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P-Stable Hybrid Super-Implicit Methods for Periodic Initial Value Problems

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Abstract

This paper deals with a class of symmetric (hybrid) P-stable methods for the numerical solution of special second order initial value problems (IVPs). For linear multistep methods, Lambert and Watson [5], had shown that a P-stable method is necessarily implicit and that the maximum order attainable by a P -stable method is at most two. P-stability is important in the case of 'periodic stiffness' as it is termed by Lambert and Watson [5], that is, when the solution consists of an oscillation of moderate frequency with a high frequency oscillation of small amplitude superimposed. In order to overcome the order-barrier on linear multistep P-stable methods, we developed a new type of implicit formulas of linear multistep methods. The formulas, which we call to be hybrid super-implicit, are of more implicitness than the so-called implicit formulas in the sense that they require the knowledge of functions not only at the past and present time-step but also at the future ones. In the cases when the right hand side of IVP is very complex, the super-implicit methods are preferred. Also, we have used off-step points which allow us to derive P-stable schemes of high order. We report numerical experiments to illustrate the accuracy and implementation aspects of this class of methods.

Keywords: Initial value problems, Super-implicit, Hybrid methods, Off-step points, P-stability.

1. Introduction

Let us consider a special class of the second-order IVPs,

$$y'' = f\left(x, y\left(x\right)\right), \quad y(0) = y_0, \quad y'(0) = y'_0. \qquad (1)$$

There is a vast literature for the numerical solution of these problems as well as for the general second-order IVPs,

$$y'' = f\left(x, y\left(x\right), y'\left(x\right)\right), \quad y(0) = y_0, \quad y'(0) = y'_0, \qquad (2)$$

see for example the excellent book by Lambert [6]. A linear multistep method of the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^2\sum_{j=0}^{k'}\beta_j f_{n+j}, \quad k \ge 2 \qquad (3)$$

for the numerical solution of (2), on the discrete point set $\{x_n \mid x_n = a + nh, n = 0,1,\ldots\}$, is characterized by the polynomials ρ and σ, where

$$\rho(\zeta) = \sum_{j=0}^{k}\alpha_j\zeta^j, \qquad \sigma(\zeta) = \sum_{j=0}^{k}\beta_j\zeta^j. \qquad (4)$$

Accordingly, (3) is refereed as the method (ρ,σ), its order is defined to be p and its error constant to be C_{p+2} if, for an adequately smooth arbitrary test function $z(x)$,

$$\begin{aligned}&\sum_{j=0}^{k}\alpha_j z(x+jh) - h^2\sum_{j=0}^{k'}\beta_j z''(x+jh)\\ &= C_{p+2}h^{p+2}z^{p+2}(x) + o\left(h^{p+3}\right),\end{aligned} \qquad (5)$$

where the error constant is given by

$$\begin{aligned}C_q &= \frac{1}{q!}\sum_{j=0}^{k} j^{q-2}\left(j^2\alpha_j - q(q-1)\beta_j\right)\\ &-\sum_{j=0}^{k'}\frac{j^{q-2}}{(q-2)!}\beta_j, \qquad q > 2.\end{aligned}$$

As in [9] the method is assumed to satisfy the following,

1) $\alpha_k = 1, \quad |\alpha_0| + |\beta_0| \ne 0,$
2) ρ and σ have no common factor,
3) $\rho(1) = \rho'(1) = 0, \ \rho''(1) = 2\sigma(1)$
4) the method is zero-stable.

The method is called symmetric if $\alpha_i = \alpha_{k-i}$, for $i = 1,\ldots,k$, and similarly, for β_i.

The definition of P-stability is based on the application of the method characterized by ρ,σ to the periodic IVP,

$$y'' + q^2 y = 0 \qquad (6)$$

Definition 1. (see [5]) The method (ρ,σ) is said to have interval of periodicity $\left(0, H_0^2\right)$ if, for all H^2 in the interval, the roots of

$$P\left(z, H^2\right) = \rho\left(z\right) + H^2\sigma\left(z\right) = 0, \quad H = qh,$$

satisfy

$$z_1 = e^{i\theta(H)}, \quad z_2 = e^{-i\theta(H)}, \quad |z_s| \leq 1, s > 2,$$

where $\theta(H)$ is a real function.

Definition 2. ([5]) The method (ρ, σ) is said to be P-stable if its interval of periodicity is $(0, \infty)$.

In [5] it has proven that a method described by _; _ has a nonvanishing interval of periodicity only if it is symmetric and, for P-stability, the order cannot exceed two. Fukushima [2] has proved that the condition is also sufficient. The search to improve the accuracy and finding a high accurate and high efficient P-stable methods, is carried out in the two main directions:

• Using higher derivatives of $f\ (x, y)$.

• Throw in additional stages, off-step points, super-future points and like (see [1, 7, 8, 9, 11]). This leads to different kind of the linear methods.

