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Stability of switched stochastic nonlinear systems by an improved average dwell time method

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Abstract

This paper investigates the stability of switched stochastic continuous-time nonlinear systems in two cases:

(1) all subsystems are global asymptotically exponentially stable in the mean (GASiM);

(2) both GASiM subsystems and unstable subsystems coexist, and proposes a number of new results on the stability analysis.

Firstly, an improved average dwell time (ADT) method is established for the stability of switched stochastic system by extending our previous dwell time method. Especially, an improved mode-dependent average dwell time (MDADT) method for the switched stochastic systems whose subsystems are quadratically stable in the mean is also obtained. Secondly, based on the improved ADT and MDADT methods, several new results on the stability analysis are provided. It should be pointed out that the obtained results have two advantages over the existing results, one is the conditions of the improved ADT method are simplified, the other is that the obtained lower bound of ADT (τ_a^*) is also smaller than those obtained by other methods. Finally, two illustrative examples with simulation are given to show the correctness and the effectiveness of the proposed results.

Keywords: Switched stochastic nonlinear system, stability in the mean, unstable subsystems, average dwell time, mode-dependent dwell time.

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1. Introduction

Switched systems arise in various fields of real life world, such as communication networks, manufacturing, auto pilot design, automotive engine control, computer synchronization, traffic control, and chemical processes, and so on. In the past two decades, increasing attention has been paid to the analysis and synthesis of switched systems due to their significance in both theory and applications, and many significant results have been obtained for the analysis and control design of switched systems, see [1–4, 8–11, 13, 14, 25–28] and references therein.

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For the switched system, there are several important problems to be investigated, such as stability analysis, stabilization and control design, etc. Stability analysis has been a hot issue since the switched system came into being, and many efforts and work have been done. Generally speaking, there are two famous methods to the stability analysis of switched systems, i.e., common Lyapunov function (CLF) method [13, 14], and multiple Lyapunov functions (MLF) method [1]. About the CLF method, it is usually very difficult to determine whether all the subsystems share a CLF or not, even for the switched linear systems. About the MLF method, it is well known that the switched system is globally stable for any switching signal if the time between consecutive switching (i.e., dwell time) is sufficiently large when all subsystems are stable. Also, some results have appeared in recent works to compute lower bounds of the dwell time [7, 12, 15, 24, 30]. However, it seems very hard to obtain the minimal dwell time for a given switched system, even for the switched linear systems. Notice that the ADT scheme characterizes a large class of stable switching signals than the dwell time scheme, and its extreme case is the arbitrary switching. Therefore, the ADT method is very important not only in theory, but also in practice, and considerable attention has been paid to take advantage of the ADT method to investigate the stability and stabilization problems for linear and nonlinear systems. As pointed out in [12], the ADT switching is a class of restricted switching signals which means that the number of switches in a finite time interval is bounded and the average dwell time between consecutive switching is not less than a constant τ_a^*, where τ_a^* is called a lower bound of ADT. In practice, we cannot obtain the exact minimum ADT which can guarantee the stability for an switched system, but we can do our best to estimate the lower bound of ADT by using particular method.

It should be pointed out that most of existing results are concerned with the stability of the switched linear or nonlinear systems with stable subsystems, see [16, 18, 20–22] and the references therein. Although the results in [24] dealt with the switched linear system with stable and unstable subsystems, the ADT method used in [24] has the following two disadvantages. One is the ADT method is only applied to the switched linear system, the other is the given lower bound (i.e., τ_a^*) has much conservativeness. Therefore, it is necessary to develop an improved ADT method for the stability analysis of switched nonlinear systems with stable and unstable subsystems. On the other hand, as the authors in [29] pointed out the lower bound of ADT τ_a^* is independent of the system modes, thus it is probably still not anticipated. In order to solve this problem, they proposed a MDADT method to reduce the conservative property of the ADT method.

In practical control problems, the involved systems are usually nonlinear and inevitably subjected to more or less stochastic disturbances. Usually, this implies that the parameter changes of systems possibly takes place by some abrupt and continual manner on some time instant sequence tending to infinity. Aiming at such cases, the models of so-called switched stochastic nonlinear systems are introduced and extensively studied. In this paper, we will extend our previous results in [5, 6] to the stability of switched stochastic nonlinear system in both cases: one is all subsystems are GASiM, the other is both GASiM and unstable subsystems coexist. Firstly, we develop an improved ADT method for the stability of such switched stochastic nonlinear system, and propose an improved MDADT method for a class of switched stochastic nonlinear systems which have quadratic stability property. Secondly, based on the proposed methods, some new stability results for the switched stochastic nonlinear system are established, which have some advantages over the existing results. Finally, two examples are given to illustrate the main results of this paper, which shows that the proposed methods in this paper are effective in the stability analysis of switched stochastic nonlinear systems.

The rest of the paper is organized as follows. Section 2 presents the problem formulation of this paper, and Section 3 gives the main results. In Section 4, illustrative examples are given to support our new results, which is followed by the conclusion in Section 5.

