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Statistical Convergence of Double Sequence in Paranormed Spaces

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Abstract

In this article we define and investigate statistical convergence and Cauchy for double sequences in paranormed spaces. We also obtain a criterion for a double sequence in paranormed spaces to be a statistical Cauchy sequence.

Keywords: statistical convergence, g-statistical convergence, double sequences, paranormed spaces.

1. Introduction

The studies on paranormed spaces were initiated by Nakano [14] and Simon [18] at the initial stage. Later on it was further studies by Maddox [10], Lascarides [7], it has become an active area of research in recent years [1,3,17].

The concept of statistical convergence was introduced over nearly the last fifty years (Fast [3] in (1951), Schoenberg in (1959)), and since then several generalization and applications of this notion have been investigated by various authors [12,13,16] . Throughout this paper ℕ will denote the set of positive integers.

Let $(X, \langle .,. \rangle)$ be a normed space. Let E be subset of positive integers ℕ and j ∈ N.The quotient $d_j(E) = \frac{|E \cap \{1,\ldots,j\}|}{j}$ is called the j'th partial density of E. Note that d_j is a probability measure on $P(N)$, with support $\{1,2,\ldots,j\}$.

$d(E) = \lim_{j\to\infty} d_j(E)$ is called the natural density $E \subseteq N$ (if exists)[2].

Recall that a sequence (x_k) of elements of X is said to be statistically convergent to $l \in X$ if the set $A(\varepsilon) = \{n \in N : \|x_n - l\| \geq \varepsilon\}$ for each $\varepsilon > 0$ has natural density zero .In other words for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \|x_k - l\| \geq \varepsilon\}| = 0$$

We write $st - \lim_{n\to\infty} x_n = l$.

The sequence (x_n) is called to be statistically Cauchy sequence if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \|x_k - x_n\| \geq \varepsilon\}| = 0$$

In [4] Fridy prove that a sequence (x_n) is statistically convergence if and only if it is statistically Cauchy.

By the convergence of a double sequence we mean the convergence in Pringsheim's sense [17].

A double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ is called to be convergence in the Pringsheim's sense if for each $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that for all $j, k \geq N$ implies $\|x_{jk} - l, z\| < \varepsilon$. l is called the Pringsheim limit of (x_{jk}) .

Let $A \subseteq N \times N$ be a set of positive integers and let $A(n, m)$ be the set of (j, k) in A such that $j \leq n$ and $k \leq m$. Then the two-dimensional concept of natural density can be defined as follows :

The lower asymptotic density of a set $A \subseteq N \times N$ is defined as $\underline{d_2}(A) = \liminf_{n,m} \frac{\text{card}\,(A(n,m))}{nm}$

If the sequence $\{\frac{\text{card}\,(A(n,m))}{nm}\}$ has a limit in Pringsheim's sense then we say that A has a double natural density and is defined as $d_2(A) = \lim_{n,m} \frac{\text{card}\,(A(n,m))}{nm}$.

In this paper we define these concepts for double sequence $x = (x_{jk})_{j,k\in\mathbb{N}}$ in paranormed spaces and prove some results.

2. Preliminaries Notes

The concept of paranorm is a generalization of absolute value [9,11]. Let X be a linear space ,and g be a real subadditive function on X satisfying the following conditions:

P1)$g(x) = 0 \quad \text{if} \quad x = 0$

P2) $g(-x) = g(x)$ for all $x \in X$.

P3)$g(x + y) \leq g(x) + g(y) \quad \text{for all } x, y \in X$

P4)if (α_n) is a sequence of scalars with $\alpha_n \to \alpha_0$ as $n \to \infty$ and $x_n \to a \quad (n \to \infty)$ .in the sense that $g(x_n - a) \to 0(n \to \infty)$, then $\alpha_n x_n \to \alpha_0 a \quad (n \to \infty)$ in the sense that $g(\alpha_n x_n - \alpha_0 a) \to 0$,$(n \to \infty)$.

g is called a paranorm on X and a paranormed space (X, g) is a topological linear space in which the topology is given by the paranorm g . A paranormed space (X, g) is total paranormed space for which $g(x) = 0$ implies $x = 0$

Note that each seminorm on X is a paranorm but converse need not be true . Note that a first countable topological vector space is a paranormed space. A Fréchet space is a total and complete paranormed space.

