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Fixed points for Quasi contraction maps on complete metric spaces

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Abstract

The present paper deals with unique fixed point results for quasi contraction mappings on a metric space satisfying some generalized inequality conditions in first section and unique common fixed point result for asymptotically regular mappings of certain type and satisfying a generalized contraction condition in another section. The results obtained generalize the earlier results of Fisher (1979), Hardy and Roger (1973) and others in turn.

Keywords: Quasi contraction, complete metric space, asymptotically regular, generalized contraction, fixed point.
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1. Introduction

Definition 1.1 ([1]). A mapping T on a metric space X into itself is said to be a quasi − contraction if and only if there exist a number c, with $0 \leq c < 1$, such that

$$d(Tx, Ty) \leq c \max\{d(x, y)d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all x, y in X.

Definition 1.2 ([1]). X is said to be T orbitally complete if and only if every Cauchy sequence which is contained in the sequence $\{x, Tx, \ldots, T^n x, \ldots\}$ for some x in X converges in X.

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He then established the following basic results for such mappings:

Theorem 1.3. *Let T be a quasi contraction on the metric space X into itself and let X be T orbitally complete. Then T has a unique fixed point in X.*

Theorem 1.4. *Let T be a continuous mapping on the complete metric space X into itself satisfying the inequality:*

$$d(T^px, T^qy) \leq c \max\{d(T^rx, T^sy), d(T^rx, T^{\gamma'}x), d(T^sy, T^{s'}y)\} : 0 \leq r, r' < p \text{ and } 0 \leq s, s' \leq q\}$$

for all x, y in X, where $0 \leq c < 1$, for some positive integer p and q, then T has a unique fixed point.

For a continuous quasi contraction mapping the following result is proved.

Theorem 1.5. *Let T be a quasi contraction on the metric space X into itself and let T be continuous. Then T has a unique fixed point in X.*

It may be noted that in case when T is a quasi contraction for which $q(\text{or } p) = 1$, the continuity condition of T is unnecessary. We then have,

Theorem 1.6. *Let T be a mapping on the complete metric space X into itself satisfying the inequality*

$$d(T^px, Ty) \leq c \max\{d(T^rx, T^sy), d(T^rx, T^{\gamma'}x), d(T^sy, T^{s'}y)\} : 0 \leq r, r' < p \text{ and } s = 0, 1\}$$

for all x, y in X, where $0 \leq c < 1$, for some positive integer p . Then T has a unique fixed point in X.

In the next section we obtain some fixed point results for such mappings, which satisfy a more general inequality conditions.

2. Results for Quasi contraction mappings

Theorem 2.1. *Let T be a quasi contraction on the complete metric space X into itself satisfying the inequality*

$$d(Tpx, T^qy) \leq c \max\{d(T^\gamma x, T^sy), d(T^\gamma x, T^{\gamma'}x), d(T^sy, T^{s'}y), d(T^{\gamma'}x, T^{s'}y)\}, \tag{2.1}$$

$0 \leq \gamma$, $\gamma < p$ and $0 \leq s, s' \leq q$ for all x, y in X where $0 \leq c < 1$ and for some fixed positive integers p and q. Then T has a unique fixed point in X.

Proof. Without loss of generality we assume that $\frac{1}{2} \leq$ c < 1. Inequality (2.1) will still hold but we will then have $\frac{c}{1-c} \geq 1$. We assume that p $\geq$ q. Let for an arbitrary point x in X the sequence $\{T^n x\}$ is unbounded.
Then the sequence $\{d(T^n x, T^q x) : n = 1, 2, \dots\}$ is unbounded and so there exists an integer n such that

$$(T^nx, T^qx) > (\frac{c}{1-c}) \max\{d(T^ix, T^qx) : 0 \leq i < p\}.$$

Suppose n is the smallest such integer satisfying the above inequality and since $\frac{c}{1-c} \geq 1$, we must have $n > p \geq q$.

Thus

$$\begin{aligned} d(T^n x, T^q x) &> (\frac{c}{1-c}) \max\{d(T^i x, T^q x) : 0 \le i < p\} \\ &\ge \max\{d(T^\gamma x, T^q x) : 0 \le \gamma < n\}. \end{aligned} \tag{2.2}$$

It now follows from inequality (2.2) that

$$\begin{aligned} (1-c) d(T^n x, T^q x) &> c \max\{d(T^i x, T^q x) : 0 \le i \le p\} \\ &\ge c \max\{d(T^i x, T^\gamma x) - d(T^\gamma x, T^q x) : 0 \le i \le p \text{ and } 0 \le \gamma < n\} \\ &\ge c \max\{d(T^i x, T^\gamma x) - d(T^n x, T^q x) : 0 \le i \le p \text{ and } 0 \le \gamma < n\} \end{aligned} \tag{2.3}$$

and so

$$d(T^n x, T^q x) > c \max\{d(T^i x, T^\gamma x) : 0 \le i \le p \text{ and } 0 \le \gamma < n\}. \tag{2.4}$$

