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A Note on t-Derivations of B-Algebras

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Abstract

In this paper, we introduce the notion of t-derivation on B-algebras and obtain some of its related properties.

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1. Introduction

Imai and Is´eki [4, 5] introduced two classes of logical algebras: BCK and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Neggers and Kim [7] introduced the notion of B-algebras which is related to several classes of algebras such as BCI/BCK-algebras. Abujabal and Al-Shehrie [1] defined and studied the notion of left derivation of BCI-algebras. Further, Al-Shehrie [2] has applied the notion of left-right derivation in BCI-algebra to B-algebra and obtained some of its properties. Furthermore This logical algebra Have been studied by another authors, see for example [3], [6], [8], [9]. In this paper, we introduce the notion of t-derivation on B-algebras and investigate some properties of 0-commutative B-algebras.

(see [2], [3], [6], [7], [9]) A B-algebra is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

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$(B1)\ x * x = 0;$

$(B2)\ x * 0 = x;$

$(B3)\ (x * y) * z = x * (z * (0 * y))$ for all $x, y, z \in X$.

In any B-algebra X the following properties satisfied for all $x, y, z \in X$,

(1) $(x * y) * (0 * y) = x$.

(2) $x * y = z * y$ implies $x = z$.

(3) $x * (y * z) = (x * (0 * z)) * y$.

(4) $x * y = 0$ implies $x = y$.

(5) $x = 0 * (0 * x)$.

(6) $x * y = x * z$ implies $y = z$.

(7) $0 * (x * y) = y * x$.

(8) $(x * y) * (z * y) = x * z$.

A B-algebra $(X, *, 0)$ is said to be 0-commutative if $x * (0 * y) = y * (0 * x)$, for any $x, y \in X$.

For any 0-commutative B-algebra X and all $x, y, z, u \in X$, the following properties hold:

(9) $(0 * x) * (0 * y) = y * x$.

(10) $(x * y) * (z * u) = (u * y) * (z * x)$.

(11) $(x * y) * (x * z) = z * y$.

(12) $(x * y) * z = (0 * y) * (z * x)$.

(13) $x * (y * z) = z * (y * x)$.

(14) $(x * y) * z = (x * z) * y$.

(15) $x * (x * y) = y$.

Let X be a B-algebra. Then X is called associative if for all $x, y, z \in X$,

(16) $(x * y) * z = x * (y * z)$.

For a B-algebra X, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$.

2. t-Derivation of B-Algebras

In this section we investigate the notion of t-derivation for a B-algebra and study some of its properties.

Definition 2.1. Let X be a B-algebra. For any $t \in X$, we define a self map $d_t: X \to X$ by $d_t(x) = x * t$ for all $x \in X$.

Lemma 2.2. Let d_t be a self map of a B-algebra X. Then the following hold:

(i) d_t is one-one.

(ii) $d_t(x) * d_t(y) = x * y$ for all $x, y \in X$.

Proof. It is sufficient to prove (ii). By applying (8) we obtain

$d_t(x) * d_t(y) = (x * t) * (y * t) = x * y.$

Definition 2.3. A self map d_t of a B-algebra X is said to be t-regular if $d_t(0) = 0$.

Lemma 2.4. Let d_t be a self map of a 0-commutative B-algebra X. Then the following hold:

(i) $d_t(x * y) = d_t(x) * y$ for all $x, y \epsilon X$.

(ii) If d_t is t-regular, then it is an identity.

Proof. (i) Since d_t is a self map of a B-algebra X, by (14),

$d_t(x * y) = (x * y) * t = (x * t) * y = d_t(x) * y.$

(ii) Let d_t be t-regular and $x \epsilon X$. Then $0 = d_t(0)$ and by $(i), o = d_t(x) * x$. Hence by (4) $d_t(x) = x$ for all $x \epsilon X$. Therefore d_t is an identity. This completes the proof.

Definition 2.5. Let X be a B-algebra. Then for any $t \epsilon X$, the self map $d_t : X \to X$ is called a left-right t-derivation (or briefly (l,r)-t-derivation) of X if it satisfies the identity $d_t(x * y) = (d_t(x) * y) \Lambda (x * d_t(y))$ for all $x, y \epsilon X$.

Similarly, if d_t satisfies the identity $d_t(x * y) = (x * d_t(y)) \Lambda (d_t(x) * y)$ for all $x, y \epsilon X$, then it is called a right-left t-derivation (or briefly (r,l)-t-derivation) of X.

Moreover, if d_t is both a (l,r)- and a (r,l)-t-derivation of X, then d_t is a t-derivation of X.

Example 2.6. Let X be a B-algebra of all real numbers except for a negative integer $-n$, with a binary operation $*$ on X by $x * y = \frac{n(x-y)}{n+y}$.

