Journal of mathematics and computer science 10 (2014), 247-252

Contents list available at JMCS

Journal of Mathematics and Computer Science

Journal Homepage: www.tjmcs.com

Detour Distance And Self Centered Graphs

K. R. Sandeep Narayan[1,*], M. S. Sunitha[2]

[1]*K R Sandeep Narayan, Assistant Professor, Dept of Mathematics, Jyothi Engineering College, Cheruthuruthy ,Thrissur, Kerala, India 679531*

[2]*M S Sunitha, Associate Professor, Dept of Mathematics, National Institute of Technology, Calicut*

[*] (Corresponding Author) **appunaran@yahoo.com,**

sunitha@nitc.ac.in

Article history:
Received March 2014
Accepted April 2014
Available online May 2014

Abstract

This paper evaluates the detour distance of a graph and associated problems. We study about the detour eccentricity and average detour eccentricity of graphs and derive some of the properties that relates self centered graphs and detour distance. A characterization of tree is also discussed.

Keywords: Graphs, Detour distance in graphs, Centre of graphs, Detour eccentricity of graphs, Average detour eccentricity, Self centered graphs.

1. Introduction

This paper deals with different types of distances in graphs, foremostamong them being the geodesic distance. The second type of distance is the detour distance. In this paper we study and analyze the self-centerednessproperty of graphs using the tool - Detour distance.

We extend the concept of average distance, average eccentricity and manyrelated ideas to detour distance.

A graph is a pair *G:(V, E)* where *V* is a finite non empty set and *E* is asymmetric binary relation on *V*.The following definitions are taken from the book by Harary [3].

In a graph$G = (V, E)$, V (or (G)) and E (orE (G)) denote the vertex set and the edge

Set of G, respectively. A graph $G = (V, E)$ is trivial, if it has only one vertex,i.e., $|V(G)| = 1$; otherwise G is nontrivial. The number of edges incidentwith a vertex u is called degree of a vertex and is denoted by $\deg_G(u)$.

K_n denotes the complete graph on n verticesand it is that graphin which there exists an edge between each pair of vertices u, v in G. For a set S of V (G), G [S] is the sub graph induced by S. A connected acyclic graph is called a tree. Complement of G is the graph $G^C = (V,E^C)$, where $E^C = \{e:e \notin E (G)\}$. A graph G is said to be regular, if every vertex of Ghas the same degree. If this degree is equal to***r***, then G is$r-$ *regular*orregular of degree ***r***.

If G is a graph and u, v are two vertices of G, length of a shortest path between$u$ and v in G is called the geodesic distance between u and v and is denoted by***d (u, v)***. The eccentricity of the vertex v is denoted by $e (v)$ and is defined as $e (v) = Max [d (u, v)/u \in G]$. If a graph G is disconnected, then $e (v) = \infty$ for all vertices $v \in G$.

Radius of G is defined as $r (G) = Min [e_g(v)/v \in G]$.

Diameter of G is defined as $d (G) = Max [e (v)/v \in G]$

Average eccentricity of G is $avec(G) = \frac{1}{n}[e (v_1) + e(v_2) + \cdots + e(v_n)]$ where$v_1, v_2, . . . v_n$are vertices of G. If $e (v) = r (G)$ then v is called a central vertexof G and the graph induced by all central vertices of G is called Centre of G and is denoted by $C (G)$. A vertex v of G is called an eccentric vertex of G [5] if there exists a vertex u in G such that $d (u, v) = e (u)$. This means that if the vertex v is farthest from another vertex u then v is an eccentric vertex of u(as well as G) and is denoted by $u^* = v$. If $G=C (G)$ then it is called self-centered graph. Medha and Chithra [4] studied about edge extension for cycles, with respect to the self-centeredness (of cycles). After an edge set is added to a self-centered graph the resultant graph is also a self-centered graph A graph is eccentric if all vertices of G are eccentric vertices.

Consider a subset S of $V (G)$, then $d(v,S) = Min[d(u,v)/u \in S$.

Distance of S, $\sigma(S) = \sum_{v\in V} d(v,S)$

It has been proved by Dankelmann [2] that for a graph $G avec (G \leq \frac{1}{n}\left[\sigma\left(C(G)\right) + r(G)\right]$and equality holds in the case of trees.

