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Existence and multiplicity of solutions for a Robin problem

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Abstract

In this article we study the nonlinear Robin boundary-value problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda f(x,u), & in\ \ \Omega\ ; \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = 0, & on\ \ \partial\Omega. \end{cases}$$

Using the variational method, under appropriate assumptions on f, we obtain a result on existence and multiplicity of solutions.

Keywords: $p(x)$-Laplace operator, variable exponent Lebesgue space, variable exponent Sobolev space, Ricceri's variational principle.

1. Introduction

The purpose of this article is to study the existence of solutions for the following problem:

$$\begin{cases} -\Delta_{p(x)} u = \lambda f(x,u), & in\ \ \Omega\ ; \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = 0, & on\ \ \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, λ is a positive number, p is a continuous function on $\overline{\Omega}$ with $p^- := \inf_{x\in\overline{\Omega}} p(x) > 1$, and $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega} \beta(x) > 0$. The main interest in studying such problems arises from the presence of the

$p(x)$-Laplace operator $div(|\nabla u|^{p(x)-2}\nabla u)$, which is a natural extension of the classical p-Laplace operator $div(|\nabla u|^{p-2}\nabla u)$ obtained in the case when p is a positive constant. However, such generalizations are not trivial since the $p(x)$- Laplace operator possesses a more complicated structure than p Laplace operator; for example, it is inhomogeneous.

Nonlinear boundary value problems with variable exponent has been received considerable attention in recent years. This is partly due to their frequent appearance in applications such as the modeling of electro-rheological fluids [16, 18, 22, 23] and image processing [7], but these problems are very interesting from a purely mathematical point of view as well. Many results have been obtained on this kind of problems; see for example [6, 8, 9, 11, 12, 19, 20, 21]. In [8], the authors have studied the case $f(x,u) = |u|^{p(x)-2}u$, they proved that the existence of infinitely many eigenvalue sequences. Unlike the p-Laplacian case, for a variable exponent p(x) ($\neq$ constant), there does not exist a principal eigenvalue and the set of all eigenvalues is not closed under some assumptions. Finally, they presented some sufficient conditions for the infimum of all eigenvalues is zero and positive, respectively.

We make the following assumptions on the function f:

(F1) $|f(x,s)| \leq a(x) + b|s|^{\alpha(x)-1}$, for all $(x,s) \in \Omega \times \mathbb{R}$, where $a(x)$ is in $L^{\frac{\alpha(x)}{\alpha(x)-1}}(\Omega)$, $b \geq 0$ is a constant, $\alpha(x) \in C(\Omega)$, $1 < \alpha^- := \inf_{x\in\overline{\Omega}}\alpha(x) \leq \alpha^+ := \sup_{x\in\overline{\Omega}}\alpha(x) < p^-$ and $p(x) > N$.

(F2) $f(x,t) < 0$, when $|t| \in (0,1)$, $f(x,t) \geq m > 0$, when $t \in (t_0,\infty)$, $t_0 > 1$.

In [4], the authors obtained the existence and multiplicity of solutions for Navier problems under the following conditions:

$\sup_{(x,s)\in\Omega\times\mathbf{R}} \frac{|f(x,s)|}{1+|s|^{t(x)-1}} < +\infty$, where $t \in C(\overline{\Omega})$ and $t(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and there exist two positive constants ρ, ϑ and a function $\gamma(x) \in C(\overline{\Omega})$ with $1 < \gamma^- \leq \gamma^+ < p^-$, such that

(I1) $F(x,s) \geq 0$ for a.e. $x \in \Omega$ and all $s \in]0,\rho]$;

(I2) there exist $p_1(x) \in C(\overline{\Omega})$ and $p^+ < p_1^- \leq p_1(x) < p^*(x)$, such that

$$\limsup_{s\to 0}\sup_{x\in\Omega} \frac{F(x,s)}{|s|^{p_1(x)}} < +\infty;$$

(I3) $|F(x,s)| \leq \vartheta\left(1 + |s|^{\gamma(x)}\right)$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$.

There are many functions which do not satisfy the above conditions (I1), (I2). For instance the function below does not satisfy (I1), (I2).

$$f(x,t) = \frac{t-1}{t^2+1} + Arctan(t) - 1, \tag{1.2}$$

where

$$F(x,t) = \int_0^t f(x,s)ds = (t-1)Arctan(t) - t.$$

But it is easy to see the above function (1.2) satisfies our conditions.

Remark 1.1. *Let* $\Omega = \mathbb{R}$, $p(x) = p \equiv 2$ *and* $F(t) = (t-1)Arctan(t) - t$.

