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A note on the p-adic gamma function and q-Changhee polynomials

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Abstract

In the present work, we consider the fermionic p-adic q-integral of p-adic gamma function and the derivative of p-adic gamma function by using their Mahler expansions. The relationship between the p-adic gamma function and q-Changhee numbers is obtained. A new representation is given for the p-adic Euler constant. Also, we study on the relationship between q-Changhee polynomials and p-adic Euler constant using the fermionic p-adic q-integral techniques the idea that the q-Changhee polynomial.

Keywords: p-Adic number, p-adic gamma function, the fermionic p-adic q-integral, Mahler coefficients, p-adic Euler constant, q-Changhee Polynomials.

2010 MSC: 11S80, 05A30, 33E50, 05A19, 26C05.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper by $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ we denote the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let q be indeterminate with $|1-q|_p < p^{-\frac{1}{p-1}}$. Recently, the q-calculus (Quantum Calculus) has a great interest and has been studied by many scientists. Many generalizations of special functions with a q-parameter recently were obtained using p-adic q-integral on $\mathbb{Z}_p$ (see, e.g., [1, 8, 9, 11]).

For $f \in C(\mathbb{Z}_p \to \mathbb{C}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) := \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{j=0}^{p^N-1} f(j) q^j (-1)^j, \tag{1.1}$$

where $[x]_{-q} = \frac{1-(-q)^x}{1+q}$ (see [5, 7, 6]). For any $f \in C(\mathbb{Z}_p \to \mathbb{C}_p)$, by (1.1), the relation

$$q^n I_{-q}(f(x+n)) + (-1)^{n-1} I_{-q}(f(x)) = [2]_q \sum_{j=0}^{p^N-1} f(j) q^j (-1)^{n-1-j}, \tag{1.2}$$

where $[x]_q = \frac{1-q^x}{1-q}$ and $n \in \mathbb{N}$ holds.

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The Changhee numbers and polynomials which are derived umbral calculus are defined by Kim et al. as the generating function to be

$$\frac{2}{t+2}(1+t)^{x}=\sum_{n=0}^{\infty}Ch_{n}(x)\frac{t^{n}}{n!}.$$

When $x=0$, $Ch_n(0)=Ch_n$ are called Changhee numbers see [10] for a summary. In [11], Kim et al. defined the degenerate Changhee polynomials and in [12], Kim et al. considered q-Changhee polynomials, $Ch_{n,q}(x)$, which are given by the generating function to be

$$\frac{1+q}{q(1+t)+1}(1+t)^{x}=\sum_{n=0}^{\infty}Ch_{n,q}(x)\frac{t^{n}}{n!}\quad\text{for } |t|_p<p^{-\frac{1}{p-1}}.$$

When $q=1$, $Ch_{n,1}(x)=Ch_n(x)$. When $x=0$, $Ch_{n,q}(0)=Ch_{n,q}$ are called q-Changhee numbers and when $q=1$ and $x=0$, $Ch_{n,1}(0)=Ch_n$. Kim also introduced the q-Changhee numbers of the second kind by

$$\widehat{Ch}_{n,q}=\int_{\mathbb{Z}_p}(-x)_n\,d\mu_{-q}(x),\quad n\geqslant 0,\quad \text{(see [12])},$$

and the q-Changhee numbers of the second kind by

$$\widehat{Ch}_{n,q}(x)=\int_{\mathbb{Z}_p}(-x-y)_n\,d\mu_{-q}(y),\quad n\geqslant 0.$$

The generating function for such polynomials is given by

$$\sum_{n\geqslant 0}\widehat{Ch}_{n,q}(x)\frac{t^{n}}{n!}=\frac{1+q}{1+q+t}(1+t)^{1-x}. \tag{1.3}$$

In [12], Kim et al. obtained the following theorems.