2. Detour Distance and Average detour eccentricity

The concept of detour distance and related properties are discussed in [1]. If u, v are two vertices in the graph G the detour distance between these vertices denoted by $D (u, v)$ is the length of a longest $u - v$- path in G [1]. Note that there exist graphs in which $D (u, v) = d (u, v)\ \forall u, v \in V$. The detour eccentricity of the vertex v is denoted by $e_D(v)$ and is defined as $e_D(v) = Max\ [D\ (u,v)/u \in G$. If a graph G is disconnected, then $e_D(v) = \infty\ \forall\ v \in G$.

Detour radius of G is defined as$r_D(G) = Min\ [e_D(v)/v \in G]$.

Detour diameter of G is defined as $d_D(G) = Max\ [e_D(v)/v\ \in G]$.

Average detour eccentricity of G is$Davec(G) = \frac{1}{n}[e_D(v_1) + e_D(v_2) + \cdots + e_D(v_n)]$where $v_1, v_2 \ldots v_n$are verticesof G. If $e_D(v) = r_D(G)$then v is called a detour central vertex of G and Thesub graph induced by all detour central vertices of G is called detour center of G and is denoted by $C_D (G)$. A vertex v of G is called a detour eccentric vertex of u(and that of G) if there exists a vertex u in G such that$D(u,v) = e_D(u)$. If $G = C_D (G)$ then it is called a detour self-centered graph. Consider a subset S of $V (G)$, then $d(v,S) = Min[d(u,v)/u \in S]$.

Distance of S, $\sigma_D(S) = \sum_{v\in V} D(v,S)$.

In the following results the relation between detour distance and geodesicdistance in a cycle is discussed. A relation connecting average detour eccentricity and detour radius is derived. We also characterize trees in terms ofgeodesic distance and detour distance.

3. Results

Proposition: 1
Let *G: (V, E)* be a cycle on ***n*** vertices, then$D(u, v) = n - d(u, v) \forall u, v \in V$.

Proof
Since *G* is cycle on ***n*** vertices, it has ***n*** edges.
Suppose *u*, *v* are two vertices in *G*. Since *G* is cycle there exists exactly two paths $P_1$and P_2 between *u*, *v*. If one is the shortest path the other one will be the longest path. Longest path will be a detour path.Also *G* being a cycle, *Length* $P_1 = n$ - *Length* P_2. If $P_1$is the longest path then clearly P_2will be shorter and then$D(u, v) = Length\ P_1 = n - Length\ P_2 = n - d(u, v)$.

Proposition:2
Let *G: (V, E)* be a cycle with n vertices, then *G* is detour self-centered and the detour eccentricity $e_D(v) = n - 1 \forall v \in V$.

Proof
Suppose the vertices of *G* are $v_1, v_2, \ldots v_n$where $v_n = v_1$.
Let $v_i$be an arbitrary vertex of G. Then the two detour eccentric vertices of $v_i$are v_{i-1} and v_{i+1}.
From Proposition 1 we get $D(v_i, v_{i-1}) = n - d(v_i, v_{i-1})$ and $D(v_i, v_{i+1}) = n - d(v_i, v_{i+1})$
This gives $D(v_i, v_{i-1}) = D(v_i, v_{i+1}) = n-1$. Thus we see that $e_D(v_i) = n - 1$. Since is arbitrary we get$e_D(v) = n - 1 \forall v \in V$.
∴ G is detour self-centered.

Corollary
LetGbeacyclewith$n \geq 3$vertices,then$e(v) < e_D(v) \forall v \in V$ and $r(G) < r_D(G)$.

Proof
G is a cycle with ***n*** vertices. $e(v) = \lfloor \frac{n}{2} \rfloor$ from [3] and $e_D(v) = n - 1 \forall v \in V$.
As$n \geq 3$,$\lfloor \frac{n}{2} \rfloor < n-1$. Hence$e(v) < e_D(v) \forall v \in V$. Since*G*isself-centered as well as detour self-centered we get $r(G) < r_D(G)$.