So we have

$$p^- = p^+ = 2, \ and\ p^* = +\infty.$$

Moreover

$$F(t) \leq 0 \quad for\ all \quad t \in (0,1), \ and\ F(t) > 0 \quad for\ all \quad t < 0.$$

And

$$\forall p_1 > p = 2, \quad \limsup_{s\to 0} \frac{F(s)}{|s|^{p_1}} = +\infty.$$

The main result of this paper is as follows.

Theorem 1.2. *If* (F1), (F2) *hold, then there exist an open interval* $\Lambda \subset (0,\infty)$ *and a positive real number* ρ *such that each* $\lambda \in \Lambda$, *(1.1) has at least three solutions whose norms are less than* ρ.

This article is organized as follows. First, we will introduce some basic preliminary results and lemmas in Section 2. In Section 3, we will give the proof of our main result.

2. Preliminary

For completeness, we first recall some facts on the variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. For more details, see [13, 14]. Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p \in C_+(\overline{\Omega})$ where

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \quad and \quad \inf_{x\in\overline{\Omega}} p(x) > 1\}.$$

Denote by $p^- := \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$. Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \{u: \Omega \to \mathbb{R} \quad is\ measurable\ and \quad \int_\Omega |u|^{p(x)} dx < +\infty\},$$

with the norm

$$|u|_{p(x)} = \inf\{\tau > 0;\ \int_\Omega |\frac{u}{\tau}|^{p(x)} dx \leq 1\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega): |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\| u \| = \inf\{\tau > 0;\ \int_\Omega (|\frac{\nabla u}{\tau}|^{p(x)} + |\frac{u}{\tau}|^{p(x)}) dx \leq 1\},$$

$$\| u \| = |\nabla u|_{p(x)} + |u|_{p(x)}.$$

We refer the reader to [13, 14] for the basic properties of the variable exponent Lebesgue and Sobolev spaces.

Lemma 2.1 (cf. [14]) . *Both $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|)$ are separable and uniformly convex Banach spaces.*

Lemma 2.2 (cf. [14]). *Hölder inequality holds, namely*

$$\int_\Omega |uv| dx \leq 2|u|_{p(x)}|v|_{p'(x)} \quad \forall u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Lemma 2.3 (cf. [13]). *Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ and $1 \leq q(x) < p^*(x)$ for $x \in \overline{\Omega}$, then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & if \;\; p(x) < N; \\ +\infty, & if \;\; p(x) \geq N. \end{cases}$$

Now, we introduce a norm, which will be used later.

Let $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega}\beta(x) > 0$ and for $u \in W^{1,p(x)}(\Omega)$, define

$$\| u \|_\beta = \inf\{\tau > 0; \int_\Omega (|\frac{\nabla u}{\tau}|^{p(x)} dx + \int_{\partial\Omega} \beta(x)|\frac{u}{\tau}|^{p(x)}) d\sigma \leq 1\}.$$

Then, by Theorem 2.1 in [10], $\|.\|_\beta$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|.\|$.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the mapping defined by the following.

Lemma 2.4 (cf. [10]). *Let $I(u) = \int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)} d\sigma$ with $\beta^- > 0$. For $u \in W^{1,p(x)}(\Omega)$ we have*

- $\| u \|_\beta < 1 (= 1, > 1) \Leftrightarrow I(u) < 1 (= 1, > 1).$
- $\| u \|_\beta \leq 1 \Rightarrow \| u \|_\beta^{p^+} \leq I(u) \leq \| u \|_\beta^{p^-}.$
- $\| u \|_\beta \geq 1 \Rightarrow \| u \|_\beta^{p^-} \leq I(u) \leq \| u \|_\beta^{p^+}.$

Lemma 2.5 (cf. [5,15,17]). *Let X be a separable and reflexive real Banach space, $\phi: X \to \mathbb{R}$ is a continuous Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi: X \to \mathbb{R}$ is a continuous Gâteaux differentiable functional whose Gâteaux derivative is compact, assume that:*

(i) $\lim_{\|u\|_X \to \infty} (\phi(u) + \lambda\psi(u)) = \infty$ for all $\lambda > 0$,

(ii) there exist $r \in \mathbb{R}$ and $u_0, u_1 \in X$ such that $\phi(u_0) < r < \phi(u_1)$,

(iii)

$$\inf_{u\in\phi^{-1}(-\infty,r]} \psi(u) > \frac{(\phi(u_1)-r)\psi(u_0)+(r-\phi(u_0))\psi(u_1)}{\phi(u_1)-\phi(u_0)}.$$

Then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive constant $\rho > 0$ such that for any $\lambda \in \Lambda$ the equation $\phi'(u) + \lambda\psi'(u) = 0$ has at least three solutions in X whose norms are less than ρ.