Definition 2.1: Let (X, g) be paranormed space .The sequence $x = (x_k)$ in X is said to be g-convergent to l in X if forall $\varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \geq N \quad g(x_k - l) < \varepsilon$

We write it as $g - \lim x = l$

Definition 2.2: A sequence $x = (x_k)$ is said to be bounded if for each $k \in \mathbb{N}$ there exists $M > 0$ such that $g(x_k) < M$.

Note that a g-convergence sequence need not be bounded.

Definition 2.3: Let (X, g) be paranormed space .The sequence $x = (x_k)$ is a called g-Cauchy sequence if

$\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$, $\forall n, m \geq N_0 \quad g(x_n - x_m) < \varepsilon$

Definition 2.4: Let (X, g) be paranormed space .The sequence $x = (x_k)$ is a called g-statistically convergent to l in X if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \ g(x_k - l) \geq \varepsilon\}| = 0$$

We write

$$g(St) - \lim_k x_k = l$$

Definition 2.5: Let (X, g) be paranormed space .The sequence $x = (x_k)$ is a called g-statistically Cauchy convergence in X if for each $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : g\left(x_k - x_{n_0}\right) \ge \varepsilon \}\right| = 0$$

3. Main Results

In [1] introduced the concepts of convergence and statistically convergence for single sequence in paranormed space. Now, we introduce the idea of convergence and statistically convergence for double sequence in paranormed space.

We inspire the follow definition of Pringsheim's sense[18].

Definition 3.1: Let (X, g) be a paranormed space . A double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ is said to be g-convergent to $l \in X$ if

$\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$, $\forall j, k \ge N$ $g(x_{jk} - l) < \varepsilon$

We write it as $x_{jk} \xrightarrow{g} l$

A double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ is said to be bounded if for all $j, k \in \mathbb{N}$ there exists $M > 0$ such that $g(x_{jk}) < M$

Definition 3.2: A double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ in paranormed space (X, g) is said to be g-statistical convergent to $l \in X$,for each $\varepsilon > 0$, the set $\{(j, k): g(x_{jk} - l) \ge \varepsilon \}$ has double natural density zero , in other words

$$\lim_{n,m\to\infty} \frac{1}{nm} \left|\{(j, k): j \le n, k \le m \;\; g(x_{jk} - l) \ge \varepsilon \}\right| = 0$$

In this case we write it as

$$g(St_2) - \lim_{j,k} x_{jk} = l$$

Corollary 3.3 : Let $(x_{jk})_{j,k\in\mathbb{N}}$ be a g-convergence double sequence in paranormed space (X, g) and $l_1 , l_2 \in X$.If $g(St_2) - \lim_{j,k} x_{jk} = l_1$ and $g(St_2) - \lim_{j,k} x_{jk} = l_2$ Then $l_1 = l_2$.

proof: Define the following sets and given $\varepsilon > 0$

$A_1(\varepsilon) = \{ (j, k): g(x_{jk} - l_1) \ge \frac{\varepsilon}{2}\}$, $A_2(\varepsilon) = \{ (j, k): g(x_{jk} - l_2) \ge \frac{\varepsilon}{2}\}$

By assume , $d_2\left(A_1(\varepsilon)\right) = 0$ and $d_2\left(A_2(\varepsilon)\right) = 0$

Now , let $A(\varepsilon) = A_1(\varepsilon) \cup A_2(\varepsilon)$ then $d_2\left(A(\varepsilon)\right) = 0$.

