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On the Existence of Multiple Solutions of a Class of Second-Order Nonlinear Two-Point Boundary Value Problems

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Abstract

A general approach is presented for proving existence of multiple solutions of the second-order nonlinear differential equation

$$u''(x) + f(u(x)) = 0, \quad x \in [0,1],$$

subject to given boundary conditions: $u(0) = B_1, u(1) = B_2$ or $u'(0) = B_1', u(1) = B_2$. The proof is constructive in nature, and could be used for numerical generation of the solution or closed-form analytical solution by introducing some special functions. The only restriction is about $f(u)$, where it is supposed to be differentiable function with continuous derivative. It is proved the problem may admit no solution, may admit unique solution or may admit multiple solutions.

Keywords: Closed-form solution; exact analytical solution; special function; unique solution; multiple solutions.

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1. Introduction

We consider here the challenge of proving existence of unique or multiple solutions to the second order nonlinear two-point boundary value problems of the type

$$\text{Problem1:} \quad u''(x) + f(u(x)) = 0, \tag{1}$$

$$u(0) = B_1, u(1) = B_2. \tag{2}$$

$$\text{Problem2}: \quad u''(x) + f(u(x)) = 0, \tag{3}$$

$$u'(0) = B_1', u(1) = B_2. \tag{4}$$

where B_1, B_1' and B_2 are finite real numbers and the function $f(u)$ is continuous. Our method, to prove existence of multiple solutions, can be applied to generate all branches of solutions as closed-form by introducing some special functions in the resolution process. Our work is motivated by two factors. The first is the frequent occurrence of specific instances of (1)-(2) and (3)-(4) in problems of interest. To illustrate this factor, consider the following set of sample problems:

1. The strongly nonlinear Bratu's problem [1-5]

$$u'' + \exp(u) = 0, \quad x \in (0,1) \tag{5}$$

$$u(0) = u(1) = 0. \tag{6}$$

2. The nonlinear problem arising in heat transfer [5-10]

$$\frac{d^2\theta}{dx^2} - \psi^2\theta^{n+1} = 0, \quad x \in (0,1) \tag{7}$$

$$\frac{d\theta}{dx}(0) = 0, \quad \theta(1) = 1. \tag{8}$$

where ψ is the convective-conductive parameter, θ is temperature profile, and n which is real positive or negative depends on the heat transfer mode.

3. The nonlinear two-point so-called Troesch's boundary value problem [11-13]

$$u'' = \mu \sinh(\mu u), \quad x \in (0,1) \tag{9}$$

$$u(0) = 0, \quad u(1) = 1. \tag{10}$$

where μ is a positive constant

4. The problem of catalytic reaction in a flat particle [14-17]

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \lambda y \exp\left[\frac{\gamma\beta(1-y)}{1+\beta(1-y)}\right] = 0, \tag{11}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(0) = 0, \quad u(1) = 1, \tag{12}$$

where y is dimensionless concentration, x is dimensionless coordinate ($0 \le x \le 1$), λ is the square of Thiele modulus, γ is a dimensionless energy of activation, and β is a dimensionless parameter describing heat evolution. The readers are refered to [18-25] to see more problems.

All these are instances of the problem (1)-(2) or (3)-(4). The second factor motivating our work is the lack of theoretical framework capable of obtaining solutions. The great numbers of methods which involve upper and lower solutions being based on fixed-point theory [26-37] illustrate only the existence of some classes of solution without providing a real procedure to obtain them. Of course this is difficult task and sometimes impossible, however the present paper gives a proof, which is constructive in nature, for existence of multiple solutions of the problems (1)-(2) and (3)-(4), and obtain all branches of solutions (if they exist) at the same time.

2. Existence and uniqueness results for corresponding initial value problem

Consider corresponding initial value problem of (1)-(2) or (3)-(4), which is

$$u''(x) + f(u(x)) = 0, \tag{13}$$

$$u(0) = B_1,\, u'(0) = B_1'. \tag{14}$$

It can be reformulated as a system of two first-order equations by introducing

$$y_1(x) = u(x),\ \ y_2(x) = u'(x). \tag{15}$$

Then equivalent initial value problem for a system of first-order equations is

$$\begin{cases} y_1'(x) = y_2(x) & y_1(0) = B_1 \\ y_2'(x) = -f(y_1(x)) & y_2(0) = B_1'. \end{cases} \tag{16}$$

Definition 2.1 *Consider a two dimensional vector-valued function* $\mathbf{F}$ *defined for* (x, y) *in some set* S *(* x *real,* y *in* $\mathbb{R}^2$ *). We say that* $\mathbf{F}$ *satisfies a Lipschitz condition on* $S \subseteq \mathbb{R}^3$ *if there exists a constant* $K > 0$ *such that*

$$\|\mathbf{F}(x, y) - \mathbf{F}(x, z)\| \le K \|y - z\| \tag{17}$$

for all (x, y), (x, z) in S, where $\|\cdot\|$ denotes L_1-norm defined by $\|y\| = |y_1| + |y_2|$.