We now recall the super-implicit methods developed by Fukushima [3]

$$y_{n+1} + \sum_{j=1}^{k} \alpha_j y_{n+1-j} = h^2 \sum_{j=0}^{k'} \beta_j f_{n+1+m-j}. \qquad (7)$$

The method is explicit when $m < 0$, implicit when $m = 0$, and super-implicit when $m > 0.$ Also m is called the degree of implicitness. Once $\alpha -$coefficients are given, the corresponding $\beta -$ coefficients are automatically determined if the parameters m and k' are fixed and if one requires that the maximum order is achieved by the method. Fukushima [2] has developed a Stormer-Cowel type formulas, namely $k = 2$ and $\alpha_1 = -2, \alpha_2 = \alpha_0 = 1.$

In that case, one can get methods of order up to $k' + 2.$ Clearly, the methods require additional formulas to treat the additional starting and final values. Solving the nonlinear system one obtains the solution for a block of points. Fukushima [2] suggested the use of Picard iteration.

2. Construction of P-stable hybrid super-implicit methods

We start by writing the hybrid super-implicit symmetric k -step methods in the form

$$\sum_{j=0}^{\frac{k}{2}} \alpha_j \left(y_{n+j} + y_{n-j}\right) = h^2 \sum_{j=0}^{\frac{k'}{2}} \beta_j \left(f_{n+j} + f_{n-j}\right), \\ + h^2\gamma\left(f_{n+\theta} + f_{n-\theta}\right) \qquad (8)$$

we have $\alpha_{\frac{k}{2}} = 1,$ and $|\alpha_0| + |\beta_0| \neq 0.$ It is better to choose $\alpha_0 = 1,$ since it appears only once. Clearly, k and k' are even and $k' \geq k$ for a super-implicit. Applying the method (8) to (6), we have

$$\sum_{j=0}^{\frac{k}{2}} \left(\alpha_j + h^2 q^2 \beta_j\right)\left(y_{n+j} + y_{n-j}\right) + h^2 \sum_{j=\frac{k}{2}+1}^{\frac{k'}{2}} h^2 q^2 \beta_j \left(y_{n+j} + y_{n-j}\right) + h^2 q^2 \gamma \left(y_{n+\theta} + y_{n-\theta}\right) = 0. \qquad (9)$$

Now, substitute $y_n = e^{iqx_n}$ we have

$$\sum_{j=0}^{\frac{k}{2}} \left(\alpha_j + H^2 \beta_j\right)\cos\left(jH\right) + \sum_{j=\frac{k}{2}+1}^{\frac{k'}{2}} H^2 \beta_j \cos\left(jH\right) + H^2 \gamma \cos\left(\theta H\right) = 0. \qquad (10)$$

where $H = hq.$ We can use one of the parameters to ensure P-stability. For example, suppose we take $k = 4, k' = 8,$

then the method becomes

$$\begin{aligned} & y_{n+2} - 2y_{n+1} + 2y_n - 2y_{n-1} + y_{n-2} = \\ & h_2\left[\beta_4\left(f_{n+4} + f_{n-4}\right) + \beta_3\left(f_{n+3} + f_{n-3}\right)\right] \\ & +\beta_2\left(f_{n+2} + f_{n-2}\right) + \beta_1\left(f_{n+1} + f_{n-1}\right) \\ & +2\beta_0 f_n + \gamma\left(f_{n+\theta} + f_{n-\theta}\right) \end{aligned} \qquad (11)$$

Choose β_0 to satisfy the P-stability condition (10), i.e.,

$$\begin{aligned} & 1 + H^2\beta_0 + \left(-2 + H^2\beta_1\right)\cos\left(H\right) + \left(1 + H^2\beta_2\right)\cos\left(2H\right) \\ & + H^2\beta_3\cos\left(3H\right) + H^2\beta_4\cos\left(4H\right) + H^2\gamma\cos\left(\theta H\right) = 0 \end{aligned}$$

Using MAPLE, we found the following values

$$\begin{aligned} & \gamma = 0.1923699508, && \theta = 1.691868661, \\ & \beta_1 = 0.6594580678, && \beta_2 = -0.01110806195, \\ & \beta_3 = 0.0001141672948, && \beta_4 = -0.000001885624571, \end{aligned}$$

and the method is of order fourteen with an error constant $2.560808279 \times 10^{-9}.$ The choice for β_0 should be $0.1591677616 + O\left(h^{14}\right).$

3. Störmer-Cowell type P-stable hybrid super-implicit methods

Störmer-Cowell type methods have left-hand side of the form

$$y_{n+1} - 2y_n + y_{n-1}.$$

For example, the following method will be of order fourteen

$$\begin{aligned}
&y_{n+1} - 2y_n + y_{n-1} = \\
&h_2\left[\beta_4\left(f_{n+4} + f_{n-4}\right) + \beta_3\left(f_{n+3} + f_{n-3}\right)\right] \\
&+\beta_2\left(f_{n+2} + f_{n-2}\right) + \beta_1\left(f_{n+1} + f_{n-1}\right) \\
&+2\beta_0 f_n + \gamma\left(f_{n+\theta} + f_{n-\theta}\right)
\end{aligned}$$

The coefficients can be found using MAPLE,

$\gamma =$ 0.2403960726, θ=0.54057y6211,
$\beta_1 = 0.01318456055$, $\beta_2 = -0.00002642103972$,
$\beta_3 = 5.674806468 \times 10^{-7}$, $\beta_4 = -1.199255345 \times 10^{-8}$.