2. Notations and preliminary results

Throughout this paper, $\mathbb{R}_+$ denotes the set of all nonnegative real numbers, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, n-dimensional real space and $n \times m$ dimensional real matrix space. All the vectors are column vectors unless otherwise specified. The transpose of vectors and matrices are denoted by superscript T. $\mathcal{C}^i$ denotes all the i-th continuous differential functions. A function $\varphi(u)$ is said to belong to the class $\mathcal{K}$ if $\varphi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, $\varphi(0) = 0$ and $\varphi(u)$ is strictly increasing in u. $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded.

Consider the following switched stochastic nonlinear systems:

$$dx = f_{\sigma(t)}(x)dt + g_{\sigma(t)}(x)dw, \quad t \geqslant 0, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the system state, w is an r-dimensional independent standard Wiener process (or Brownian motion), $\sigma(\cdot) : [0, \infty) \to \mathcal{I} = \{1, 2, \cdots, N\}$ is the switching law and is right-hand continuous and piecewise constant on t. For every $i \in \mathcal{I}$, $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times r}$ is continuous, uniformly locally Lipschitz and satisfies $f_i(0, 0) = g_i(0, 0) = 0$, initial data $x_0 \in \mathbb{R}^n$.

Consider the following switched stochastic linear systems:

$$dx = A_{\sigma(t)}x(t)dt + C_{\sigma(t)}x(t)dw(t), \tag{2.2}$$

where $x \in \mathbb{R}^n$ is the state, w is an r-dimensional standard Brownian motion, A, C are constant matrices with appropriate dimensions, $\sigma(.)$ is the switching law as in (2.1).

For an arbitrary switching path $\sigma(t) = i_m \in \mathcal{I}$ $(t \in [t_m, t_{m+1}), m = 0, 1, 2, 3, \cdots)$, $\{t_m\}_{m=0}^{+\infty}$ is called the switching time sequence, which is assumed to satisfy

$$t_0 < t_1 < t_2 < \cdots < t_m < \cdots < +\infty.$$

Let $\tau_k = t_k - t_{k-1}$denote the dwell time, $k = 1, 2, \cdots$.

For the development of this paper, we introduce a notation used in [7], i.e., for any switching signal $\sigma(t)$ and any $t \geqslant \tau$, let $N_\sigma(\tau, t)$ denote the number of switching of $\sigma(t)$ over the interval $[\tau, t)$. For given $N_0, \tau_a > 0$, let $S_a[\tau_a, N_0]$ denote the set of all switching signals satisfying

$$N_\sigma(\tau, t) \leqslant N_0 + \frac{t - \tau}{\tau_a}, \tag{2.3}$$

where τ_a is called average dwell time and N_0 denotes the chatter bound.

Next, we present some definitions.

Definition 2.1. For any given $V(x) \in C^2(\mathbb{R}^n; \mathbb{R}_+)$, associated with the switched stochastic nonlinear system (2.1), we define the differential operator $\mathcal{L}$ to every $i \in \mathcal{I}$ as follows:

$$\mathcal{L}V = \frac{\partial V}{\partial x} f_i(x) + \frac{1}{2}\mathrm{Tr}\left\{ g_i^T(x) \frac{\partial^2 V}{\partial x^2} g_i(x) \right\}.$$

Similarly, for the above $V(x)$, associated with the switched stochastic linear system (2.2), we define the differential operator $\mathcal{L}$ to every $i \in \mathcal{I}$ as follows:

$$\mathcal{L}V = x^T \left\{ A_i^T P_i + P_i A_i + \mathrm{Tr}(C_i P_i C_i^T) \right\} x.$$

Then, we recall the definitions of quadratic stability and mode-dependent average dwell time.

Definition 2.2 ([19, 23])**.** The nonlinear system (2.1) with $N = 1$ is called quadratic stability in the mean, if there is a Lyapunov function $V = x^T P x$ which ensure this system stable in the mean where $P > 0$.

Definition 2.3 ([29])**.** For a switching signal $\sigma(t)$ and any $T \geqslant t \geqslant 0$, let $N_{\sigma i}(t, T)$ be the switching numbers that the i-th subsystem is activated over the interval $[t, T)$ and $T_i(t, T)$ denote the total running time of the i-th subsystem over the interval $[t, T)$, $i \in \mathcal{I}$. We say that $\sigma(t)$ has a MDADT τ_{ai} if there exist positive numbers N_{0i} (we call N_{0i} the mode-dependent chatter bounds here) and τ_{ai} such that

$$N_{\sigma i}(t, T) \leqslant N_{0i} + \frac{T_i(t, T)}{\tau_{ai}}, \quad \forall\, T \geqslant t \geqslant 0.$$

The objective of this paper is to investigate the stability of switched stochastic nonlinear systems in two cases: all subsystems are GASiM, and both GASiM subsystems and unstable subsystems coexist.

3. Main results

3.1. All subsystems of the switched system are GASiM

Firstly, we investigate the stability of switched stochastic nonlinear system (2.1) whose subsystems are GASiM, and propose the following results.