For any $t \epsilon X$, define a self map $d_t : X \to X$ by $d_t(x) = x * t$ for all $x \epsilon X$. First, we show that X is a 0-commutative B-algebra. For any $x, y \epsilon X$:

$x * (0 * y) = x * \frac{n(0-y)}{n+y} = x * \frac{-ny}{n+y} = \frac{nx + xy + ny}{n}.$

Also, $y * (0 * x) = y * \frac{n(0-x)}{n+x} = y * \frac{-nx}{n+x} = \frac{ny + yx + nx}{n}.$

Hence X is a 0-commutative B-algebra. Next for all $x, y, t \epsilon X$,

$(x * y) * t = n\frac{n(x-y) - t(n+y)}{(n+y)(n+t)}$, and $(x * t) * y = n\frac{n(x-y) - t(n+y)}{(n+y)(n+t)}$.

Since X is a 0-commutative B-algebra, by (15) for all $x, y, t \epsilon X$,

$(d_t(x) * y) \Lambda (x * d_t(y)) = (x * (y * t)) * ((x * (y * t)) * ((x * t) * y))$

$$= (x * t) * y = n\frac{n(x - y) - t(n + y)}{(n + y)(n + t)} = (x * y) * t = d_t(x * y).$$

So d_t is a (l,r)-t-derivation of X. It is easy to check that d_t is not a (r,l)-t-derivation of X.

Example 2.7 Let $X := \{0, 1, 2\}$ be a B-algebra with the following table,

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

For any $t \epsilon X$, define a self map $d_t : X \rightarrow X$ by $d_t(x) = x * t$ for all $x \epsilon X$. Then it is easy to check that d_t is a (l, r)-t-derivation of X, which is not a (r, l)-t-derivation of X. If we set $x := 0, y := 2$ and $t := 1$, then

$d_t(x * y) = (x * y) * t = 0 \neq 2 = ((x * t) * y) * (((x * t) * y) * (x * (y * t)))$

$= (x * d_t(y)) \wedge (d_t(x) * y)$.

But if for any $t \epsilon X$, define a self map $d_t : X \rightarrow X$ by $d_t(x) = x * t = x$

then X is a t-derivation of X, which is t-regular.

Theorem 2.8. Let d_t be a self map of a B-algebra X. Then

(i) If d_t is a (l, r)-t-derivation and t-regular of X, then $d_t(x) = d_t(x) \wedge x$ for all $x \epsilon X$.

(ii) If d_t is a (r, l)-t-derivation of X, then $d_t(x) = x \wedge d_t(x)$ for all $x \epsilon X$ if and only if d_t is t-regular.

Proof. (i) If d_t is a (l, r)-t-derivation and t-regular of X, then by (B2)

$d_t(x) = d_t(x * 0) = (d_t(x) * 0) \wedge (x * d_t(0)) = d_t(x) \wedge (x * 0) = d_t(x) \wedge x$.

(ii) Let d_t be a (r, l)-t-derivation of X. If d_t is t-regular, then by (B2)

$d_t(x) = d_t(x * 0) = (x * d_t(0)) \wedge (d_t(x) * 0) = (x * 0) \wedge d_t(x) = x \wedge d_t(x)$.

Conversely, suppose that $d_t(x) = x \wedge d_t(x)$ for all $x, y \epsilon X$, then

$d_t(0) = 0 \wedge d_t(0) = d_t(0) * (d_t(0) * 0) = d_t(0) * d_t(0) = 0$.

So d_t is t-regular.

3. t-Derivation of 0-Commutative B-Algebras

In this section, we investigate the notion of t-derivation for 0-commutative B-algebras.

Theorem 3.1. Let d_t be a self map of an associative 0-commutative B-algebra X. Then d_t is a t-derivation of X.

Proof. Since X is an associative 0-commutative B-algebra, we have

$d_t(x * y) = (x * y) * t$

$= (x * (y * t)) * 0$ [by $(B2)$]

$= (x * (y * t)) * ((x * (y * t)) * (x * (y * t)))$ [by $(B1)$]

$= (x * (y * t)) * ((x * (y * t)) * ((x * y) * t))$ [by (16)]

$= (x * (y * t)) * ((x * (y * t)) * ((x * t) * y))$ [by (14)]

$= ((x * t) * y) \wedge (x * (y * t))$

$= (d_t(x) * y) \wedge (x * d_t(y))$.

Again,

$d_t(x * y) = (x * y) * t = ((x * t) * y) * 0$ [by (14) and $(B2)$]

$= ((x * t) * y) * (((x * t) * y) * ((x * t) * y))$ [by (B1)]

$= ((x * t) * y) * (((x * t) * y) * ((x * y) * t))$ [by (14)]

$= ((x * t) * y) * (((x * t) * y) * (x * (y * t)))$ [by (16)]

$= (x * (y * t)) \wedge ((x * t) * y) = \big(x * d_t(y)\big) \wedge (d_t(x) * y).$

Lemma 3.2. Let d_t be a (r,l)-t-derivation of a 0-commutative B-algebra X. Then $d_t(x * y) = x * d_t(y)$ for all $x, y \in X$.