Theorem 2.6. *Let $X = W^{1,p(x)}(\Omega)$ and $f: \Omega \times \mathbb{R} \to \mathbb{R}$ be a carathéodory function with primitive $F(x,u) = \int_0^u f(x,t)dt$. If the following condition hold:*

- $|f(x,s)| \le a(x) + b|s|^{\alpha(x)-1}$, for all $(x,s) \in \Omega \times \mathbb{R}$,

where $a(x) \in L^{\frac{\alpha(x)}{\alpha(x)-1}}(\Omega)$ and $b \ge 0$ is a constant, $\alpha(x) \in C_+(\Omega)$ such that for all $x \in \Omega$,

$$\alpha(x) < \begin{cases} \frac{Np(x)}{N-p(x)}, & if\ \ p(x) < N; \\ +\infty, & if\ \ p(x) \ge N, \end{cases} \tag{2.1}$$

then $\psi(u) = -\int_\Omega F(x,u(x))dx \in C^1(X,\mathbb{R})$ and $D\psi(u,\varphi) =< \psi'(u),\varphi >= -\int_\Omega f(x,u(x))\varphi dx$, moreover, the operator $\psi': X \to X^*$ is compact.

Proof. It is easily adapted from Theorem 2.9 in [3].

Let $X = W^{1,p(x)}(\Omega)$ and

$$\phi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u|^{p(x)}\, d\sigma,$$

$$\psi(u) = -\int_\Omega F(x,u)dx,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

Obviously $\phi \in C^1(X,\mathbb{R})$ and

$$(\phi'(u), v) = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv d\sigma,$$

$$(\psi'(u), v) = -\int_\Omega f(x,u)v dx.$$

Definition 2.7. *We say that $u \in X$ is a weak solution of (1.1) if*

$$\int_\Omega \left|\nabla u\right|^{p(x)-2}\nabla u \nabla v \, dx + \int_{\partial\Omega} \beta(x)\left|u\right|^{p(x)-2} uv \, d\sigma = \lambda \int_\Omega f(x,u)v dx \quad for\ all\ v \in X.$$

3. Proof of main result

[Proof of theorem 1.2]. For proving our result we use lemma 2.5. It is well known that ϕ is a continuous convex functional, then it is weakly lower semicontinuous and its inverse derivative is continuous, from theorem 2.6 the precondition of lemma 2.5 is satisfied. In following we need to verify that the conditions (i), (ii) and (iii) in lemma 2.5 are fulfilled.

For $u \in X$ such that $\| u \|_\beta \geq 1$, we have

$$\psi(u) = -\int_\Omega F(x,u)dx = -\int_\Omega [\int_0^{u(x)} f(x,t)dt]dx$$

$$\leq \int_\Omega [a(x)|u(x)| + \frac{b}{\alpha(x)}|u|^{\alpha(x)}]dx$$

$$\leq 2|a|_{\frac{\alpha(x)}{\alpha(x)-1}}|u|_{\alpha(x)} + \frac{b}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx$$

$$\leq 2C|a|_{\frac{\alpha(x)}{\alpha(x)-1}} \| u \|_\beta + \frac{b}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx.$$

By the embedding theorem, we have $u \in L^{\alpha(x)}(\Omega)$; therefore,

$$\int_\Omega |u|^{\alpha(x)}dx \leq \max\{|u|^{\alpha^+}_{\alpha(x)}, |u|^{\alpha^-}_{\alpha(x)}\} \leq C' \| u \|_\beta^{\alpha^+}.$$

Then

$$|\psi(u)| \leq 2C|a|_{\frac{\alpha(x)}{\alpha(x)-1}} \| u \|_\beta + \frac{b}{\alpha^-}C' \| u \|_\beta^{\alpha^+}.$$

On the other hand,

$$\phi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} \, dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)} \, d\sigma \geq \frac{1}{p^+} \| u \|_\beta^{p^-}.$$

Which implies that for any $\lambda > 0$,

$$\phi(u) + \lambda\psi(u) \geq \frac{1}{p^+} \parallel u \parallel_\beta^{p^-} - 2\lambda C|a|_{\frac{\alpha(x)}{\alpha(x)-1}} \parallel u \parallel_\beta - \frac{\lambda b C'}{\alpha^-} \parallel u \parallel_\beta^{\alpha^+}.$$

From $p^- > \alpha^+$ we obtain

$$\lim_{\|u\|_\beta \to \infty} (\phi(u) + \lambda\psi(u)) = \infty,$$

then (i) of lemma 2.5 is verified.

lt remains to show (ii) and (iii) of this lemma (Ricceri). By (F2), it is clear that F(x,t) is increasing for $t \in (t_0, \infty)$ and decreasing for $t \in (0,1)$ uniformly for $x \in \Omega$, and $F(x,0) = 0$ is obvious, $F(x,t) \to +\infty$ when $t \to +\infty$ because ($F(x,t) \geq mt$ uniformly on x). Then, there exists a real number $\delta > t_0$ such that

$$F(x,t) \geq 0 = F(x,0) \geq F(x,\tau) \quad \forall u \in X, t > \delta, \tau \in (0,1).$$

Let a, b be two real numbers such that $0 < a < min\{1, c_1\}$ where c_1 is a constant which satisfies

$$\parallel u \parallel_{C(\overline{\Omega})} \leq c_1 \parallel u \parallel_\beta,$$

where $\parallel u \parallel_{C(\overline{\Omega})} := \sup_{x \in \overline{\Omega}} |u(x)|$.