Theorem 3.1. *Consider the switched stochastic nonlinear system (2.1), if there exist $\mathcal{C}^1$ functions $V_i(x)$: $\mathbb{R}^n \to [0,\infty)$, $i \in \mathcal{I}$, and functions $\alpha, \beta \in \mathcal{K}_\infty$ such that*

$$\alpha(\|x\|) \leqslant V_i(x) \leqslant \beta(\|x\|), \quad \forall\, i \in \mathcal{I}, \tag{3.1}$$

and

$$\mathcal{L}V_i(x)|_{(i)} \leqslant -\lambda_i V_i(x), \tag{3.2}$$

where $V_i(x)$ is a Lyapunov function for the i-th subsystem, and $\lambda_i > 0$, $i \in \mathcal{I}$, then the system (2.1) is GASiM under any switching signal with ADT

$$\tau_a > \tau_a^* = \frac{a}{\lambda_{\min}},$$

where

$$a = \ln\mu, \quad \mu = \sup_{x\neq 0}\frac{\beta(\|x(t)\|)}{\alpha(\|x(t)\|)}, \quad \lambda_{\min} = \min_{i\in\mathcal{I}}\lambda_i. \tag{3.3}$$

Proof. Let $t_1, t_2, \cdots$, denote the time points at which switching occurs, write p_k for the value of $\sigma(t)$ on $[t_{k-1}, t_k)$. Consider the continuously differentiable function

$$W(t) := e^{\lambda_{p_k} t} V_{p_k}(x(t)), \quad t \in [t_{k-1}, t_k),$$

$$W(t) = W(t_k) + \int_{t_{k-1}}^{t} e^{\lambda_{p_k} s}\frac{\partial V_{p_k}}{\partial x} g_{p_k}(s, x(s))dw(s) + \int_{t_{k-1}}^{t} e^{\lambda_{p_k} s}[\mathcal{L}V_{p_k}(x(s)) + \lambda_{p_k} V_{p_k}(x(s))]ds. \tag{3.4}$$

If t is replaced by $t_r = \min\{t, \tau_r\}$ in the above, where $\tau_r = \inf\{s \geqslant 0 : |x(s)| \geqslant r\}$, then the stochastic integral (first integral) in (3.4) defines a martingale (with r fixed and t varying), not just a local martingale. Thus, on taking expectations in (3.4) with t_r in place of t and then using (3.2) on the right, we obtain

$$EW(t_r) \leqslant EW(t_{k-1}).$$

On letting $r \to \infty$ and using Fatou's lemma on the left and monotone convergence on the right, we obtain

$$EW(t) \leqslant EW(t_{k-1}),$$

and then

$$EV_{p_k}(x(t_k^-) \leqslant e^{-\lambda_{p_k}(t_k - t_{k-1})} EV_{p_k}(x(t_{k-1})) = e^{-\lambda_{p_k}\tau_k} EV_{p_k}(x(t_{k-1})).$$

According to inequality (3.1), we obtain that

$$\begin{aligned} E\alpha(\|x_k\|) &\leqslant EV_{p_k}(x(t_k^-)) \leqslant e^{-\lambda_{p_k}\tau_k} EV_{p_k}(x(t_{k-1})) \leqslant e^{-\lambda_{p_k}\tau_k} E\beta(\|x_{k-1}\|) \\ &\leqslant e^{-\lambda_{p_k}\tau_k}\frac{E\beta(\|x_{k-1}\|)}{E\alpha(\|x_{k-1}\|)} E\alpha(\|x_{k-1}\|) \leqslant \mu e^{-\lambda_{p_k}\tau_k} E\alpha(\|x_{k-1}\|). \end{aligned}$$

Then, for any t satisfying $t_0 < t_1 < \ldots < t_i \leqslant t < t_{i+1}$, we obtain

$$\begin{aligned} E\alpha(\|x_t\|) &\leqslant EV_{p_t}(x(t^-)) \leqslant e^{-\lambda_{p_t}(t-t_i)} EV_{p_t}(x_{t_i}) \leqslant e^{-\lambda_{p_t}(t-t_i)} E\beta(\|x_{t_i}\|) \\ &\leqslant \mu e^{-\lambda_{p_t}(t-t_i)} E\alpha(\|x_{t_i}\|) \\ &\;\;\vdots \\ &\leqslant \mu^{i+1} e^{-\lambda_{\min}(t-t_0)} E\alpha(\|x_0\|) = e^{(i+1)a-\lambda_{\min}(t-t_0)} E\alpha(\|x_0\|) \\ &= c e^{aN_\sigma(t_0,t)-\lambda_{\min}(t-t_0)} E\alpha(\|x_0\|), \end{aligned}$$

where $c = e^a$.

When $a = 0$, i.e, $\mu = 1$, we can obtain that

$$E\alpha(\|x_t\|) \leqslant e^{-\lambda_{\min}(t-t_0)} E\alpha(\|x_0\|),$$

which implies that the switched system (2.1) is GASiM under arbitrary switching signal.