We will now prove that

$$d(T^n x, T^q x) > c \max\{d(T^i x, T^\gamma x) : 0 \le i, \gamma < n\}. \tag{2.5}$$

For, if not so then we have

$$d(T^n x, T^q x) \le c \max\{d(T^i x, T^\gamma x) : 0 \le i, \gamma < n\}$$

i.e.,

$$d(T^n x, T^q x) \le c \max\{d(T^i x, T^\gamma x) : p < i, \gamma < n\}. \tag{2.6}$$

In view of inequality (2.4), we can apply inequality (2.1) indefinitely to inequality (2.6), since whenever terms of the form d(T^ix, T^γx) appear with $0 \le i \le$ p, they can be omitted because of inequality (2.4). This means that

$$d(T^n x, T^q x) \le c^k \max\{d(T^i x, T^\gamma x) : p < i, \gamma < n\}$$

for k = 1, 2, ... and on letting limit k tending to infinity it follows that d(T^nx, T^qx) = 0, which gives a contradiction. So inequality (2.5) now follows. However, on using inequality (2.1), we now have

$$\begin{aligned} d(T^n x, T^q x) &\le c \max\{d(T^\gamma x, T^s x), d(T^\gamma x, T^{\gamma'} x), d(T^s x, T^{s'} x), (T^{\gamma'} x, T^{s'} y) \\ &\quad : n - p \le \gamma, \gamma' \le n \text{ and } 0 \le s, s' \le q\} \\ &\le c \max\{d(T^\gamma x, T^s x) : 0 \le \gamma, s \le n\}, \end{aligned}$$

which is impossible, because of inequality (2.5). This contradiction further implies that the sequence {T^nx : n = 1, 2,...} must be bounded.
Now taking

$$M = sup\{d(T^\gamma x, T^s x) : \gamma, s = 0, 1, 2, \ldots\} < \infty$$

and then for arbitrary $\epsilon > 0$, choosing N such that c^NM $< \epsilon$, it follows that for m, n $\ge N$ max{p, q} and on using inequality (2.1) N times we get

$$d(T^m x, T^n x) \le c^N M < \epsilon.$$

Thus the sequence {T^nx: n = 1,2,...} is a Cauchy sequence in the complete metric space X and so has a limit z in X. Since T is continuous it follows that $Tz = z$ and so z is a fixed point of T. The uniqueness of z follows easily from the inequality (2.1).This completes the proof of the Theorem. □

Our next generalization goes as follows.

Theorem 2.2. *Let T be a mapping on the complete metric space X into itself satisfying the inequality*

$$d(T^p x, Ty) \leq c \max\{d(T^\gamma x, T^s y), d(T^\gamma x, T^{\gamma'} x), d(y, Ty), d(T^{\gamma'} x, Ty) : 0 \leq \gamma, \gamma' \leq p \text{ and } s = 0, 1\}$$

for all x, y in X where $0 \leq c < 1$, for some fixed positive integer p. Then T has a unique fixed point in X.

Proof. Let x be an arbitrary point in X. Then, as in the proof of 2.1, the sequence $\{T^n x\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X. For n $\geq$ p, we now have

$$(T^n x, Tz) \leq c \max\{d(T^\gamma x, T^s x), d(T^\gamma x, T^{\gamma'} x), d(z, Tz), (T^{\gamma'} x, Tz) : n - p \leq \gamma, \gamma' \leq n and s = 0, 1\}.$$

Taking n tends to infinity it follows that

$$\begin{aligned} d(z, Tz) &\leq c \max\{d(z, T^s z) : s = 0, 1\} \\ &= cd(z, Tz). \end{aligned}$$

Since c < 1, we see that z is a fixed point of T. This completes the proof of the theorem. □

The following corollary is immediate when p = 1.

Corollary 2.3. *Let T be a mapping on the complete metric space X into itself satisfying the inequality*

$$d(Tx, Ty) \leq c \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all x, y in X, where $0 \leq c < 1$. Then T has a unique fixed point in X.

We now note that the condition that T be continuous when p,q $\geq$ 2 is necessary in Theorem 1.5 This is easily seen by considering X be the closed interval [0, 1] with the usual metric. Define a discontinuous mapping T on X by

$$Tx = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{2}x & \text{if } x \neq 0. \end{cases}$$

We then have

$$d(T^p x, T^q y) = \frac{1}{2} d(T^{p-1} x, T^{q-1} y)$$

for all x, y in X and so T is a quasi contraction with $c = \frac{1}{2}$. T however has no fixed point.
We now prove a fixed point theorem on a compact metric space.

