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Connections of Linear Operators defined by Analytic Functions with Q_p Spaces

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Abstract

This paper is concerned mainly with the linear operators $I_f^{\gamma,\alpha}$ and $J_f^{\gamma,\alpha}$ of analytic function f.The norm of $I_f^{\gamma,\alpha}$ and $J_f^{\gamma,\alpha}$ on some analytic function spaces is computed in this paper. We study the relation between $I_f^{\gamma,\alpha}$ and $J_f^{\gamma,\alpha}$ operators, the $B(\lambda)$spaces and Q_p spaces $(0 < p < \infty)$.

Keywords: Integral operator, $\mathrm{Q_p}$ spaces, Pre-Schwarzian derivative.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$denote the class of analytic functions fon the open unit disc$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also denote by $\mathcal{A}$ the subclass of $\mathcal{H}(\mathbb{D})$ consisting of functions normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and $\mathcal{S}$ the class of all univalent functions in $\mathcal{A}$.

A function $f \in \mathcal{H}(\mathbb{D})$ is a Bloch function if

$$\|f\|_B := |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions is denoted by $\mathbb{B}$. A classical source for Bloch functions is [18, 19].

For $\alpha > 0$, the α-Bloch space, denoted by $\mathbb{B}^\alpha$, is the space of all functions f in $\mathbb{D}$, for which

$$\|f\|_{\mathbb{B}^\alpha} := |f(0)| + \sup_{z\in\mathbb{D}}|f'(z)|(1 - |z|^2)^\alpha < \infty.$$

Obviously, $\mathbb{B}^{\alpha_1} \subset \mathbb{B} \subset \mathbb{B}^{\alpha_2}$for $0 < \alpha_1 < 1 < \alpha_2 < \infty$.

The Hardy space $H^p(0 < p < \infty)$ is the class of all functions f analytic in $\mathbb{D}$ such that

$$\|f\|_p := \lim_{r\to 1^-} M_p(r,f) < \infty,$$

where

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right)^{\frac{1}{p}}, \qquad (0 < p < \infty)$$

and

$$M_\infty(r,f) = \max_{|z|\leq r}|f(z)|.$$

we mention [4] as a general reference for the theory of Hardy spaces.

The space $BMOA$ consists of those functions $f\ in\ H^1$ whose boundary values have bounded mean oscillation in the unit circle $\partial\mathbb{D}$ as defined by F. John and L. Nirenberg [10]. We mention [5, 6, 14], as general references for these space. Let us recall that

$$H^\infty \subset BMOA \subset \bigcap_{0<p<\infty} H^p, \qquad H^\infty \subset BMOA \subset \mathbb{B}.$$

Even though the inclusion $BMOA \subset \mathbb{B}$ is strict, a result of Pommerenke [13] implies that

$$\boldsymbol{S} \cap \mathbb{B} = \boldsymbol{S} \cap \boldsymbol{BMOA} \qquad (1.1)$$

Let $d\sigma$ denote the normalized Lebesgue area measure in $\mathbb{D}$ and $g(a,z)$ the Green function with logarithmic singularity at a, $i.e.$, $g(a,z) = -\log|\varphi_a(z)|$, where $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ is the Mobius transformation of $\mathbb{D}$.

For $0 < p < \infty$, Q_p is the space of all functions $f \in \mathcal{H}(\mathbb{D})$, for which

$$\|f\|_{Q_p}^2 = |f(0)|^2 + \sup_{a\in\mathbb{D}} \int_D |f'(z)|^2\left(1-|\varphi_a(z)|^2\right)^p d\sigma(z) < \infty.$$

This space was introduced by Aulaskari and Lappan in [1] while looking for new characterizations of Bloch functions, and we mention the books [16] and [17] as general references for the space Q_p. Let us just mention here that $Q_p = \mathbb{B}$ for all $p > 1$; $Q_1 = BMOA$; and that whenever $0 < p < 1$, Q_p is a proper subspace of $BMOA$.Aulaskari, Lappan, Xiao, and Zhao[2] extended (1.1) showing that

$$\boldsymbol{S} \cap \boldsymbol{Q_p} = \boldsymbol{S} \cap \mathbb{B}, \qquad (0 < p < \infty). \qquad (1.2)$$

For any $f \in \mathcal{H}(\mathbb{D})$, the next two integral operators on $\mathcal{H}(\mathbb{D})$ are induced as follows:

$$I_f^{\gamma,\alpha}(h)(z) = \int_0^z h'(w) f^\gamma(w) w^{\alpha-1} dw, \qquad (z \in \mathbb{D}),$$

and

$$J_f^{\gamma,\alpha}(h)(z) = \int_0^z h(w) f'^\gamma(w) w^{\alpha-1} dw, \qquad (z \in \mathbb{D}),$$

where $\gamma, \alpha > 0$.