Theorem 1.1. *Let* $(x)_n=x(x-1)\cdots(x-n+1)$. *For* $n\geqslant 0$,

$$\int_{\mathbb{Z}_p}(x)_n\,d\mu_{-q}(x)=Ch_{n,q}.$$

Theorem 1.2. *Let* $(x)_n=x(x-1)\cdots(x-n+1)$. *For* $n\geqslant 0$, *the following relation holds:*

$$\int_{\mathbb{Z}_p}(x+y)_n\,d\mu_{-q}(y)=Ch_{n,q}(x).$$

Theorem 1.3. *For* $n\geqslant 0$,

$$\int_{\mathbb{Z}_p}\binom{x}{n}d\mu_{-q}(x)=\left(\frac{-q}{1+q}\right)^{n}.$$

Theorem 1.4. *For* $n\geqslant 0$

$$\widehat{Ch}_{n,q}=(-1)^{n}\frac{2+q}{(1+q)^{n}}.$$

Theorem 1.5. *For* $n\geqslant 0$

$$\widehat{Ch}_{n,q}(x)=Ch_{n,q}(1-x).$$

Note that $\binom{x}{n}=\frac{(x)_n}{n!}$.

p-Adic numbers introduced by the German mathematician Kurt Hensel (1861–1941), are widely used in mathematics: in number theory, algebraic geometry, representation theory, algebraic and arithmetical dynamics, and cryptography. p-Adic numbers have been used in applied fields with successfully applying in superfield theory of p-adic numbers by Vladimirov and Volovich. In addition, p-adic model of the universe, p-adic quantum theory, p-adic string theory such as areas occurred in physics (for detail see [18, 17]).

In 1975, Morita [14] defined the p-adic gamma function Γ_p by the formula

$$\Gamma_p(x) = \lim_{n\to x}(-1)^n \prod_{\substack{1\leqslant j<n\\(p,j)=1}} j,$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. The p-adic gamma function Γ_p is analytic on $\mathbb{Z}_p$ and satisfies the functional relation

$$\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x), & |x|_p = 1, \\ -\Gamma_p(x), & |x|_p < 1. \end{cases}$$

The p-adic Euler constant γ_p is defined by the formula

$$\gamma_p := -\frac{\Gamma_p'(1)}{\Gamma_p(1)} = \Gamma_p'(1) = -\Gamma_p'(0). \tag{1.4}$$

The p-adic gamma function $\Gamma_p(x)$ has a great interest and has been studied by Barsky (1981) [2], Diamond (1977) [3], Dwork (1983) [4] and others.

For $x \in \mathbb{Z}_p$, the symbol $\binom{x}{n}$ is defined by $\binom{x}{0} = 1$ and

$$\binom{x}{n} := \frac{x(x-1)\dots(x-n+1)}{n!}, \quad (n = 1, 2, \cdots).$$

The functions $x \to \binom{x}{n}$ $(x \in \mathbb{Z}_p, n \in \mathbb{N})$ form an orthonormal base of the space $C(\mathbb{Z}_p \to \mathbb{C}_p)$ with respect the norm $\|\cdot\|_\infty$. This orthonormal base has the following property:

$$\binom{x}{n}' = \sum_{j=0}^{n-1} \frac{(-1)^{n-j-1}}{n-j}\binom{x}{j}, \quad \text{(see, [16, p.168])}. \tag{1.5}$$

In 1958, Mahler introduced an expansion for continuous functions of a p-adic variable using special polynomials as binomial coefficient polynomial [13]. It means that for any $f \in C(\mathbb{Z}_p \to \mathbb{C}_p)$, there exist unique elements $a_0, a_1, \dots$ of $\mathbb{C}_p$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \quad (x \in \mathbb{Z}_p).$$

The base $\left\{\binom{*}{n} : n \in \mathbb{N}\right\}$ is called *Mahler base* of the space $C(\mathbb{Z}_p \to \mathbb{C}_p)$, and the elements $\{a_n : n \in \mathbb{N}\}$ in $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ are called Mahler coefficients of $f \in C(\mathbb{Z}_p \to \mathbb{C}_p)$.

The Mahler expansion of the p-adic gamma function Γ_p and its Mahler coefficients are determined by the following proposition.

Proposition 1.6 ([15, 16]). *Let*

$$\Gamma_p(x+1) = \sum_{n=0}^{\infty} a_n \binom{x}{n}, \qquad (x \in \mathbb{Z}_p),$$

and

$$\exp\left(x+\frac{x^p}{p}\right)\frac{1-x^p}{1-x}=\sum_{n=0}^{\infty} b_n x^n, \qquad (x\in E).$$

Then, $a_n=(-1)^{n+1}n!b_n$ *for all* n*, where* E *is the region of convergence of the power series* $\sum \frac{x^n}{n!}$.

