a \leq \frac{\sigma^2}{2r}\left[(\frac{S_j}{\Delta S})^2-\frac{r}{\sigma^2}(\frac{S_j}{\Delta S})+\frac{r^2}{4\sigma^4}-\frac{r^2}{4\sigma^4}\right], \qquad (10)$$

$$\Leftrightarrow a \leq \frac{\sigma^2}{2r}\left[(\frac{S_j}{\Delta S}-\frac{r}{2\sigma^2})^2-\frac{r^2}{4\sigma^4}\right],$$

$$\Leftrightarrow a \leq \frac{\sigma^2}{2r}(\frac{S_j}{\Delta S}-\frac{r}{2\sigma^2})^2-\frac{r}{8\sigma^2},$$

now, the last inequality in (10) shows sufficiency of $a \leq -\frac{r}{8\sigma^2}$ for (6), (as a consequence (8) holds too), on the other hand from (9) we have $\frac{1}{\Delta t}\geq(\frac{\sigma S_j}{\Delta S})^2+r(1-2a)$, from which $\Delta t<\frac{1}{(\sigma j)^2+r(1-2a)}$ and this completes the proof.

Theorem 2. Under the conditions (6), the proposed scheme is stable and convergent with local truncation error $O(\Delta t, \Delta S^2)$.

Proof. Using the Fourier stability method [1] put

$$V_j^n = e^{\alpha n \Delta t} e^{i\beta j \Delta S} \qquad (11)$$

Where $i=\sqrt{-1}$, and β is an arbitrary real. Substituting of (11) into (5) we obtain

$$\frac{1}{\Delta t}e^{\alpha(n+1)\Delta t}e^{i\beta j\Delta S}=(-\frac{rS_j}{2\Delta S}+\frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2-ra)e^{\alpha n\Delta t}e^{i\beta(j-1)\Delta S}$$

$$+(\frac{1}{\Delta t}-(\frac{\sigma S_j}{\Delta S})^2-r(1-2a))e^{\alpha n\Delta t}e^{i\beta j\Delta S}$$

$$+(\frac{rS_j}{2\Delta S}+\frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2-ra)e^{\alpha n\Delta t}e^{i\beta(j+1)\Delta S}, \qquad (12)$$

division by $e^{\alpha n\Delta t}e^{i\beta j\Delta S}$ leads to

$$e^{\alpha\Delta t}=\frac{(-\frac{rS_j}{2\Delta S}+\frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2-ra)e^{-i\beta\Delta S}+(\frac{1}{\Delta t}-(\frac{\sigma S_j}{\Delta S})^2-r(1-2a))+(\frac{rS_j}{2\Delta S}+\frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2-ra)e^{i\beta\Delta S}}{\frac{1}{\Delta t}}, \qquad (13)$$

and taking the real part it is seen that the absolute value of the amplification factor $e^{\alpha\Delta t} \le 1$.

Therefor the scheme is stable and convergent with local truncation error:

$$T_j^n = -\frac{V(\mathrm{S}_j, \mathrm{t}_{n+1}) - V(\mathrm{S}_j, \mathrm{t}_n)}{\Delta t} + rS_j \frac{V(\mathrm{S}_{j+1}, \mathrm{t}_n) - V(\mathrm{S}_{j-1}, \mathrm{t}_n)}{2\Delta S}$$

$$+\frac{1}{2}(\sigma S_j)^2 \frac{V(\mathrm{S}_{j-1}, \mathrm{t}_n) - 2V(\mathrm{S}_j, \mathrm{t}_n) + V(\mathrm{S}_{j+1}, \mathrm{t}_n)}{\Delta S^2}$$

$$-ra(V(\mathrm{S}_{j+1}, \mathrm{t}_n) + V(\mathrm{S}_{j-1}, \mathrm{t}_n)) - r(1-2\mathrm{a})V(\mathrm{S}_j, \mathrm{t}_n), \tag{14}$$

by Taylor's expansion, we have

$$V(\mathrm{S}_j, \mathrm{t}_{n+1}) = V(\mathrm{S}_j, \mathrm{t}_n) + \Delta t\left(\frac{\partial V}{\partial t}\right)_j^n + \frac{1}{2}\Delta t^2\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n + \frac{1}{6}\Delta t^3\left(\frac{\partial^3 V}{\partial t^3}\right)_j^n + \cdots,$$

$$V(\mathrm{S}_{j+1}, \mathrm{t}_n) = V(\mathrm{S}_j, \mathrm{t}_n) + \Delta S\left(\frac{\partial V}{\partial S}\right)_j^n + \frac{1}{2}\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n + \frac{1}{6}\Delta S^3\left(\frac{\partial^3 V}{\partial S^3}\right)_j^n + \cdots,$$

$$V(\mathrm{S}_{j+1}, \mathrm{t}_n) = V(\mathrm{S}_j, \mathrm{t}_n) - \Delta S\left(\frac{\partial V}{\partial S}\right)_j^n + \frac{1}{2}\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n - \frac{1}{6}\Delta S^3\left(\frac{\partial^3 V}{\partial S^3}\right)_j^n + \cdots,$$

substitution into the expression for T_j^n then gives

$$T_j^n = \left(-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV\right)_j^n - \frac{1}{2}\Delta t\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n - ra\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n + \cdots. \tag{15}$$

But V is the solution of the Black-Scholes equation so

$$\left(-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV\right)_j^n = 0.$$

Therefor the principle part of the local truncation error is

$$-\frac{1}{2}\Delta t\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n - ra\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n.$$

Hence $T_j^n = O(\Delta \mathrm{t}, \Delta \mathrm{S}^2)$.