Lemma 2.2 *Suppose* $\mathbf{F}$ *is a two dimensional vector-valued function as*

$$\mathbf{F}(x, y) = \left(y_2, -f(y_1)\right)^T, \tag{18}$$

defined for (x, y) on a set S of the form

$$|x| < a,\ \ \|y\| < \infty. \tag{19}$$

If f' exits, and it is continuous on $\mathbb{R}$, then $\mathbf{F}$ satisfies a Lipschitz condition on S.

Proof. *Let* (x, y), (x, z) *be fixed points in* S *, and define the vector-valued function* F *for real* s *,* $0 \le s \le 1$, *by*

$$F(s) = \mathbf{F}(x, z + s(y - z))$$

$$=\begin{pmatrix} z_2+s(y_2-z_2) \\ -f\left(z_1+s(y_1-z_1)\right) \end{pmatrix} \tag{20}$$

This is a well-defined function since the points $(x, z+s(y-z))$ are in S for $0 \le s \le 1$. Clearly $|x|< a$, and if

$$\|y\|<\infty, \|z\|<\infty,$$

then

$$\|z+s(y-z)\| \le (1-s)\|z\| + s\|y\| \le \|z\| + \|y\| < \infty, \tag{21}$$

We now have

$$F^{'}(s)=\left(y_2-z_2, q(s)\right)^T, \tag{22}$$

where

$$q(s)=-(y_1-z_1)f^{'}\left(z_1+s(y_1-z_1)\right)$$

$$. \tag{23}$$

Using continuity of f and $f^{'}$ on $\mathbb{R}$, there exists $M \in \mathbb{R}^+$ such that $|f(t)|< M$ and $|f^{'}(t)|< M$ with $|t|<\infty$, then

$$|q(s)| \le M|y_1-z_1|,$$

therefore

$$\begin{aligned} &\|F^{'}(s)\| = |y_2-z_2| + |q(s)| \\ &\le |y_2-z_2| + M|y_1-z_1| \\ &\le |y_2-z_2|K + |y_1-z_1|K \\ &= K\|y-z\|. \end{aligned} \tag{24}$$

with $K = max\{M,1\}$. Thus, since

$$\mathbf{F}(x,y)-\mathbf{F}(x,z) = F(1)-F(0) = \int_0^1 F^{'}(s)\mathrm{d}s, \tag{25}$$

we have

$$\|\mathbf{F}(x,y)-\mathbf{F}(x,z)\| \le K\|y-z\|, \tag{26}$$

which was to be proved.

Suppose $y_0 = (B_1, B_1^{'})^T$ and consider a successive approximations $\Phi_0(x)$, $\Phi_1(x)$, $\Phi_2(x)$,..., where

$$\Phi_0(x) = y_0,$$

$$\Phi_{k+1}(x) = y_0 + \int_0^x \mathbf{F}(t, \Phi_k(t))\mathrm{d}t, \quad k = 0,1,2,\ldots. \tag{27}$$

Now since $\mathbf{F}(x, y)$ defined by (18) is continuous on

$$S : |x| < a, \|y\| < \infty, \tag{28}$$

it is bounded there, that is, there is a positive constant M such that

$$\|\mathbf{F}(x, y)\| \leq M.$$

On the other hands, Lemma 1 reveals that $\mathbf{F}$ satisfies a Lipschitz condition on S. All these confirm that the hypotheses of the following theorem hold.

Theorem 2.3 *Let $\mathbf{F}(x, y)$ be a real-valued continuous function on S defined by (28) such that*

$$\|\mathbf{F}(x, y)\| \leq M.$$

Suppose there exists a constant $K > 0$ such that

$$\|\mathbf{F}(x, y) - \mathbf{F}(x, z)\| \leq K \|y - z\|, \tag{29}$$

for all (x, y) and (x, z) in S. Then the successive sequence (27) converges to $\Phi(x)$ as the solution of

$$y^{'} = \mathbf{F}(x, y),$$

on the S, which satisfies $\Phi(x_0) = y_0$. Moreover, this solution is unique.

Proof. *Please see the Ref. [38].*

Therefore, we conclude that there exists one, and only one, solution for the initial valve problem (13)-(14). The same results hold for initial value problem corresponding to (1)-(2) and (3)-(4).