If $\gamma = \alpha = 1$, then $I_f^{1,1}(h) = I_f(h)$ and $J_f^{1,1}(h) = J_f(h)$, which are the Alexander operators and Both integral operators have been studied by many authors. See [12, 13, 16, 18, 19] and the references therein.

Norm of composition operator, weighted composition operator and some integral operators have been studied extensively by many authors, see [7, 15].

2. Main Results

In this section, we state and prove our main results. In order to formulate our main results, we need the following lemma from ([11] , Lemma 2.1).

Lemma 2.1. *Let $0 < p < 1$. For any $z_0 \in \mathbb{D}$, the function*

$$g_{z_0}(z) = \frac{z_0 - z}{1 - \bar{z}_0 z} - z_0$$

is analytic in $\mathbb{D}$ and $\|g_{z_0}\|_{Q_p} = 1/(p+1)^{1/2}$.

Theorem 2.1. *Let $0 < p < 1$ and $\alpha + \gamma \geq 1$. If $f \in \mathcal{A}$, then $I_f^{\gamma,\alpha}$ is bounded on Q_p space if and only if $f \in H^\infty$. Moreover, $\left\|I_f^{\gamma,\alpha}\right\|_{Q_p} = \|If^\gamma\|_{H^\infty}$ where $I(z) = z^{\alpha-1}$.*

Proof. Let $f \in H^\infty$ and $M := \|f\|_{H^\infty}$. Then by Showarz lemma, for any $z \in \mathbb{D}$, we have

$$\frac{|f(z)|}{M} < |z| \Longrightarrow$$

$$\left|z^{\alpha-1} f^\gamma(z)\right| < M|z|^{\alpha+\gamma-1} \leq M.$$

Therefore $z^{\alpha-1} f^\gamma(z) \in H^\infty$ and there exists $C > 0$ such that $\|If^\gamma\|_{H^\infty} = C$.

Now for any $\|h\|_{Q_p} = 1$, we have

$$\left\|I_f^{\gamma,\alpha} h\right\|^2 = \sup_{a\in\mathbb{D}} \int_D \left|h'(z) z^{\alpha-1} f^\gamma(z)\right|^2 \left(1 - |\varphi_a(z)|^2\right)^p d\sigma(z)$$

$$\leq \|If^\gamma\|_{H^\infty}^2 \sup_{a\in\mathbb{D}} \int_D |h'(z)|^2 \left(1 - |\varphi_a(z)|^2\right)^p d\sigma(z)$$

$$\leq C^2 \|h\|_{Q_p}$$

$$= C^2.$$

Then $\left\|I_f^{\gamma,\alpha}\right\|_{Q_p} \leq C$. To prove the converse, for given any $\epsilon > 0$, there exists $z_0 \in \mathbb{D}$ such that

$|I(z_0) f^\gamma(z_0)| > C - \epsilon$. Let

$$h(z) = \frac{g_{z_0}(z)}{\|g_{z_0}\|_{Q_p}},$$

where

$$g_{z_0}(z) = \frac{z_0 - z}{1 - \bar{z}_0 z} - z_0.$$

It is easy to see that

$$\|h\|_{Q_p} = 1, \quad |h'(z_0)|\left(1 - |z_0|^2\right) = 1/\|g_{z_0}\|_{Q_p}.$$

Therefore we have

$$\left\|I_f^{\gamma,\alpha}\right\|^2 \geq \left\|I_f^{\gamma,\alpha} h\right\|_{Q_p}^2 = \sup_{a\in\mathbb{D}} \int_D \left|h'(z)z^{\alpha-1}f^{\gamma}(z)\right|^2 \left(1-|\varphi_a(z)|^2\right)^p d\sigma(z)$$

$$= \sup_{a\in\mathbb{D}} \int_D \left|h'(\varphi_a(w))\varphi_a^{\alpha-1}(w)f^{\gamma}(\varphi_a(w))\varphi'_a(w)\right|^2 \left(1-|w|^2\right)^p d\sigma(w).$$