The error constant is $2.130816914 \times 10^{-11}$.

4. Numerical results

Before we solve presented test problems, we are going to make some remarks about implementation of new method. In order to implement such formula, a special predictor to estimate $y_{n+\theta}$ and $y_{n-\theta}$ is necessary. Then, we create a set of equations relating y_n to past, present and future values. Thus we have to solve a system of nonlinear equations. Fukushima [2] suggested to solve this nonlinear system using Picard iteration. Clearly the number of unknown in each equation of the system will increase when increasing the order of the numerical method.

Example 1. Consider the stiff initial value problem

$y''(x) + qy(x) = 8\left(\cos(x) + \frac{2}{3}\cos(3x)\right)$, $y(0) = 1$, $y'(0) = 0$ where $q = 5.$ The exact solution is

$$y(x) = \frac{1}{3}\left(\cos(x) + \cos(3x) + \cos(5x)\right)$$

whose complex oscillatory pattern can be seen in Fig.1. See the plot of the numerical solutions and the exact solution in Fig. 2 with stepsize $h = \frac{\pi}{8}$.

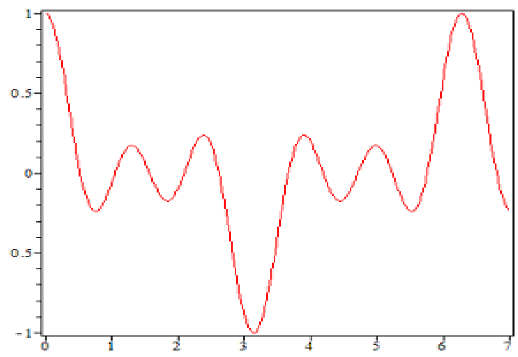

Figure 1: Exact solution over one period.

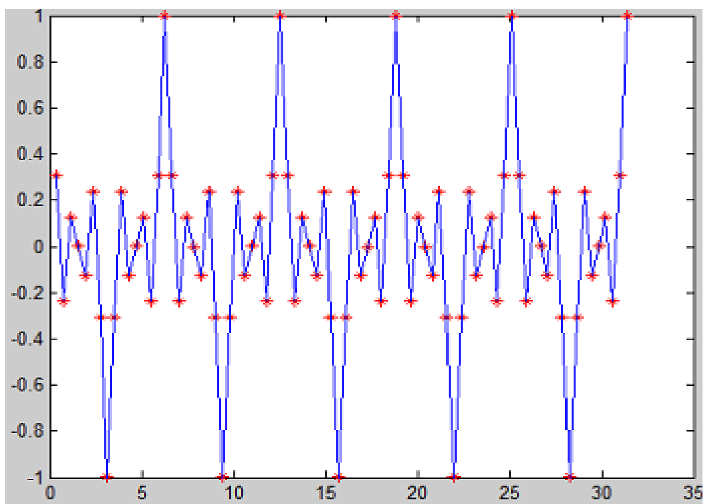

Figure 2: Exact solution and numerical solution over five periods.

Example 2. Consider the nonlinear undamped Duffing equation

$$y''(x)+y(x)+y^3(x)=B\cos(qx),$$
$$y(0)=0.200427,\ \ y'(0)=0,$$

where $B=0.002$ and $q=1.01$. The exact solution is

$$y(x)=\sum_{i=0}^{3}K_{2i+1}\cos((2i+1)qx),$$
$$\{K_1,K_2,K_5,K_7\}=\{0.200179477536,$$
$$0.246946143\times10^{-3},0.304016\times10^{-6},0.374\times10^{-9}\}.$$

In the numerical experiment, we take the step length $h=\frac{\pi}{5}$. The following table gives the error at point $x=\pi, 2\pi, 4\pi$.

Table1. Absolute errors in $y(x)$ with $h=\pi/5$.

x	The new method
π	2.13E-12
2π	4.21E-13
4π	3.76E-12

5. Conclusion

This paper has been able to develop the implicit formulas of linear multistep methods to hybrid super-implicit formulas for the solution of periodic initial value problems. In the case the right hand side is complex, we may prefer hybrid super-implicit methods. We do not claim that our numerical results demonstrate the superiority of our approach over any of the more conventional approaches. However, we do feel that hybrid super-implicit methods are more accurate than Obrechkoff schemes of the same order (our interest for future research). The disadvantage of these methods is that in general, the number of unknowns in system of nonlinear equations will increase when increasing the order of the numerical method.

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