When $a > 0$, according to (2.3), we obtain

$$aN_\sigma(t_0, t) - \lambda_{\min}(t - t_0) \leqslant aN_0 + (\frac{a}{\tau_a} - \lambda_{\min})(t - t_0),$$

and then

$$E\alpha(\|x_t\|) \leqslant ce^{aN_0} e^{(\frac{a}{\tau_a} - \lambda_{\min})(t-t_0)} E\alpha(\|x_0\|).$$

If $\tau_a > \tau_a^* = \frac{a}{\lambda_{\min}}$, then the switched system (2.1) is GASiM under any switching signal with the above ADT. □

Remark 3.2. From the proof of Theorem 3.1, when $a = 0$, a CLF can be taken, the switched system (2.1) is GASiM under arbitrary switching signal, while $a > 0$, means that MLFs have been taken, the switched system (2.1) is GASiM under any switching signal with $\tau_a > \tau_a^* = \frac{a}{\lambda_{\min}}$. In addition, if $V_i(x) = x^T P_i x$, where $P_i > 0, i \in \mathcal{I}$, then we can obtain that

$$\mu = \max_{i \in \mathcal{I}} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_i)}. \tag{3.5}$$

Moreover, comparing with the corresponding existed results in [7, 16, 24, 30], Theorem 3.1 has two advantages. One is the conditions for Theorem 3.1 are less than those, i.e., there are only two inequalities (3.1) and (3.2) are needed for this theorem, but there is another condition (i.e., $V_i \leqslant \mu V_j,\ \ \mu \geqslant 1,\ i \neq j,\ i, j \in \mathcal{I}$) for the existed results besides these common conditions (inequalities (3.1) and (3.2). The other is the obtained lower bound of ADT τ_a^* is smaller than the corresponding $\tau_a^{*\prime}$ obtained by the ADT method in [7] for switched linear system, i.e.,

$$\tau_a^* = \max_{i \in \mathcal{I}} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_i)} / \lambda_{\min} \leqslant \max_{i,j \in \mathcal{I}} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)} / \lambda_{\min} = \tau_a^{*\prime}.$$

With Theorem 3.1, we can obtain the following corollary.

Corollary 3.3. *Consider the switched stochastic linear system* (2.2)*, if there exist a compact set of matrices* $\mathcal{P} = \{P_i,\ i \in \mathcal{I}\}$ *and number* $\lambda > 0$ *such that*

$$(A_i + \lambda I)^T P_i + P_i(A_i + \lambda I) + \operatorname{tr}(CP_iC^T) + P_iP_i^T < 0,$$

then the system (2.2) *is GASiM under any switching signal with ADT*

$$\tau_a > \tau_a^* = \frac{a}{\lambda},$$

where a, μ *are given as* (3.3) *and* (3.5)*, respectively.*

As the authors in [29] pointed out that the lower bound of ADT τ_a^* is independent of the system modes, and thus it is probably still not anticipated. Then, they obtain a MDADT method, which can reduces the conservative property of the ADT method. Inspired by the ideas in [29], we extend our results to obtain an improved MDADT method for a class of switched stochastic nonlinear systems which have quadratic stability property.

Then, we present the result in the following.

Theorem 3.4. *Consider the switched stochastic nonlinear system (2.1), if there exist $\mathcal{C}^1$ functions $V_i(x)$: $\mathbb{R}^n \to [0, \infty)$, $i \in \mathcal{I}$, and a class of real numbers $\alpha_i > 0, \beta_i > 0$ such that*

$$\alpha_i\|x\|^2 \leqslant V_i(x) \leqslant \beta_i\|x\|^2, \quad \forall\, i \in \mathcal{I},$$

and

$$\mathcal{L}V_i(x)|_{(i)} \leqslant -\lambda_i V_i(x),$$

where $\lambda_i > 0$, $i \in \mathcal{I}$, then the switched system (2.1) is GASiM for any switching signal with MDADT

$$\tau_{ai} > \tau^*_{ai} = \frac{a_i}{\lambda_i},$$

where

$$a_i = \ln \mu_i, \quad \mu_i = \frac{\beta_i}{\alpha_i}.$$

Proof. For any t satisfying $t_0 < t_1 < \cdots < t_i \leqslant t < t_{i+1}$, we obtain

$$\begin{aligned} E\|x_t\|^2 &\leqslant \frac{1}{\alpha_{\sigma(t_i)}} EV_{\sigma(t)}(x_t) \leqslant \frac{1}{\alpha_{\sigma(t_i)}} e^{-\lambda_{\sigma(t_i)}(t-t_i)} EV_{\sigma(t)}(x_{t_i}) \leqslant \frac{\beta_{\sigma(t_i)}}{\alpha_{\sigma(t_i)}} e^{-\lambda_{\sigma(t_i)}(t-t_i)} E\|x_{t_i}\|^2 \\ &= \mu_{\sigma(t_i)} e^{-\lambda_{\sigma(t)}(t-t_i)} E\|x_{t_i}\|^2 \leqslant \mu_{\sigma(t_i)}\mu_{\sigma(t_{i-1})} e^{-\lambda_{\sigma(t)}(t-t_i)-\lambda_{\sigma(t_i)}(t_i-t_{i-1})} E\|x_{t_{i-1}}\|^2 \\ &\;\vdots \\ &\leqslant \mu \prod_{i=1}^{N} e^{a_i N_{\sigma i}(t_0,t)-\lambda_i T_i(t_0,t)} E\|x_0\|^2. \end{aligned} \tag{3.6}$$

When $a_i = 0$, i.e., $\mu_i = 1, i \in \mathcal{I}$, we conclude from (3.6) that

$$E\|x_t\|^2 \leqslant \prod_{i=1}^{N} e^{-\lambda_i T_i(t_0,t)} E\|x_0\|^2 \leqslant e^{-\lambda_{\min}(t-t_0)} E\|x_0\|^2,$$

where $\lambda_{\min}$ is given as (3.3), which implies that the switched system (2.1) is GASiM under any switching signal with the above MDADT.