Now if $(i,j) \in A^c(\varepsilon)$, we have :

$$g(l_1 - l_2) = g\big(l_1 - x_{jk} + x_{jk} - l_2\big) \leq g\big(l_1 - x_{jk}\big) + g(x_{jk} - l_2) < \varepsilon$$

We get $g(l_1 - l_2) = 0$ and so $l_1 = l_2$

Corollary 3.4: If $(x_{jk})_{j,k\in\mathbb{N}}$, $(y_{jk})_{j,k\in\mathbb{N}}$ be double sequences in paranormed space (X,g) and

$$g(St_2) - \lim_{j,k} x_{jk} = a \qquad g(St_2) - \lim_{j,k} y_{jk} = b$$

then

i) $g(St_2) - \lim_{j,k}(x_{jk} + y_{jk}) = a + b$

ii) $g(St_2) - \lim_{j,k} \alpha x_{jk} = \alpha a$

Proof: The proof is easy.

In [1], has defined the concept of g-statistically Cauchy single sequence in paranormed space.We introduce the g-statistically Cauchy for double sequences in paranormed space and prove some analogues.

Definition 3.5: A double sequence$(x_{jk})_{j,k\in\mathbb{N}}$ in paranormed space (X,g) is said to be g-statistically Cauchy if for every $\varepsilon > 0$, there exits N and M such that for all $j,p \geq N$, $k,q \geq M$, the set

$$\{(j,k), j \leq n, k \leq m : g(x_{jk} - x_{pq}) > \varepsilon\}$$

has double natural density zero.

Theorem 3.6: Let (X,g) be paranormed space.A double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ is g-statistically convergent to $l \in X$ if and only if there exists a subset $A = \{(j,k)\} \subseteq \mathbb{N}\times\mathbb{N}$, such that $d_2(A) = 1$ and $x_{jk} \xrightarrow{g} l$ on $(j,k) \in A$.

Proof: Let $g(St_2) - \lim_{j,k} x_{jk} = l$.Put $A_r = \{(j,k) \in \mathbb{N}\times\mathbb{N} : g(x_{jk} - l) \geq \frac{1}{r}\}$

And $M_r = \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : g\big(x_{jk} - l\big) < \frac{1}{r}\right\}$ (where $r = 1,2,\ldots$)

Then$d_2(A_r) = 0$ and $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq M_{i+1} \supseteq \cdots$ and $d_2(M_r) = 1$, $r = 1,2,\ldots$

Suppose that for $(j,k) \in M_r$,$(x_{jk})_{j,k\in\mathbb{N}}$ is not g-convergent to l.Hence there exist, $\varepsilon > 0$ such that $g\big(x_{jk} - l\big) \geq \varepsilon$ for infinitely many terms.

Let $M_\varepsilon = \{(j,k) : g\big(x_{jk} - l\big) < \varepsilon\}$ and $\varepsilon > \frac{1}{r}$ $r = 1,2,\ldots.$ $M_r \subseteq M_\varepsilon$.

Therefore $d_2(M_r) \neq 1$which is contradicts. Hence $(x_{jk})_{j,k\in\mathbb{N}}$ is g $-$convergent to l.

Conversely, let that there exists $A = \{(j,k)\} \subseteq \mathbb{N}\times\mathbb{N}$, such that

i) $d_2(A) = 1$ ii) $x_{jk} \overset{g}{\to} l$ on $(j,k) \in A$

By (ii) for each $\varepsilon > 0$,there is $N_0 \in \mathbb{N}$ such that $g(x_{jk} - l) < \varepsilon$ for all $j,k \geq N_0$

Now $A_\varepsilon = \{(j,k): g(x_{jk} - l) \geq \varepsilon\} \subseteq \mathbb{N} \times \mathbb{N} - \{(j_{N_{0+1}}, k_{N_{0+1}}), (j_{N_{0+2}}, k_{N_{0+2}}), \ldots\}$

Hence $d_2(A_\varepsilon) \leq 1 - 1 = 0$. Therefore $g(St_2) - \lim_{j,k} x_{jk} = l$.