Proposition:3
Let *G: (V, E)* be a connected graph on ***n*** vertices, and let ***k*** be the number of pendant vertices of *G* then
$D(u, v) \leq n - k$ if *u*, *v* are not pendant
$D(u, v) \leq n - k + 1$ if either *u* or *v* is pendant
$D(u, v) \leq n - k + 2$ if both *u* and *v* are pendant

Proof

Given *G: (V, E)* is a graph with ***n*** vertices and ***k*** pendant vertices. Suppose *u,v* are not pendant and $v_1, v_2, \ldots, v_k$ be the pendant vertices. Any $u - v$ detour will exclude these ***k*** vertices and the edges incident with them. Hence the maximum length of any $u - v$ detour will be less than or equal to $n-k$. Hence $D(u,v) \leq n-k$.
Suppose one among *u* or *v* (say u) be pendant. Then each $u - v$ detour will start at *u*. These detours will pass through the edge adjacent with *u*. They will not pass through the remaining $k - 1$ vertices and edges incident on them. Thus each of these detours will have a maximum of $n - k + 1$ edges. Hence $D(u, v) \leq n - k + 1$.

Suppose both u and v are pendant. Then each u − v detour will start at u and end at v. These detours will pass through the edges adjacent with both *u* and *v*. They will not pass through the remaining $k - 2$ pendant vertices.
Hence $D(u, v) \leq n - k + 2$.

Theorem: 1
Let *G* be a connected graph having n vertices where $n \geq 2$. Then *G* is a tree if and only if $D(u, v) = D(u, v)$ for all vertices *u, v* in *G*.

Proof
If G is a tree then there exist unique path between any two vertices of G and the longest and shortest path will be same. Hence $d(u, v) = D(u, v)$
Conversely assume that *G* is a graph such that $d(u, v) = D(u, v)$ for all vertices *u, v* in *G*. We have to prove that *G* is a tree. For *G* with *2* vertices the result is trivial.
Let $n > 3$ and suppose *G* is not a tree. Then it must contain a cycle *C* having ***m*** vertices where $3 \leq m \leq n$. Suppose *u, v* be two adjacent vertices in *G* which are in *C*. Then $d(u, v) = 1$.
As *G* does not contain multiple edges, $D(u, v) = m-1$ where $m-1 > 1$. This is a contradiction because *G* is a graph such that $d(u, v) = D(u, v)$.
∴*G* is a tree.

Corollary
If *G* is a tree then the detour center of *G* is K_1 or K_2.

Theorem: 2
Let G: (V, E) be a connected graph with n vertices. Then $Davec(G) \leq r_D + \frac{1}{n}[\sigma_D(C(G))]$

Proof
Let $u_1, u_2, \ldots u_n$ be the vertices of *G*. Some of the $u_i - u_j$ detour will pass through the detour center and some may not. Let *P* be a $u_i - u_j$ detour in *G* passing through a detour central vertex u_k such that $D(u_i, u_k)$ is minimum so that $D(u_i, u_k) \leq r_D(G)$.
Then $e_D(u_i) \leq r_D(G) + D(C_D(G), u_i)$. Suppose $u^*_1, u^*_2, \ldots, u^*_n$ be the eccentric vertices of $u_1, u_2 \ldots u_n$ respectively. Then
$e_D(u_i) \leq r_D(G) + D(C_D(G), u^*_i)$.

$$Davec(G) = \frac{1}{n}[e_D(u_1) + e_D(u_2) + \cdots + e_D(u_n)]$$

$$Davec(G) \leq \frac{1}{n}[r_D(G) + D(C_D), u_1{}^*) + \cdots + r_D(G) + D(C_D), u_n{}^*]$$
$$Davec(G) \leq r_D(G) + \frac{1}{n}[\sigma_D(C_D(G)]$$
$$\therefore Davec(G) \leq r_D + \frac{1}{n}[\sigma_D(C(G))]$$

Corollary
If *G: (V, E)*isdetour self centeredor a tree then, $Davec(G) = r_D(G) + \frac{1}{n}[\sigma_D(C_D(G)]$

Proof
If *G* is a tree then *d (u,v) = D(u,v)* $\forall u,v \in V$ and the equality follows asthere exist exactly one path between any two vertices of G [3].
Suppose *G* be a detour Self Centered graph. Then $\frac{1}{n}[\sigma_D(C(G))] = 0$
And$e_D(v) = r_D(G)\ \forall v \in V$. Hence *Davec (G)* = $r_D(G)$ and equality holds.