3. Existence of multiple solutions for the boundary value problem

Consider the boundary value problems (1)-(2) and (3)-(4) and define function $\mathrm{Ef} : \mathbb{R} \to \mathbb{C}$ as

$$\mathrm{Ef}\,(z; F(\tau); z_1, z_2) = \int_{z_1}^{z} \frac{\mathrm{d}\tau}{\sqrt{z_2^2 - 2F(\tau)}}, \tag{30}$$

where $F(\tau)$ is given continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and more, z_1 and z_2 are constants. We now give two theorems which discuss about multiplicity of solutions of the problems (1)-(2) and (3)-(4).

Theorem 3.1 *Consider the boundary value problem (1)-(2) and suppose that* $f^{'}$ *exits, and it is continuous on* $\mathbb{R}$. *Moreover, define* $F_1(u) = \int_{B_1}^{u} f(t)\mathrm{d}t$. *If there exists the number of* n *real roots for the equation*

$$\mathrm{Ef}\,(B_2; F_1(\tau); B_1, \gamma) = 1, \tag{31}$$

while it is solved respect to γ with $\gamma = u^{'}(0)$, then the problem (1)-(2) admits exactly the number of n solutions.

Proof. *One easily sees equation(1) admits the first integral by multiplying* $u^{'}$ *to the both sides as follows*

$$\frac{1}{2}u^{'2} + F_1(u) = C_1, \tag{32}$$

where C_1 is a constant of integration which should be determined. Since $f(t)$ is continuous then $F_1(u)$ is well-defined. Taking into account $F_1(B_1) = 0$, the boundary conditions at $x = 0$ give for the integration constant C_1 the value

$$C_1 = \frac{1}{2}\gamma^2, \tag{33}$$

So, Eq. (32) is converted to the following

$$F_1(u) + \frac{1}{2}u^{'2} - \frac{1}{2}\gamma^2 = 0. \tag{34}$$

after simplification Eq. (34) becomes

$$\mathrm{d}x = \frac{\mathrm{d}u}{\sqrt{\gamma^2 - 2F_1(u)}}. \tag{35}$$

Using $u(0) = B_1$ and by integration of (35), we can derive the following relation between x and u

$$x = \int_{B_1}^{u} \frac{\mathrm{d}\tau}{\sqrt{\gamma^2 - 2F_1(\tau)}} \tag{36}$$

Now, by definition (30) the above equation becomes

$$x = \mathrm{Ef}\,(u; F_1(\tau); B_1, \gamma). \tag{37}$$

Applying the boundary condition (2) at $x = 1$ i.e. $u(1) = B_2$, yields

$$1 = \mathrm{Ef}\,(B_2; F_1(\tau); B_1, \gamma). \tag{38}$$

Since f' exits, and it is continuous on $\mathbb{R}$, therefore by Lemma 1 and Theorem 1 the corresponding initial value problem has one, and only one solution. Then, we conclude that the number of solutions of the problem (1)-(2) equals to the number of real roots of Eq. (38) when it is solved with respect to γ, and then the proof is completed.

Theorem 3.2 *Consider the boundary value problem (3)-(4) and suppose that* f' *exits, and it is continuous on* $\mathbb{R}$. *Moreover, define* $F_2(u;\delta)=\int_\delta^u f(t)\mathrm{d}t$ *with unknown but fixed* δ. *If there exists the number of* n *real roots for the equation*

$$\mathrm{Ef}\,(B_2;F_2(\tau;\delta);\delta,B_1')=1, \tag{39}$$

while it is solved with respect to δ with $\delta=u(0)$, then the problem (3)-(4) admits exactly the number of n solutions.

Proof. *One easily sees that equation (3) admits the first integral*

$$\frac{1}{2}u'^2+F_2(u;\delta)=D_1, \tag{40}$$

where $F_2(u;\delta)=\int_\delta^u f(t)\mathrm{d}t$ and D_1 is an integral constant. Since $f(t)$ is continuous then $F_2(u;\delta)$ is well-defined. Using $F_2(\delta;\delta)=0$, the boundary conditions at $x=0$ gives for the integration constant D_1 the value

$$D_1=\frac{1}{2}B_1'^2. \tag{41}$$

Eq. (40) can be rewritten as

$$F_2(u;\delta)+\frac{1}{2}u'^2-\frac{1}{2}B_1'^2=0, \tag{42}$$

and after some simplifications Eq. (42) becomes

$$\mathrm{d}x=\frac{\mathrm{d}u}{\sqrt{B_1'^2-2F_2(u;\delta)}}. \tag{43}$$

Using $u(0)=\delta$ and by integration of (43), we can obtain the following relation between x and u

$$x=\int_\delta^u\frac{\mathrm{d}\tau}{\sqrt{B_1'^2-2F_2(\tau;\delta)}}. \tag{44}$$