2. Main results

In this paper, we consider p-adic gamma function with the fermionic p-adic q-integral. We derive the relationship between q-Changee polynomials and p-adic gamma function. We obtain the fermionic p-adic q-integral of p-adic gamma function and the derivative of p-adic gamma function. For the p-adic Euler constant. A new representation is obtained. Also, we study on the Changhee polynomials and p-adic Euler constant.

In what follows, we indicate the fermionic p-adic q-integral with Mahler coefficients of p-adic gamma function.

Theorem 2.1. *For* $x\in\mathbb{Z}_p$*, the following equality holds:*

$$\int_{\mathbb{Z}_p}\Gamma_p(x+1)d\mu_{-q}(x)=\sum_{n=0}^{\infty}a_n\frac{Ch_{n,q}}{n!},$$

where a_n *is defined by Proposition 1.6.*

Proof. Let $x\in\mathbb{Z}_p$. We have

$$\int_{\mathbb{Z}_p}\Gamma_p(x+1)d\mu_{-q}(x)=\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}a_n\binom{x}{n}d\mu_{-q}(x)=\sum_{n=0}^{\infty}a_n\int_{\mathbb{Z}_p}\binom{x}{n}d\mu_{-q}(x). \tag{2.1}$$

Note that $\binom{x}{n}=\frac{(x)_n}{n!}$. From Theorem 1.1, we get

$$\int_{\mathbb{Z}_p}\Gamma_p(x+1)d\mu_{-q}(x)=\sum_{n=0}^{\infty}\frac{a_n}{n!}Ch_{n,q}.$$ □

Using Theorem 1.3 we can rewrite (2.1) and we have the following corollary.

Corollary 2.2. *For* $n\in\mathbb{N}$,

$$\int_{\mathbb{Z}_p}\Gamma_p(x+1)d\mu_{-q}(x)=\sum_{n=0}^{\infty}a_n\left(\frac{-q}{1+q}\right)^n,$$

where a_n *is defined by Proposition 1.6.*

Under condition of Proposition 1.6 and using (1.5), derivative of p-adic gamma functions, Γ_p' is obtained as

$$\Gamma_p'(x+1)=\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^{n-j-1}}{n-j}\binom{x}{j}, \tag{2.2}$$

where a_n is defined by Proposition 1.6.

Theorem 2.3. *The relationship between the* q*-Changhee polynomials and the* p*-adic Euler constant is given by*

$$\gamma_p=\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}a_n\frac{(-1)^{n-j}}{(n-j)\,j!}\frac{\left(qCh_{j,q}+Ch_{j,q}(-1)\right)}{[2]_q}.$$

Proof. When $f(x) = \Gamma_p'(x)$ and $n = 1$ in (1.2), we get

$$q\int_{\mathbb{Z}_p} \Gamma_p'(x+1)\, d\mu_{-q}(x) + \int_{\mathbb{Z}_p} \Gamma_p'(x)\, d\mu_{-q}(x) = [2]_q\, \Gamma_p'(0).$$

From (2.2) and (1.4), we can write

$$q\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1}}{n-j} \int_{\mathbb{Z}_p} \binom{x}{j} d\mu_{-q}(x) + \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1}}{n-j} \int_{\mathbb{Z}_p} \binom{x-1}{j} d\mu_{-q}(x) = -[2]_q\, \gamma_p. \tag{2.3}$$

Using Theorem 1.1 and Theorem 1.2 we can rewrite (2.3) as

$$\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1}}{(n-j)\, j!} \left(q\mathrm{Ch}_{j,q} + \mathrm{Ch}_{j,q}(-1)\right) = -[2]_q\, \gamma_p.$$