Theorem: 3
Let G : (V, E) be a connected graph having n vertices. Then G is detourself-centered if and only if *Davec(G)* = $r_D(G)$.

Proof
Since G is detour self-centered we have$e_D(v) = r_D(G) \forall v \in V$. Then bydefinition
$$Davec(G) = \frac{1}{n}[e_D(v_1) + e_D(v_2) \ldots + e_D(v_n)] = \frac{1}{n}[n.r_D(G)] = r_D(G)$$

Thus *Davec(G)* = $r_D(G)$.

Conversely assume that *Davec(G)* = $r_D(G)$. We prove that *G* is detour self-centered.
We have$Davec(G) \leq r_D + \frac{1}{n}[\sigma_D(C(G))]$. Since *Davec(G)* = $r_D(G)$ we get, $0 \leq \frac{1}{n}[\sigma(C_D(G)]$.

Case- 1
$$0 = \frac{1}{n}[\sigma(C_D(G)]$$
$$\sigma[C_D(G)] = 0 \Rightarrow D(v_1, (C_D(G)) + D(v_2, (C_D(G)) + \cdots + D(v_n, (C_D(G)) = 0$$
This implies $D(v_i, (C_D(G)) = 0 \forall\ i = 1,2, \ldots, n.\ ie\ v_i \in C_D(G) \forall i.$
Therefore *G* is detour Self Centered.

Case- 2
Suppose $0 < \frac{1}{n}[\sigma_D(C_D(G)]$. Then $\sigma_D[(C_D(G)] > 0]$.
$\Rightarrow D(v_1, (C_D(G)) + D(v_2, (C_D(G)) + \cdots + D(v_n, (C_D(G)) > 0$
Among the vertices $v_1, v_2, \ldots, v_n$, let $v_1, v_2, \ldots, v_k$ be the vertices such that $D(v_i, (C_D(G)) = 0, i = 1,2, \ldots, k$.
Let $v_{k+1}, v_{k+2}, \ldots v_n$ be the vertices with $D(v_i, (C_D(G)) > 0, i = k+1, k+2, \ldots, n$
As $D[(v_i, (C_D(G))] = 0, i = 1,2, \ldots, k$ clearly $v_1, v_2, \ldots, v_k \in C_D(G)$.

As $D[(v_i, (C_D(G))] > 0, i = k+1, k+2, .., n$ clearly $v_{k+1}, v_{k+2}, \dots v_n \notin C_D(G)$.

$\therefore e_D(v_i) > r_D(G), i = k+1, k+2, \dots, n$ and $e_D(v_i) = r_D(G), i = 1, 2, 3, \dots k$.
Then clearly $Davec(G) = \frac{1}{n}[e_D(v_1) + e_D(v_2) \dots + e_D(v_n)] > r_D(G)$.

This is a contradiction to the assumption that $Davec(G) = r_D(G)$

Therefore there does not exist $v_{k+1}, v_{k+2}, \dots v_n$ such that $D(v_i, C_D(G)) > 0$ for $i=k+1, k+2, \dots,, n$.

$\Rightarrow D[v_i, C_D(G)] = 0 \ for \ all \ i$

$\therefore v_i \in C_D(G) for \ all \ i = 1,2, \dots, n$

Hence G is detour self-centered.

References

[1] Chartrand and P. Zhang, Distance in graphs - taking the long view, AKCE Journal Of Graphs and Combinatorics 1 (2004), pp. 1–13.

[2] Dankelmann, P., W. Goddard and C. S. Swart, The average eccentricity of a graph and its subgraphs, UtilitasMathematica 65 (2004), pp. 41–51.

[3] Harary, F., “Graph Theory,” Addison Wesley, 1969.

[4] Huilgol, M. I. and C. Ramaprakash, Cyclic edge extensions - self centered graphs, Journal of Mathematics and Computer Science 10 (2014), pp. 131–137.

[5] Tong, L.-D. and H.-T. Wang, Eccentric spectrum of a graph, Taiwanese Journal Of Mathematics 12 (2008), pp. 969–977.