The above inequality is well defined due to compactly embedding from $W^{1,p(x)}(\Omega)$ to $C(\overline{\Omega})$.

We choose $b > \delta$ satisfying $b^{p^-}\beta^-|\partial\Omega| > 1$. When $t \in [0, a]$ we have

$$F(x,t) \leq F(x,0) = 0.$$

Then

$$\int_\Omega \sup_{0<t<a} F(x,t)dx \leq \int_\Omega F(x,0)dx = 0.$$

Furthermore, since $b > \delta$ we have

$$\int_\Omega F(x,b)dx > 0.$$

Moreover,

$$\frac{1}{c_1^{p^+}} \cdot \frac{a^{p^+}}{b^{p^-}} \int_\Omega F(x,b)dx > 0.$$

Which implies

$$\int_\Omega \sup_{0<t<a} F(x,t)dx \leq 0 < \frac{1}{c_1^{p^+}} \frac{a^{p^+}}{b^{p^-}} \int_\Omega F(x,b)dx.$$

Let $u_0, u_1 \in X$, $u_0(x) = 0$ and $u_1(x) = b$ for any $x \in \overline{\Omega}$. We define $r = \frac{1}{p^+}(\frac{a}{c_1})^{p^+}$. Clearly $r \in (0,1)$, $\phi(u_0) = \psi(u_0) = 0$,

$$\phi(u_1) = \int_{\partial\Omega} \frac{\beta(x)}{p(x)} b^{p(x)} d\sigma \geq \frac{\beta^-}{p^+} b^{p^-} |\partial\Omega| > \frac{1}{p^+} 1 > \frac{1}{p^+}(\frac{a}{c_1})^{p^+} = r,$$

and

$$\psi(u_1) = -\int_\Omega F(x, u_1(x))dx = -\int_\Omega F(x, b)dx < 0.$$

So we have $\phi(u_0) < r < \phi(u_1)$. Then (ii) of lemma 2.5 is verified.

On the other hand, we have

$$-\frac{(\phi(u_1) - r)\psi(u_0) + (r - \phi(u_0))\psi(u_1)}{\phi(u_1) - \phi(u_0)} = -r\frac{\psi(u_1)}{\phi(u_1)}$$

$$= r\frac{\int_\Omega F(x, b)dx}{\int_{\partial\Omega} \frac{\beta(x)}{p(x)} b^{p(x)} d\sigma} > 0.$$

Let $u \in X$ be such that $\phi(u) \leq r < 1$.

Set $I(u) = \int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)} d\sigma$.

Since $\frac{1}{p^+} I(u) \leq \phi(u) \leq r$, for $u \in X$, we obtain

$$I(u) \leq p^+.r = (\frac{a}{c_1})^{p^+} < 1.$$

It follows that $\| u \|_\beta < 1$ by Lemma 2.4. We have

$$\frac{1}{p^+} \| u \|_\beta^{p^+} \leq \frac{1}{p^+} I(u) \leq \phi(u) \leq r.$$

Then

$$|u(x)| \leq c_1 \| u \|_\beta \leq c_1(p^+.r)^{\frac{1}{p^+}} = a \quad \forall u \in X, x \in \overline{\Omega}, \phi(u) \leq r.$$

The above inequality shows that

$$-\inf_{u \in \phi^{-1}(-\infty, r]} \psi(u) = \sup_{u \in \phi^{-1}(-\infty, r]} -\psi(u) \leq \int_\Omega \sup_{0<t<a} F(x, t)dx \leq 0.$$

Then

$$\inf_{u \in \phi^{-1}(-\infty, r]} \psi(u) > \frac{(\phi(u_1) - r)\psi(u_0) + (r - \phi(u_0))\psi(u_1)}{\phi(u_1) - \phi(u_0)}.$$

Which means that condition (iii) in lemma 2.5 is obtained. Since the assumptions of lemma 2.5 are verified, there exist an open interval $\Lambda \subset (0, \infty)$ and a positive constant $\rho > 0$ such that for any $\lambda \in \Lambda$ the equation $\phi'(u) + \lambda\psi'(u) = 0$ has at least three solutions in X whose norms are less than ρ.

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