Taking $w = re^{i\theta}$ and by the subharmonicityof $\left|h'(\varphi_a(w))\varphi_a^{\alpha-1}(w)f^{\gamma}(\varphi_a(w))\varphi'_a(w)\right|^2$, we obtain

$$\left\|I_f^{\gamma,\alpha}\right\|^2 \geq \sup_{a\in\mathbb{D}} \int_0^1 \frac{1}{\pi}\int_0^{2\pi} \left|h'\left(\varphi_a(re^{i\theta})\right)\varphi_a^{\alpha-1}(re^{i\theta})f^{\gamma}\left(\varphi_a(re^{i\theta})\right)\varphi'_a(re^{i\theta})\right|^2 \left(1-r^2\right)^p r dr d\theta$$

$$\geq \sup_{a\in\mathbb{D}} \left|h'(a)f^{\gamma}(a)a^{\alpha-1}\right|^2 \left(1-|a|^2\right)^2 \int_0^1 2\left(1-r^2\right)^p r dr$$

$$= \frac{1}{1+p} \sup_{a\in\mathbb{D}} \left|h'(a)f^{\gamma}(a)a^{\alpha-1}\right|^2 \left(1-|a|^2\right)^2$$

$$\geq \frac{1}{1+p} \left|h'(z_0)f^{\gamma}(z_0){z_0}^{\alpha-1}\right|^2 \left(1-|z_0|^2\right)^2$$

$$\geq \frac{1}{1+p} \frac{\left|f^{\gamma}(z_0)z_0^{\alpha-1}\right|^2}{\left\|g_{z_0}\right\|_{Q_p}^2}.$$

By Lemma 2.1 we obtain

$$\left\|I_f^{\gamma,\alpha}\right\| \geq \left|f^{\gamma}(z_0)z_0^{\alpha-1}\right| > C - \epsilon.$$

Since ϵ is arbitrary, we have $\left\|I_f^{\gamma,\alpha}\right\| \geq \sup_{z\in\mathbb{D}} \left|f^{\gamma}(z)z^{\alpha-1}\right|$ and the proof is complete. □

Theorem 2.2. *Let $\gamma \leq 1$ and $\alpha \geq 1$. If $f \in Q_p$, then $J_f^{\gamma,\alpha}$ is bounded on H^∞. Moreover $\left\|J_f^{\gamma,\alpha}\right\|_{Q_p} \leq \|f\|_{Q_p}$.*

Proof. If $\|h\|_{H^\infty} = 1$, then we have

$$\left\|J_f^{\gamma,\alpha}\, h\right\|_{Q_p}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|h(z)z^{\alpha-1}f'^{\gamma}(z)\right|^2 \left(1-|\varphi_a(z)|^2\right)^p d\sigma(z)$$

$$\leq \|h\|_{H^\infty}^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \left(1-|\varphi_a(z)|^2\right)^p d\sigma(z)$$

$$\leq \|h\|_{H^\infty}^2 \|f\|_{Q_p}^2 = \|f\|_{Q_p}^2.$$

Therefore $J_f^{\gamma,\alpha}$ is bounded on H^∞ and $\left\|J_f^{\gamma,\alpha}\right\|_{Q_p} \leq \|f\|_{Q_p}$. The proof is complete. □

Theorem 2.3. *Assume that $0 < p < 1, \gamma + \alpha \geq 1$ and $f \in \mathcal{A}$. Thenthe integral operator $I_f^{\gamma,\alpha}$ is compact from Q_p space to Q_p space if and only if $f \in H^\infty$.*

Proof. If $I_f^{\gamma,\alpha}$ is compact, then it is bounded, and by Theorem 2.1 it follows that $f \in H^\infty$. Now assume $f \in H^\infty$ and (h_n) is a sequence in Q_p such that $h_n \to 0$. We have

$$\left\|I_f^{\gamma,\alpha} h_n\right\|_{Q_p}^2 = \sup_{a\in\mathbb{D}} \int_D \left|h_n'(z)z^{\alpha-1}f^{\gamma}(z)\right|^2 \left(1-|\varphi_a(z)|^2\right)^p d\sigma(z)$$

$$\leq \|f^{\gamma}\|_{H^\infty}^2 \|h_n\|_{Q_p}^2.$$

Since for $h_n \to 0$ on $\overline{\mathbb{D}}$ we have $\|h_n\|_{Q_p} \to 0$, and by letting $n \to \infty$ in the last inequality, we obtain that $\lim_{n\to\infty} \left\| I_f^{\gamma,\alpha} h_n \right\|_{Q_p} = 0$. Therefore, $I_f^{\gamma,\alpha}$ is compact. □