Remark: If $(x_{jk})_{j,k\in\mathbb{N}}$ g-statistically convergent to $l \in X$, then there exists a double sequence $(y_{jk})_{j,k\in\mathbb{N}}$ in X such that $y_{jk} \overset{g}{\to} l$ and $d_2(\{(j,k): x_{jk} = y_{jk}\}) = 1$

Theorem 3.7: If $(x_{jk})_{j,k\in\mathbb{N}}$ be a double sequence in paranormed space (X,g).The sequence $(x_{jk})_{j,k\in\mathbb{N}}$ is g-statistically convergent if and only if $(x_{jk})_{j,k\in\mathbb{N}}$is g-statistically Cauchy.

Proof: Let $g(St_2) - \lim_{j,k} x_{jk} = l$.Then for each $\varepsilon > 0$,the set $\{(j,k): j \leq n, k \leq m \quad g(x_{jk} - l) \geq \varepsilon\}$ has double natural density zero.

Choose two numbers N and M such that $g(x_{NM} - l) \geq \varepsilon$.

Put $A(\varepsilon) = \{(j,k): j \leq n, k \leq m \quad g(x_{jk} - x_{NM}) \geq \varepsilon\}$

$B(\varepsilon) = \{(j,k): j \leq n, k \leq m \quad g(x_{jk} - l) \geq \varepsilon\}$

$C(\varepsilon) = \{(j,k): j = N \leq n, k = M \leq m \quad g(x_{NM} - l) \geq \varepsilon\}$

Then $A(\varepsilon) \subseteq B(\varepsilon) \cup C(\varepsilon)$.

Hence $d_2(A(\varepsilon)) \leq d_2(B(\varepsilon)) + d_2(C(\varepsilon)) = 0$. Therefor$(x_{jk})_{j,k\in\mathbb{N}}$ is g-statistically Cauchy.

Conversely, suppose that $(x_{jk})_{j,k\in\mathbb{N}}$ be g-statistically Cauchy but not g −statistically convergence.Hence there exist N and M such that the set $A(\varepsilon)$has double natural density zero .Therefor the set $E(\varepsilon)=\{(j,k): j \leq n, k \leq m \quad g(x_{jk} - x_{NM}) < \varepsilon\}$

has double natural density 1 .In particular,we can write

$$g(x_{jk} - x_{NM}) \leq 2g(x_{jk} - l) < \varepsilon \quad \text{if} \quad g(x_{jk} - l) < \frac{\varepsilon}{2} .$$

Since$(x_{jk})_{j,k\in\mathbb{N}}$ is not g-statistically convergent, the set $B(\varepsilon)$ has double natural density 1 , the set $\{(j,k): j \leq n, k \leq m \quad g(x_{jk} - l) < \varepsilon\}$ has double natural density zero . Therefor by above ,the set $\{(j,k): j \leq n, k \leq m \quad g(x_{jk} - x_{NM}) < \varepsilon\}$

has double natural density zero ,i.e.,the set $A(\varepsilon)$ has double natural density 1 which is contradiction. Therefore $(x_{jk})_{j,k\in\mathbb{N}}$ is g-statistically convergent.

From Theorems 3.6 and 3.7 we can state the following for double sequences.

Theorem 3.8: Let$(x_{jk})_{j,k\in\mathbb{N}}$ be a double sequence in paranormed space (X, g).If the sequence$(x_{jk})_{j,k\in\mathbb{N}}$ be g-statistically convergent to $l \in X$ then there exists a subsequence $(y_{jk})_{jk\in\mathbb{N}}$ of$(x_{jk})_{j,k\in\mathbb{N}}$ such that $y_{jk} \overset{g}{\to} l$.

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