When $a_i > 0, i \in \mathcal{I}$, according to (2.3), we obtain

$$a_i N_{\sigma i}(t_0, t) - \lambda_i T_i(t_0, t) \leqslant a_i N_{0i} + (\frac{a_i}{\tau_{ai}} - \lambda_i)T_i(t_0, t),$$

and then

$$E\|x_t\|^2 \leqslant \mu \prod_{i=1}^{N} e^{a_i N_{0i}} \prod_{i=1}^{N} e^{(\frac{a_i}{\tau_{ai}}-\lambda_i)T_i(t_0,t)} E\|x_0\|^2 \leqslant \mu \prod_{i=1}^{N} e^{a_i N_{0i}} e^{-\lambda_0(t-t_0)} E\|x_0\|^2,$$

where $\lambda_0 = \min_{i=1}^{N}\{\lambda_i - \frac{a_i}{\tau_{ai}}\}$. If $\tau_{ai} > \tau^*_{ai}$, $i \in \mathcal{I}$, we obtain $\lambda_0 > 0$, and then the switched system (2.1) is GASiM under any switching signal with the above MDADT. □

3.2. *Both GASiM and unstable subsystems coexist*

In the next, we consider the switched stochastic nonlinear systems in which both GASiM and unstable subsystems coexist. For the switching signal $\sigma(t)$ and any $t > \tau$, we let $T^u(\tau, t)$ (resp., $T^s(\tau, t)$) denote the total activation time of the unstable subsystems (resp., GASiM subsystems) on interval $[\tau, t)$. Then, we let $\mathcal{I} = \mathcal{I}_s \cup \mathcal{I}_u$ such that $\mathcal{I}_s \cap \mathcal{I}_u = \emptyset$, and introduce a switching law form [24]:

S1: Determine the $\sigma(t)$ satisfying $\frac{T^s(t_0,t)}{T^u(t_0,t)} \geqslant \frac{\lambda_u+\lambda^*}{\lambda_s-\lambda^*}$ holds for any $t > t_0$, where $\lambda^* \in (0,\ \lambda_s)$, λ_u and λ_s are given as (3.9).

Then, we give the main results in the following.

Theorem 3.5. *Consider the switched stochastic nonlinear system (2.1), if there exist $\mathcal{C}^1$ functions $V_i(x)$: $\mathbb{R}^n \to [0, \infty)$, $i \in \mathcal{I}$, functions $\alpha, \beta \in \mathcal{K}_\infty$ such that (3.1) and*

$$\mathcal{L}V_i(\xi)|_{(i)} \leqslant \lambda_i V_i(\xi), \quad i \in \mathcal{I}_u, \tag{3.7}$$

$$\mathcal{L}V_i(\xi)|_{(i)} \leqslant -\lambda_i V_i(\xi), \quad i \in \mathcal{I}_s, \tag{3.8}$$

where $\lambda_i > 0$, $i \in \mathcal{I}$, then under the switching law **S1** *the switched system (2.1) is GASiM for any switching signal with ADT*

$$\tau_a > \tau_a^* = \frac{a}{\lambda^*},$$

where a is given as (3.3), and $\lambda^* \in (0, \lambda_s)$ is an arbitrarily chosen number,

$$\lambda_u = \max_{i \in \mathcal{I}_u} \lambda_i, \quad \lambda_s = \min_{i \in \mathcal{I}_s} \lambda_i. \tag{3.9}$$

Proof. Let $t_1, t_2, \cdots$, denote the time points at which switching occurs, and write p_k for the value of $\sigma(t)$ on $[t_{k-1}, t_k)$. By doing similar procedure in the proof of Theorem 3.1, we can obtain that

$$EV_{p_j}(x(t_k^-)) \leqslant e^{\text{sign}(p_k)\lambda_{p_k}\tau_k} EV_{p_k}(x_{k-1}),$$

where $\text{sign}(p_k) = 1$, if $p_k \in \mathcal{I}_u$, $\text{sign}(p_k) = -1$, if $p_k \in \mathcal{I}_s$. Thus,

$$\begin{aligned}
E\alpha(\|x_k\|) &\leqslant EV_{p_k}(x(t_k^-)) \leqslant e^{\text{sign}(p_k)\lambda_{p_k}\tau_k} EV_{p_k}(x_{k-1}) \\
&\leqslant e^{\text{sign}(p_k)\lambda_{p_k}\tau_k} E\beta(\|x_{k-1}\|) \\
&\leqslant e^{\text{sign}(p_k)\lambda_{p_k}\tau_k} \frac{E\beta(\|x_{k-1}\|)}{E\alpha(\|x_{k-1}\|)} E\alpha(\|x_{k-1}\|) \\
&\leqslant \mu e^{\text{sign}(p_k)\lambda_{p_k}\tau_k} E\alpha(\|x_{k-1}\|),
\end{aligned}$$

where μ is given as (3.3). Then, for any t satisfying $t_0 < t_1 < \cdots < t_i \leqslant t < t_{i+1}$, we obtain