Now, by definition (30) the above equation becomes

$$x=\mathrm{Ef}\,(u;F_2(\tau;\delta);\delta,B_1'). \tag{45}$$

Applying the boundary condition (4) at $x=1$ i.e. $u(1)=B_2$, yields

$$1=\mathrm{Ef}\,(B_2;F_2(\tau;\delta);\delta,B_1^{'}). \tag{46}$$

Since $f^{'}$ exits, and it is continuous on $\mathbb{R}$, therefore by Lemma 1 and Theorem 1 the corresponding initial value problem has one, and only one solution. Then, we conclude that the number of solutions of the problem (3)-(4) equals to the number of real roots of Eq. (46) when it is solved with respect to δ, and then the proof is completed.

It is worth mentioning here that the theorems 3 and 4 not only give important results about multiplicity of the solutions of the boundary value problems (1)-(2) and (3)-(4) but also provide closed-form solutions for them. In fact, as soon as γ and δ are obtained from (38) and (46), the exact closed-form solutions are presented in the implicit form by

$$x=\mathrm{Ef}\,(u;F_1(\tau);B_1,\gamma), \tag{47}$$

and

$$x=\mathrm{Ef}\,(u;F_2(\tau;\delta);\delta,B_1^{'}). \tag{48}$$

for the problems (1)-(2) and (3)-(4), respectively. The main advantage of the exact analytical solutions (47) and (48) is this fact that today well-performing computer software programs like Mathemtamagica and Maple are available both for symbolic and numerical calculations involving in general the function $\mathrm{Ef}\,(z;F(\tau);A,z_1,z_2)$.

4. Illustrative example

Consider a straight fin of length L with a uniform cross-section area A. The fin surface is exposed to a convective environment at temperature T_a and the local heat transfer coefficient along the fin surface is assumed to exhibit a power-law-type dependence on the local temperature difference between the fin and the ambient fluid as

$$h=(T-T_a)^n \tag{49}$$

where a is a dimensional constant defined by physical properties of the surrounding medium, T is the local temperature on the fin surface, and the exponent n depends on the heat transfer mode. The value of n can vary in a wide range between -4 and 5. For example, the exponent n may take the values $-4,\ -0.25,\ 0,\ 2,$ and $3,$ indicating the fin subject to transition boiling, laminar film boiling or condensation, convection, nucleate boiling, and radiation into free space at zero absolute temperature, respectively. For one-dimensional steady state heat conduction, the equation in terms of dimensionless variables

$$x=\frac{X}{L},\ h=\frac{T-T_a}{T_b-T_a} \tag{50}$$

can be written as

$$\frac{d^2\theta}{dx^2} - N^2\theta^{n+1} = 0 \tag{51}$$

where the axial distance x is measured from the fin tip, T_b is the fin base temperature, and N is the convective- conductive parameter of the fin defined as

$$h = \left(\frac{h_b PL^2}{kA}\right)^{\frac{1}{2}} = \left(\frac{aPL^2}{kA}(T_b - T_a)^n\right)^{\frac{1}{2}} \tag{52}$$

In the above equation h_b, P and k represent the heat transfer coefficient at fin base, the periphery of fin cross-section, and the conductivity of the fin, respectively. For simplicity, assume the fin tip is insulated and the boundary conditions to Eq. (51) can be expressed as

$$\frac{d\theta}{dx}(0) = 0, \quad \theta(1) = 1 \tag{53}$$

The Eq. (51) with boundary conditions (53) has been considered by the first author and Abbasbandy in Ref. [9] and given the exact analytical solutions for all values of n $(-4 \le n \le 5)$ and N by the method discussed in the section 3. Moreover, they have shown the problem may admit no solution, may admit unique solution or may admit dual solutions.

5. Conclusions

There are many problems in engineering and physical sciences which can be modeled by such second-order nonlinear two-point boundary value problems as (1)-(2) and (3)-(4). Therefore, that is very consequential to know that how many solutions these problems admit and to obtain them simultaneously. Based on this regard, a general approach has been presented for proving existence of multiple solutions of these problems. The presented proof, which is constructive in nature, can be used for numerical generation of the solution or closed-form analytical solution by introducing some special functions. The only restriction in our problems is about $f(u)$, where it is supposed to be differentiable function with continuous derivative. It has been proved the problems may admit no solution, may admit unique solution or may admit multiple solutions.

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