Theorem 2.4. *Let T be a continuous mapping on the compact metric space X into itself satisfying the inequality*

$$\begin{aligned} d(T^p x, T^q y) < \max\{&d(T^\gamma x, T^s y), d(T^\gamma x, T^{\gamma'} x), d(T^s y, T^{s'} y), d(T^{\gamma'} x, T^{s'} y) \\ &: 0 \leq \gamma, \gamma' \leq p \text{ and } 0 \leq s, s' \leq q\} \end{aligned}$$

for all x, y in X for which the right hand side of the inequality is positive. Then T has a unique fixed point in X.

Proof. Suppose first of all that T is a quasi contraction. The result then follows from Theorem 2.1 If T is not a quasi- contraction and if {c_n: n = 1,2,. . . } is a monotonically increasing sequence of numbers converging to 1, then there must exist sequences. {x_n} and {y_n} in X such that

$$d(T^p x_n, T^q y_n) > c_n \max\{d(T^\gamma x_n, T^s y_n), d(T^\gamma x_n, T^{\gamma'} x_n), d(T^s y_n, T^{s'} y_n), d(T^{\gamma'} x_n, T^{s'} y_n) : \\ 0 \leq \gamma, \gamma' \leq p and 0 \leq s, s' \leq q\}$$

for n = 1, 2, ... Since X is compact, there exist subsequences {x_{nk} : k = 1, 2,..} and {y_{nk} : k = 1, 2,..} of {x_n} and {y_n} converging to x and y, respectively. We then have

$$d(T^p x_{nk}, T^q y_{nk}) > c_{nk} \max\{d(T^\gamma x_{nk}, T^s y_{nk}), d(T^\gamma x_{nk}, T^{\gamma'} x_{nk}), d(T^s y_{nk}, T^{s'} y_{nk}), d(T^{\gamma'} x_{nk}, T^{s'} y_{nk}) : \\ 0 \leq \gamma, \gamma' \leq p and 0 \leq s, s' \leq q\}$$

for k = 1, 2, ... Since T is continuous, taking limit as k tends to infinity, we get

$$d(T^p x, T^q y) \geq \max\{d(T^\gamma x, T^s y), d(T^\gamma x, T^{\gamma'} x), d(T^s y, T^{s'} y), d(T^{\gamma'} x, T^{s'} y) : \\ 0 \leq \gamma, \gamma' \leq p and 0 \leq s, s' \leq q\},$$

which leads to a contradiction unless x = y = Tx. Thus x is a fixed point of T. The uniqueness of x follows easily. This completes the proof of the theorem. □

When p = q = 1, we have the following corollary:

Corollary 2.5. *Let T be a continuous mapping of the compact metric space X into itself satisfying the inequality*

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all x, y in X for which the right hand side of the inequality is positive. Then T has an unique fixed point.

3. A fixed point result for generalized contraction

In this section we prove common fixed point theorems with the help of sequences which are not necessarily obtained as a sequence of iterates of mappings under consideration. The mappings are asymptotically regular of certain nature mentioned below. The result obtained generalizes a result due to Hardy and Roger [3].

Definition 3.1. Let A and B be two self mappings on X and {x_n}a sequence in X. Then the sequence {x_n} is said to be asymptotically A- regular with respect to B if

$$Lt_{n\to\infty} d(Bx_n, Ax_n) = 0, \quad \text{when B is identity map.}$$

Definition 3.2. Let f and g be two self mappings on X. Then the pair {f, g} is said to be a weakly commuting pair if $d(fgx, gfx) \leq d(gx, fx)$ for all x ∈ X.

Theorem 3.3. *Let (X,d) be a complete metric space. Let A, B, S, T be four self mappings of X satisfying the following conditions:*

(a) *$d(Sx,Ty) \leq a_1 d(Sx,Ax) + a_2 d(Ty,By) + a_3 d(Sx,By) + a_4 d(Ty,Ax) + a_5 d(Ax,By)$ for all x,y $\in X$ where a_i (i = 1,2,3,4,5) are non negative real and* $\max\{(a_2+a_4),(a_3+a_4+a_5)\} < 1$,

(b) *A and B are continuous,*

(c) *$\{A,S\}$ and $\{B,T\}$ are weakly commuting pairs,*

(d) *there exists a sequence which is asymptotically S- regular as well as T- regular with respect to A and B.*

Then A, B, S, T have a unique common fixed point.