$$\begin{aligned}
E\alpha(\|x_t\|) &\leqslant EV_{p_t}(x(t^-)) \leqslant e^{\text{sign}(p_t)\lambda_{p_t}(t-t_i)} EV_{p_t}(x_{t_i}) \\
&\leqslant e^{\text{sign}(p_t)\lambda_{p_t}(t-t_0)} E\beta(\|x_{t_i}\|) \\
&\leqslant \mu e^{\text{sign}(p_t)\lambda_{p_t}(t-t_i)} E\alpha(\|x_{t_i}\|) \cdots \\
&\leqslant \mu^{i+1} e^{\lambda_u T^u(t_0,t) - \lambda_s T^s(t_0,t)} E\alpha(\|x_0\|) \\
&= e^{(i+1)a + \lambda_u T^u(t_0,t) - \lambda_s T^s(t_0,t)} E\alpha(\|x_0\|) \\
&= c e^{a N_\sigma(t_0,t) + \lambda_u T^u(t_0,t) - \lambda_s T^s(t_0,t)} E\alpha(\|x_0\|).
\end{aligned}$$

According to the switching law **S1**, i.e.,

$$\lambda_u T^u(t_0, t) - \lambda_s T^s(t_0, t) \leqslant -\lambda^*(T^u(t_0, t) + T^s(t_0, t)) = -\lambda^*(t - t_0), \tag{3.10}$$

we obtain from (3.10) that

$$E\alpha(\|x_t\|) \leqslant c e^{a N_\sigma(t_0,t) - \lambda^*(t-t_0)} E\alpha(\|x_0\|). \tag{3.11}$$

When $a = 0$, i.e., $\mu = 1$, we can obtain from (3.11) that

$$E\alpha(\|x_t\|) \leqslant e^{-\lambda^*(t-t_0)} E\alpha(\|x_0\|),$$

which implies that the switched system (2.1) is GASiM for arbitrary switching paths.

When $a > 0$, according to (2.3), we obtain

$$aN_\sigma(t_0,t) - \lambda^*(t-t_0) \leqslant aN_0 + (\frac{a}{\tau_a} - \lambda^*)(t-t_0),$$

and then

$$E\alpha(\|x_t\|) \leqslant ce^{aN_0}e^{(\frac{a}{\tau_a}-\lambda^*)(t-t_0)}E\alpha(\|x_0\|).$$

If $\tau_a > \frac{a}{\lambda^*}$, then under the switching law **S1** the switched system (2.1) is GASiM under any switching signal with the above ADT. □

With Theorem 3.5, we can obtain the following corollary.

Corollary 3.6. *Consider the switched stochastic linear system* (2.2)*, if there exist a compact set of matrices* $\mathcal{P} = \{P_i,\ i \in \mathcal{I}\}$ *and numbers* $\lambda_1 > 0$, $\lambda_2 > 0$ *such that*

$$(A_i + \lambda_1 I)^T P_i + P_i(A_i + \lambda_1 I) + tr(CP_iC^T) + P_iP_i^T < 0, \quad \text{if } i \in \mathcal{I}_s,$$

$$(A_i - \lambda_2 I)^T P_i + P_i(A_i - \lambda_2 I) + tr(CP_iC^T) + P_iP_i^T < 0, \quad \text{if } i \in \mathcal{I}_u,$$

then under the switching law **S1** *the switched system* (2.2) *is GASiM under any switching signal with the ADT*

$$\tau_a > \tau_a^* = \frac{a}{\lambda^*},$$

where a is given as (3.3)*, and $\lambda^* \in (0, \lambda_s)$ is an arbitrarily chosen number, $\lambda_u = \frac{1}{2}\lambda_2$, $\lambda_s = \frac{1}{2}\lambda_1$.*

Learning form [17], we obtain another results which can deal with some subsystems of the switched system are GASiM, some subsystems are not.

Theorem 3.7. *Consider the switched stochastic nonlinear system* (2.1)*, if there exist $\mathcal{C}^1$ functions $V_i(x)$: $\mathbb{R}^n \to [0,\infty)$, $i \in \mathcal{I}$, and functions $\alpha, \beta \in \mathscr{K}_\infty$ such that* (3.1)*,* (3.7) *and* (3.8)*. If there exist constants $\tau_0, \rho \geqslant 0$ such that*

$$\rho < \frac{\lambda_s}{\lambda_s + \lambda_u},$$

$$T^u(t_0,t) \leqslant \tau_0 + \rho t, \quad \forall\, t \geqslant 0, \tag{3.12}$$

then the switched system (2.1) *is GASiM under any switching signal with ADT*

$$\tau_a > \tau_a^* = \frac{a}{\lambda_s - (\lambda_s + \lambda_u)\rho},$$

where a is given as (3.3)*, and λ_u, λ_s are given as* (3.9)*.*

Proof. The proof of Theorem 3.7 follows the lines of the proof of Theorem 3.5. Similarly to Theorem 3.5, for any t satisfying $t_0 < t_1 < \cdots < t_i \leqslant t < t_{i+1}$, we obtain

$$E\alpha(\|x_t\|) \leqslant ce^{aN_\sigma(t_0,t)+\lambda_u T^u(t_0,t)-\lambda_s T^s(t_0,t)}E\alpha(\|x_0\|),$$

where $c = e^a$.