Proof. Since d_t is a (r,l)-t-derivation of X, by (15),

$d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y) = (d_t(x) * y) * ((d_t(x) * y) * (x * d_t(y)))$

$= x * d_t(y).$

Definition 3.3. Let X be a B-algebra and d_t, $d_t^{'}$ be two self maps of X. Then we define $d_t \circ d_t^{'} : X \to X$ by $(d_t \circ d_t^{'})(x) = d_t(d_t^{'}(x))$, for all $x \in X$.

Theorem 3.4. Let X be a 0-commutative B-algebra and $d_t, d_t^{'}$ are (r,l)-t-derivations of X. Then $d_t \circ d_t^{'}$ is a t-derivation of X.

Proof. Since d_t, $d_t^{'}$ are two self maps of X, by Lemma 2.4(i) and (15) for all $x, y \in X$,

$(d_t \circ d_t^{'})(x * y) = d_t(d_t^{'}(x * y)) = d_t(d_t^{'}(x) * y) = d_t(d_t^{'}(x)) * y =$

$(x * d_t(d_t^{'}(y))) * \Big(x * d_t\big(d_t^{'}(y)\big)\Big) * \big(d_t\big(d_t^{'}(x)\big) * y\big) = (d_t(d_t^{'}(x)) * y) \wedge (x * d_t(d_t^{'}(y)))$

$= ((d_t \circ d_t^{'})(x) * y) \wedge (x * (d_t \circ d_t^{'})(y)).$

Next, since d_t, $d_t^{'}$ are (r,l)-t-derivations of X, by Lemma 3.2 and (15), for all $x, y \in X$, we have

$(d_t \circ d_t^{'})(x * y) = d_t(d_t^{'}(x * y)) = d_t(x * d_t^{'}(y)) = x * d_t(d_t^{'}(y))$

$= (d_t(d_t^{'}(x)) * y) * ((d_t(d_t^{'}(x)) * y) * (x * d_t(d_t^{'}(y)))) =$

$(x * d_t(d_t^{'}(y))) \wedge (d_t(d_t^{'}(x)) * y) = (x * (d_t \circ d_t^{'})(y)) \wedge ((d_t \circ d_t^{'})(x) * y).$

Theorem 3.5. Let X be a 0-commutative B-algebra and let d_t be a (r,l)-t-derivation and $d_t^{'}$ be a self map of X. Then $d_t \circ d_t^{'} = d_t^{'} \circ d_t$

Proof. Suppose d_t is a (r,l)-t-derivation and $d_t^{'}$ is a self map of X. By Lemmas 2.4(i) and 3.2, for all $x, y \in X$, $(d_t \circ d_t^{'})(x * y) = d_t(d_t^{'}(x * y)) = d_t(d_t^{'}(x) * y) = d_t^{'}(x) * d_t(y)$.

Again, by Lemmas 3.2 and 2.4(i), for all $x, y \in X$,

$(d_t^{'} \circ d_t)(x * y) = d_t^{'}(d_t(x * y)) = d_t^{'}(x * d_t(y)) = d_t^{'}(x) * d_t(y).$

Therefore, $(d_t \circ d_t^{'})(x * y) = (d_t^{'} \circ d_t)(x * y)$

By putting $y := 0$, for all $x \in X$, we get

$(d_t \circ d_t^{'})(x) = (d_t^{'} \circ d_t)(x)$. Hence, $d_t \circ d_t^{'} = d_t^{'} \circ d_t$.

This completes the proof.

Definition 3.6. Let X be a B-algebra and let d_t and $d_t^{'}$ be two self maps of X. Then we define $d_t * d_t^{'} : X \to X$ by $(d_t * d_t^{'})(x) = d_t(x) * d_t^{'}(x)$ for all $x \in X$.

Theorem 3.7. Let d_t, $d_t^{'}$ be two (r,l)-t-derivations of a 0-commutative B-algebra X. Then $d_t * d_t^{'} = d_t^{'} * d_t$.

Proof. Since d_t is a (r,l)-t-derivation of X, for all $x, y \epsilon X$, by Lemmas $2.4(i)$ and 3.2,

$(d_t \circ d_t^{'})(x * y) = d_t(d_t^{'}(x * y)) = d_t(d_t^{'}(x) * y) = d_t^{'}(x) * d_t(y).$

Again, since $d_t^{'}$ is a (r,l)-t-derivation of X, then by Lemmas 3.2 and $2.4(i)$,

$(d_t \circ d_t^{'})(x * y) = d_t(d_t^{'}(x * y)) = d_t(x * d_t^{'}(y)) = d_t(x) * d_t^{'}(y).$

Therefore, $d_t^{'}(x) * d_t(y) = d_t(x) * d_t^{'}(y)$. By putting $y := x$, for all $x \epsilon X$, we get

$d_t^{'}(x) * d_t(x) = d_t(x) * d_t^{'}(x)$. Hence $d_t * d_t^{'} = d_t^{'} * d_t$. This proves the theorem.

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