Proof : Let $\{x_n\}$ be a sequence as described in (b). Then using (a) we get

$$\begin{aligned}
d(Ax_n,Bx_m) &\leq d(Ax_n,Sx_n) + d(Sx_n,Tx_m) + d(Tx_m,Bx_m)\\
&\leq d(Ax_n,Sx_n) + a_1 d(Sx_n,Ax_n) + a_2 d(Tx_m,Bx_m) + a_3 d(Sx_n,Bx_m)\\
&\quad + a_4 d(Tx_m,Ax_n) + a_5 d(Ax_n,Bx_m) + d(Tx_m,Bx_m)\\
&\leq d(Ax_n,Sx_n) + a_1 d(Sx_n,Ax_n) + a_2 d(Tx_m,Bx_m) + a_3[d(Sx_n,Ax_n) + d(Ax_n,Bx_m)]\\
&\quad + a_4[d(Tx_m,Bx_m) + d(Bx_m,Ax_n)] + a_5 d(Ax_n,Bx_m) + d(Tx_m,Bx_m)
\end{aligned}$$

Therefore , $d(Ax_n, Bx_m) \leq \frac{1+a_1+a_3}{1-a_3-a_4-a_5} d(Ax_n,Sx_n) + \frac{1+a_2+a_4}{1-a_3-a_4-a_5} d(Tx_m, Bx_m)$. This shows that $\{Ax_n\}$ is a Cauchy sequence. Let $\lim_n \to\infty Ax_n = z = \lim_n \to\infty Bx_m$. Then it follows that $\lim_n \to\infty Sx_n = z = \lim_n \to\infty Tx_m$. By virtue of the continuity of A and B, we find that

$$A^2x_n \to Az, ASx_n \to Az and B^2x_m \to Bz, BTx_m \to Bz.$$

We shall show that

$$SAx_n \to Az \text{ and } TBx_m \to Bz.$$

For this, consider the inequality,

$$\begin{aligned}
d(SAx_n,Az) &\leq d(SAx_n,ASx_n) + d(ASx_n,Az)\\
&\leq d(Ax_n,Sx_n) + d(ASx_n,Az),
\end{aligned}$$

which shows that $SAx_n \to Az$.
Similarly,

$$d(TBx_m,Bz) \leq d(TBx_m,BTx_m) + d(BTx_m,Bz) \leq d(Bx,Tx_m) + d(BTx_m,Bz),$$

which shows that $TBx_m \to Bz$.
Now,

$$\begin{aligned}
d(Az,Tz) &\leq d(Az,SAx_n) + d(SAx_n,Tz)\\
&\leq d(Az,SAx_n) + a_1 d(SAx_n,A^2x_n) + a_2 d(Tz,Az)\\
&\quad + a_3 d(SAx_n,Az) + a_4 d(Tz,A^2x_n) + a_5 d(A^2x_n,Az).
\end{aligned}$$

Taking limit as n tending to infinity, we get

$$d(Az,Tz) \leq (a_2+a_4) d(Az,Tz)$$

and hence Az = Tz. Similarly we can show that Bz = Sz.
Further

$$d(SAx_n, TBx_m) \leq a_1 d(SAx_n, A^2x_n) + a_2 d(TBx_m, B^2x_m) + a_3 d(SAx_n, B^2x_m) + a_4 d(TBx_m, A^2x_n) + a_5 d(A^2x_n, B^2x_m).$$

On taking limits we have

$$\begin{aligned} d(Az, Bz) &\leq a_1 d(Az, Az) + a_2 d(Bz, Bz) + a_3 d(Az, Bz) + a_4 d(Bz, Az) + a_5 d(Az, Bz) \\ &\leq (a_3 + a_4 + a_5) d(Az, Bz) \end{aligned}$$

i.e. $(1 - a_3 - a_4 - a_5)$ d(Az,Bz) ≤ 0. So, $Az = Bz$.
Hence $Az = Bz = Sz = Tz$.
Now consider,

$$d(Sx_n, Tz) \leq a_1 d(Sx_n, Ax_n) + a_2 d(Tz, Bz) + a_3 d(Sx_n, Bz) + a_4 d(Tz_, , Ax_n) + a_5 d(Ax_n, Bz).$$

As limit n $\to \infty$,we have

$$d(z, Tz) \leq a_1 d(z, z) + a_2 d(Tz, Tz) + a_3 d(z, Tz) + a_4 d(Tz, z) + a_5 d(z, Tz)$$

or

$$d(z, Tz) \leq (a_3 + a_4 + a_5) d(z, Tz) < d(z, Tz),$$

which implies that z = Tz. Thus z is a common fixed points of A, B, S and T.

In order to prove the uniqueness of common fixed point. Let $z_1$and z_2 be any two distinct common fixed points of A, B, S and T. Then

$$\begin{aligned} d(z_1, z_2) = d(Sz_1, Tz_2) &\leq d(Sz_1, Az_1) + a_2 d(Tz_2, Bz_2) + a_3 d(Sz_1, Bz_2) \\ &\quad + a_4 d(Tz_2, Az_1) + a_5 d(Az_1, Bz_2) \\ &= (a_3 + a_4 + a_5) d(z_1, z_2) < d(z_1, z_2). \end{aligned}$$

Therefore, $z_1 = z_2$. This completes the proof.

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