□

Theorem 2.4. *The* p*-adic Euler constant has the expansion*

$$\gamma_p = \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^n}{(n-j)\,[2]_q} \left(\frac{q^{j+1}}{(1+q)^j} + \sum_{i=0}^{j} \frac{1}{(1+q)^i (j-i)!}\right),$$

where a_n *is defined by Proposition 1.6.*

Proof. Firstly, we compute $\mathrm{Ch}_{n,q}(-1)$. From Theorem 1.5, we have $\widehat{\mathrm{Ch}}_{n,q}(2) = \mathrm{Ch}_{n,q}(1-2)$. From (1.3)

$$\sum_{n\geqslant 0} \widehat{\mathrm{Ch}}_{n,q}(x) \frac{t^n}{n!} = \frac{1+q}{1+q+t} \sum_{n\geqslant 0} \binom{1-x}{n} t^n = \sum_{n\geqslant 0} \frac{(-1)^n t^n}{(1+q)^n} \sum_{n\geqslant 0} \binom{1-x}{n} t^n,$$

or

$$\sum_{n\geqslant 0} \widehat{\mathrm{Ch}}_{n,q}(x) \frac{t^n}{n!} = \sum_{n\geqslant 0}\sum_{i=0}^{n} \left(\frac{-1}{1+q}\right)^i \binom{1-x}{n-i} t^n.$$

Then, we have

$$\widehat{\mathrm{Ch}}_{n,q}(x) = n! \sum_{i=0}^{n} \left(\frac{-1}{1+q}\right)^i \binom{1-x}{n-i}.$$

Note that $x^n = (-1)^n (-x)_n$.

$$\widehat{\mathrm{Ch}}_{n,q}(x) = n! \sum_{i=0}^{n} \frac{(-1)^i}{(1+q)^i} \frac{(1-x)_{n-i}}{(n-i)!},$$

or

$$\widehat{\mathrm{Ch}}_{n,q}(x) = n! \sum_{i=0}^{n} \frac{(-1)^i}{(1+q)^i} \frac{(-1+x)^{n-i} (-1)^{n-i}}{(n-i)!},$$

or

$$\mathrm{Ch}_{n,q}(1-x) = \widehat{\mathrm{Ch}}_{n,q}(x) = n! \sum_{i=0}^{n} \frac{(-1)^n}{(1+q)^i} \frac{(-1+x)^{n-i}}{(n-i)!}. \tag{2.4}$$

When $x = 2$ in (2.4) we have $\mathrm{Ch}_{n,q}(-1) = n! \sum_{i=0}^{n} \frac{(-1)^n}{(1+q)^i (n-i)!}$. Using Theorem 1.3, Theorem 1.1 and value of $\mathrm{Ch}_{n,q}(-1)$ we can rewrite Theorem 2.3 as the following

$$\gamma_p = \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j}}{(n-j)\, j!} \frac{\left(q\left(\frac{-q}{1+q}\right)^j j! + j! \sum_{i=0}^{j} \frac{(-1)^j}{(1+q)^i (j-i)!}\right)}{[2]_q}.$$

The proof of theorem is completed with a little calculations. □

Theorem 2.5. *The following relation holds*

$$\int_{\mathbb{Z}_p} \Gamma_p(x+s)d\mu_{-q}(x) = \sum_{n=0}^{\infty} a_n \frac{Ch_{n,q}(s-1)}{n!},$$

where a_n is defined by Proposition 1.6.

Proof. Let $x \in \mathbb{Z}_p$. We have

$$\int_{\mathbb{Z}_p} \Gamma_p(x+s)d\mu_{-q}(x) = \sum_{n=0}^{\infty} a_n \int_{\mathbb{Z}_p} \frac{(x+s-1)_n}{n!} d\mu_{-q}(x).$$

By using Theorem 1.2 we can write

$$\int_{\mathbb{Z}_p} \Gamma_p(x+s)d\mu_{-q}(x) = \sum_{n=0}^{\infty} a_n \frac{Ch_{n,q}(s-1)}{n!}.$$

□

Theorem 2.6. *For $x, s \in \mathbb{Z}_p$, we have*

$$\int_{\mathbb{Z}_p} \Gamma_p'(x+s)\, d\mu_{-q}(x) = \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1} Ch_{j,q}(s-1)}{(n-j)\, j!}.$$

Proof. Let $x, s \in \mathbb{Z}_p$. From (2.2), we have

$$\int_{\mathbb{Z}_p} \Gamma_p'(x+s)\, d\mu_{-q}(x) = \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1}}{n-j} \int_{\mathbb{Z}_p} \binom{x+s-1}{j} d\mu_{-q}(x).$$

By using Theorem 1.2 we can write

$$\int_{\mathbb{Z}_p} \Gamma_p'(x+s)\, d\mu_{-q}(x) = \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1} Ch_{j,q}(s-1)}{(n-j)\, j!}.$$

□

In the case $s = 1$ in Theorem 2.6 we obtain the following conclusion.