3. Numerical Results

In this section, we present the numerical results using the proposed scheme (5). Here, we corroborate the properties of our new scheme for Eq.(1). These numerical results are obtained with σ^2 □ r and different values for Δt. (See Figure 3-4).

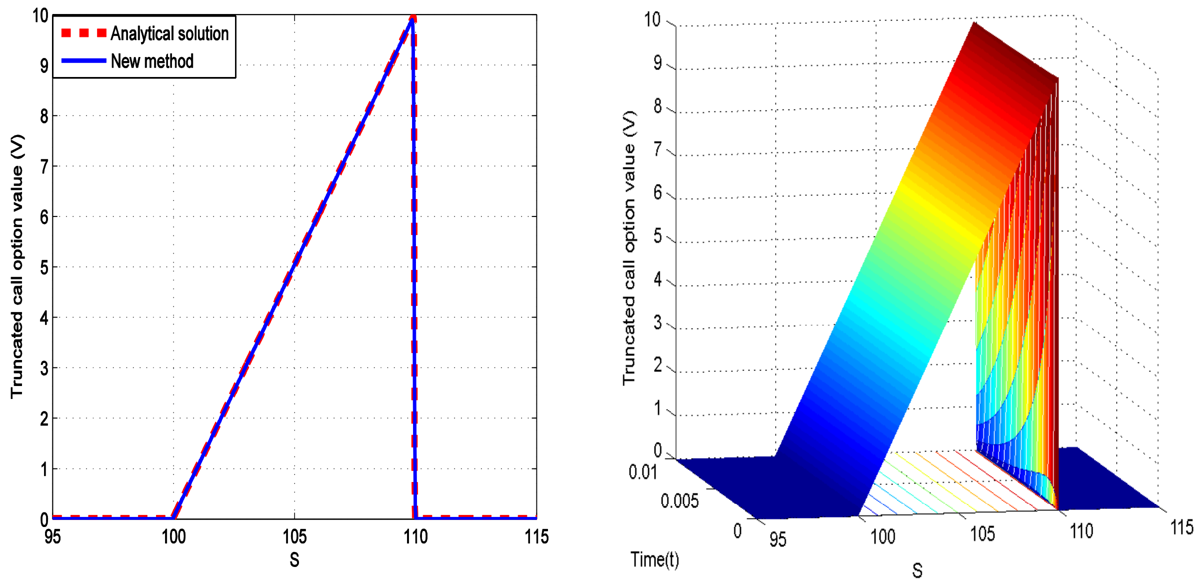

Figure 3. Truncated call option value for nonstandard explicit method with $\Delta S = 0.01$, $\Delta t = 10^{-6}$. parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 0.01$, $S_{\max} = 120$.

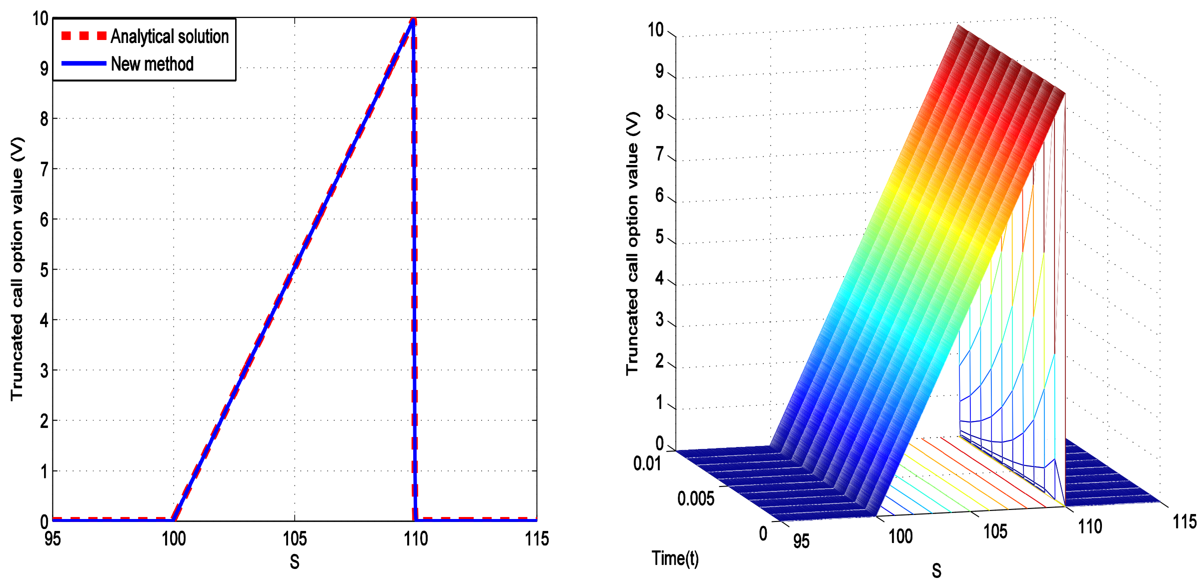

Figure 4. Truncated call option value for nonstandard explicit method with $\Delta S = 0.01$, $\Delta t = 10^{-3}$. parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 0.01$, $S_{\max} = 120$.

We have programed these methods in MATLAB.

4. Conclusions and discussion

We constructed an explicit method based on a nonstandard discretization scheme to solve option valuation problem with double barrier knock-out call option. In particular, the proposed method uses a nonstandard discretization in reaction term and the spatial derivatives are approximated using standard finite difference scheme. It is shown that the proposed nonstandard numerical scheme preserve the positivity as well as stability and consistence. Furthermore, the proposed scheme performs well with larger stepsizes. Future work will include extending the method to nonlinear Black-Scholes equation.

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