According to (3.12), we get

$$T^s(t_0,t) \geqslant (1-\rho)(t-t_0) - \tau_0. \tag{3.13}$$

We obtain from (3.13) that

$$\begin{aligned} E\alpha(\|x_t\|) &\leqslant ce^{aN_\sigma(t_0,t)+\lambda_u[\tau_0+\rho(t-t_0)]+\lambda_s[\tau_0+(\rho-1)(t-t_0)]}E\alpha(\|x_0\|) \\ &= ce^{aN_\sigma(t_0,t)+(\lambda_s+\lambda_u)\tau_0-[\lambda_s-(\lambda_s+\lambda_u)\rho](t-t_0)}E\alpha(\|x_0\|). \end{aligned} \tag{3.14}$$

When $a = 0$, i.e., $\mu = 1$, we can obtain from (3.14) that

$$E\alpha(\|x_t\|) \leqslant ce^{(\lambda_s+\lambda_u)\tau_0}e^{-[\lambda_s-(\lambda_s+\lambda_u)\rho](t-t_0)}E\alpha(\|x_0\|),$$

which implies that the switched system (2.1) is GASiM under arbitrary switching paths.

When $a > 0$, according to (2.3), we obtain

$$aN_\sigma(t_0,t) - [\lambda_s - (\lambda_s+\lambda_u)\rho](t-t_0) \leqslant aN_0 + \left\{\frac{a}{\tau_a} - [\lambda_s - (\lambda_s+\lambda_u)\rho]\right\}(t-t_0),$$

and then

$$E\alpha(\|x_t\|) \leqslant ce^{aN_0}e^{(\lambda_s+\lambda_u)\tau_0}e^{\left\{\frac{a}{\tau_a}-[\lambda_s-(\lambda_s+\lambda_u)\rho]\right\}(t-t_0)}E\alpha(\|x_0\|).$$

If $\tau_a > \tau_a^*$, then the switched system (2.1) is GASiM under any switching signal with the above ADT. □

4. Illustrative examples

In this section, we give two illustrative examples to show how to use the results obtained in this paper to analyze the stability of switched stochastic nonlinear system with stable and unstable subsystems in the mean.

Example 4.1. Consider the following switched stochastic linear system

$$dx = A_i x(t)dt + C_i x(t)dw(t), \tag{4.1}$$

where $i \in \mathcal{I} = \{1,2\}$, w is an r-dimensional standard Brownian motion, and

$$A_1 = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad C_1 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad C_2 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

It is easy to know that $V(x) = x^T x$ is a CLF for the switched system (4.1), and

$$\mathcal{L}V_1(x)|_{(1)} = x^T(A_1^T + A_1 + tr(C_1C_1^T))x = 5x_1^2 + 6x_1x_2 + 7x_2^2 \leqslant 10V_1(x),$$

$$\mathcal{L}V_2(x)|_{(2)} = x^T(A_2^T + A_2 + tr(C_2C_2^T))x = -3x_1^2 + 4x_1x_2 - 3x_2^2 \leqslant -V_2(x).$$

According to Theorem 3.5, we obtain that $\lambda_u = 10, \lambda_s = 1$ and $a = 0$. Therefore, the ADT $\tau_a^* = 0$, i.e., the ADT can be arbitrary. Next, we choose $\lambda^* = 0.1$, then the switching law **S1** will require

$$\frac{T^s(t_0,t)}{T^u(t_0,t)} \geqslant \frac{\lambda_u+\lambda^*}{\lambda_s-\lambda^*} = \frac{10.1}{0.9} \approx 11.22.$$

According to Theorem 3.5, the system (4.1) is GASiM under the above switching law **S1**.

To illustrate the correctness of the above conclusion, we carry out some simulation results with the following choices. Initial Condition: $[x_1(0), x_2(0)] = [2, -3]$, and Switching Path:

$$\sigma(t) = \begin{cases} 1, & t \in [t_{2m}, t_{2m+1}), \ t_{2m+1} - t_{2m} = 0.1 * \text{rand}, \\ 2, & t \in [t_{2m+1}, t_{2m+2}), \ t_{2m+2} - t_{2m+1} = 1.2 + 0.1 * \text{rand}, \end{cases}$$

where $m = 0, 1, 2, \cdots$, rand $\in (0,\ 1)$ is a stochastic number. The simulation result is given in Figure 1, which is the response of the state under the above path $\sigma(t)$.

It can be observed from Figure 1 that the trajectory $x(t)$ converges to origin quickly. The simulation shows that Theorem 3.5 is very effective in analyzing the stability for the switched stochastic linear systems with both unstable and exponentially stable subsystems in the mean.

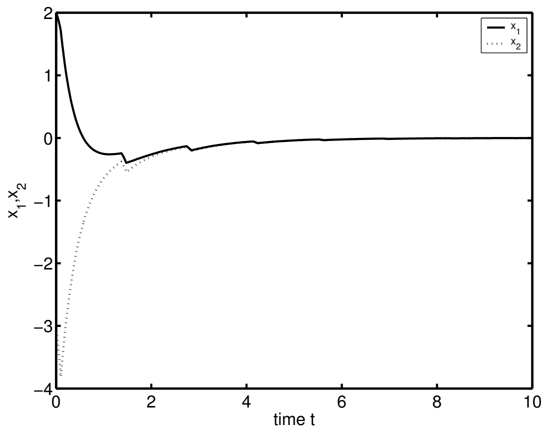

Figure 1: The state's response.