Corollary 2.7. *For $x \in \mathbb{Z}_p$, we have*

$$\int_{\mathbb{Z}_p} \Gamma_p'(x)\, d\mu_{-q}(x) = \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} a_n \frac{(-1)^{n-j-1} Ch_{j,q}}{(n-j)\, j!},$$

where $Ch_{n,q}$ are q-Changhee numbers.

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References

[1] S. Araci, D. Erdal, J. J. Seo, *A study on the fermionic p-adic q-integral representation on $\mathbb{Z}_p$ associated with weighted q-Bernstein and q-Genocchi polynomials*, Abstr. Appl. Anal., **2011** (2011), 10 pages. 1

[2] D. Barsky, *On Morita's p-adic gamma function*, Math. Proc. Cambridge Philos. Soc., **89** (1981), 23–27. 1

[3] J. Diamond, *The p-adic log gamma function and p-adic Euler constants*, Trans. Amer. Math. Soc., **233** (1977), 321–337. 1

[4] B. Dwork, *A note on the p-adic gamma function*, Study group on ultrametric analysis, 9th year: 1981/82, Marseille, (1982), Inst. Henri Poincaré, Paris, (1983), 10 pages. 1

[5] T. Kim, q-*Volkenborn integration*, Russ. J. Math. Phys., **9** (2002), 288–299. 1

[6] T. Kim, q-*Euler numbers and polynomials associated with* p-*adic* q-*integrals*, J. Nonlinear Math. Phys., **14** (2007), 15–27. 1

[7] T. Kim, *Some identities on the* q-*Euler polynomials of higher order and* q-*Stirling numbers by the fermionic* p-*adic integral on* $\mathbb{Z}_p$, Russ. J. Math. Phys., **16** (2009), 484–491. 1

[8] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic* p-*adic invariant integral on* $\mathbb{Z}_p$, Russ. J. Math. Phys., **16** (2009), 93–96. 1

[9] D. S. Kim, T. Kim, *Daehee numbers and polynomials*, Appl. Math. Sci. (Ruse), **7** (2013), 5969–5976. 1

[10] T. Kim, D. S. Kim, T. Mansour, S.-H. Rim, M. Schork, *Umbral calculus and Sheffer sequences of polynomials*, J. Math. Phys., **54** (2013), 15 pages. 1

[11] T. Kim, H.-I. Kwon, J. J. Seo, *Degenerate* q-*Changhee polynomials*, J. Nonlinear Sci. Appl., **9** (2016), 2389–2393. 1, 1

[12] T. Kim, T. Mansour, S.-H. Rim, J. J. Seo, *A note on* q-*Changhee Polynomials and Numbers*, Adv. Studies Theor. Phys., **8** (2014), 35–41. 1, 1

[13] K. Mahler, *An interpolation series for continuous functions of a* p-*adic variable*, J. Reine Angew. Math., **199** (1958), 23–34. 1

[14] Y. Morita, *A* p-*adic analogue of the* Γ-*function*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **22** (1975), 255–266. 1

[15] A. M. Robert, *A course in* p-*adic analysis*, Graduate Texts in Mathematics, Springer-Verlag, New York, (2000). 1.6

[16] W. H. Schikhof, *Ultrametric calculus*, An introduction to p-adic analysis, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, (1984). 1.5, 1.6

[17] V. S. Vladimirov, I. V. Volovich, *Superanalysis*, I, Differential calculus, (Russian) Teoret. Mat. Fiz., **59** (1984), 3–27. 1

[18] I. V. Volovich, *Number theory as the ultimate physical theory*, p-Adic Numbers Ultrametric Anal. Appl., **2** (2010), 77–87. 1