A function $f \in \mathcal{H}(\mathbb{D})$ is called uniformly locally univalent if there exists a constant $\rho > 0$ such that f is univalent on the hyperbolic disk $|(z-a)/(1-\bar{a}z)| < \tanh \rho$ of radius ρ for every $a \in D$. It is known that a non-constant analytic function f is uniformly locally univalent if and only if the norm

$$\|f'' / f'\| = \sup_{z\in\mathbb{D}} (1-|z|^2) \left| \frac{f''(z)}{f'(z)} \right|$$

of the pre-Schwarzian derivative f'' / f' of f is finite. Let

$$B(\lambda) = \{f \in \mathcal{H}(\mathbb{D});\ \|f'' / f'\| \le 2\lambda\}.$$

Kim and Sugawa in [8, 9] investigated various properties of the functions belong to the class $B(\lambda)$. The following Lemma is doue to Becker [3].

Lemma 2.2. If $f \in \mathcal{A}$ *and* $\|f'' / f'\| \le 1$ *, then* f *is univalent.*

In the next theorem we prove:

Theorem 2.4. *Let* $0 < \lambda < \alpha$ *and* $f \in B(\lambda)$*. Then* $f \in \mathbb{B}^\alpha$ *and*

$$\|f\|_{\mathbb{B}^\alpha} \le |f(0)| + |f'(0)| 2^{\lambda+\alpha}.$$

Proof. Let $0 < \lambda < \alpha$, $|z| = r$ and $f \in B(\lambda)$. Then we have

$$\begin{aligned} \log \left| \frac{f'(z)}{f'(0)} \right| &\le \left| \log \frac{f'(z)}{f'(0)} \right| \\ &= \left| \int_0^z \frac{f''(w)}{f'(w)} dw \right| \\ &\le r \int_0^1 \left| \frac{f''(tz)}{f'(tz)} \right| dt \\ &\le r \int_0^1 \frac{2\lambda}{1-r^2t^2} dt \\ &= 2\lambda \log \sqrt{\frac{1+r}{1-r}} \end{aligned}$$

This implies

$$|f'(z)| \le |f'(0)| \left(\frac{1+r}{1-r}\right)^{\lambda}. \tag{2.1}$$

Therefore we have

$$\|f\|_{\mathbb{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|(1 - |z|^2)^\alpha$$

$$\leq |f(0)| + |\boldsymbol{f}'(\mathbf{0})| \sup_{0<r<1} (1 - r^2)^\alpha \left(\frac{\mathbf{1+r}}{\mathbf{1-r}}\right)^{\boldsymbol{\lambda}}$$

$$\leq |f(0)| + 2^{\lambda+\alpha}|\boldsymbol{f}'(\mathbf{0})| \sup_{0<r<1} (1 - r)^{\alpha-\lambda}$$

$$= |f(0)| + 2^{\lambda+\alpha}|\boldsymbol{f}'(\mathbf{0})| ,$$

and the proof is complete. □

Theorem 2.5. *Let* $2\lambda < 1$ *and* $f \in \mathcal{A} \cap B(\lambda)$*. Then* $f \in Q_p$ *for all* $0 < p < \infty$.

Proof. Let $2\lambda < 1$ and $f \in \mathcal{A} \cap B(\lambda)$. *By setting* $\alpha = 1$ *in Theorem 2.4, we* have $f \in \mathbb{B}$. On the other hand, by Lemma 2.1, f is univalent in the open unit disc and so $f \in S \cap \mathbb{B}$. Then by relation (1.2), $f \in S \cap Q_p$ *for all* $0 < p < \infty$.□

Corollary 2.1. *Let* $0 < p < 1, \gamma + \alpha \geq 1$ *and* $2\lambda < 1$*. If* $f \in H^\infty$*, then for all* $g \in B(\lambda)$, $I_f^{\gamma,\alpha} g \in Q_p$ and $I_f^{\gamma,\alpha}$ is compact from $B(\lambda)$ space to Q_p space.

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