Example 4.2. Consider the following switched stochastic nonlinear system

$$dx = f_i(x)dt + g_i(x)dw(t), \tag{4.2}$$

where $i \in \mathcal{I} = \{1, 2\}$, w is an r-dimensional standard Brownian motion, and

$$f_1(x) = \begin{pmatrix} -x_1 - x_1x_2^2 \\ x_1^2x_2 - 3x_2 \end{pmatrix},\ f_2(x) = \begin{pmatrix} 2x_1 + 2x_2 \\ x_1 + 3x_2 \end{pmatrix},\ g_1(x) = \begin{pmatrix} \frac{1}{2}x_1 - \frac{1}{2}x_2 \\ -\frac{1}{2}x_1 + \frac{1}{2}x_2 \end{pmatrix},\ g_2(x) = \begin{pmatrix} -\frac{1}{2}x_1 \\ \frac{\sqrt{3}}{2}x_2 \end{pmatrix}.$$

It is easy to know that $V(x) = x^Tx$ is a CLF for the switched system (4.2), and

$$\mathcal{L}V_1(x)|_{(1)} = \frac{\partial V}{\partial x}f_i(x) + \frac{1}{2}\mathrm{Tr}\left\{g_i^T(x)\frac{\partial^2 V}{\partial x^2}g_i(x)\right\} = -x_1^2 - 5x_2^2 \leqslant -V_1(x),$$

$$\mathcal{L}V_2(x)|_{(2)} = \frac{\partial V}{\partial x}f_i(x) + \frac{1}{2}\mathrm{Tr}\left\{g_i^T(x)\frac{\partial^2 V}{\partial x^2}g_i(x)\right\} = 2x_1^2 + 4x_1x_2 + 3x_2^2 \leqslant 5V_2(x).$$

According to the above results, we obtain that $\lambda_u = 5, \lambda_s = 1$ and $a = 0$. Therefore, the ADT $\tau_a^* = 0$, i.e., the ADT can be arbitrary. Next, we choose $\lambda^* = 0.1$, then the switching law **S1** will require

$$\frac{T^s(t_0, t)}{T^u(t_0, t)} \geqslant \frac{\lambda_u + \lambda^*}{\lambda_s - \lambda^*} = \frac{5.1}{0.9} \approx 5.67.$$

According to Theorem 3.5, the switched system (4.2) is GASiM under the above switching law **S1**.

To illustrate the correctness of the above conclusion, we carry out some simulation results with the following choices. Initial Condition: $[x_1(0), x_2(0)] = [-2.5, 3]$, and Switching Path:

$$\sigma(t) = \begin{cases} 2,\ t \in [t_{2m}, t_{2m+1}),\ t_{2m+1} - t_{2m} = 0.1 * \mathrm{rand}, \\ 1,\ t \in [t_{2m+1}, t_{2m+2}),\ t_{2m+2} - t_{2m+1} = 0.6 + 0.1 * \mathrm{rand}, \end{cases}$$

where $m = 0, 1, 2, \cdots$, rand $\in (0,\ 1)$ is a stochastic number. The simulation result is given in Figure 2, which is the response of the state under the above path $\sigma(t)$.

It can be observed from Figure 2 that the trajectory $x(t)$ converges to origin quickly. The simulation shows that Theorem 3.5 is very effective in analyzing the stability for the switched stochastic nonlinear systems with both unstable and exponentially stable subsystems in the mean.

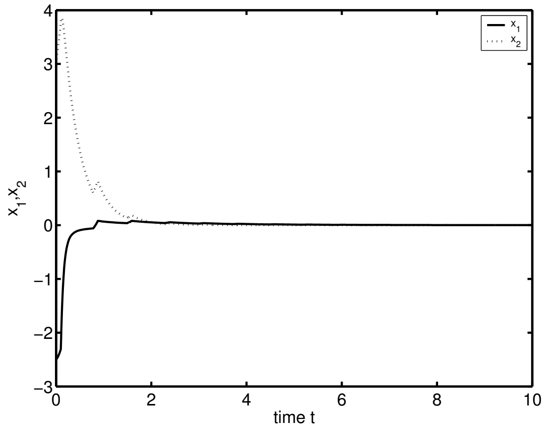

Figure 2: The state's response.

5. Conclusions

In this paper, we have investigated the stability of switched stochastic continuous-time nonlinear systems in two cases: all subsystems are GASiM, and both GASiM subsystems and unstable subsystems coexist, and proposed a number of new results on the stability analysis. An improved ADT method has been established for the stability of such switched system, and an improved MDADT method for the switched systems whose subsystems are quadratically stable in the mean also has been obtained. Based on the obtained methods several new results to the stability analysis have been obtained. Comparing with the corresponding existing results, not only the conditions of the improved ADT method are less than those, but also the obtained lower bound of ADT is smaller than the corresponding result obtained by other methods. Finally, two illustrative examples with numerical simulation have been studied by using the obtained results to show the correctness and effectiveness of the obtained results.

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