Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci., 18 (2018), 49–62
Research Article
Online: ISSN 2008-949x
Journal of Mathematics and Computer Science
Journal Homepage: www.tjmcs.com - www.isr-publications.com/jmcs

Fixed point theorems for generalized α-ψ type contractive mappings in b-metric spaces and applications

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Abstract

In this paper, we establish fixed point theorems for a new generalized α-ψ type contractive mapping in complete b-metric spaces. As applications of our results, we obtain fixed point theorems on metric space endowed with a partial order or a graph. We also obtain fixed point theorems for cyclic contractive mappings. Moreover, an application to integral equation is given here to illustrate the usability of the obtained results.

Keywords: α-ψ contractive mapping, b-metric space, fixed point theorem.

2010 MSC: 47H10, 47H09, 49J40.

1. Introduction and preliminaries

Fixed point theorems for α-ψ type contractive mappings in metric spaces were firstly obtained in 2012 by Samet et al. [29]. In this direction several authors obtained further results (see, e.g., [3–7, 16, 18, 19, 27, 31]).

Let Ψ be family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) ψ is increasing;

(ii) ψ is continuous bijective;

(iii) $\lim_{n\to+\infty} \psi^n(t) = 0$, for all $t \geqslant 0$, where ψ^n is the n-th iterate of ψ.

It is easy to see that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.

In this paper we denote $G(t) = t - \lambda s\psi(t)$, $\lambda s \in (0, 1]$. We easily obtain that G is increasing continuous bijective, hence G^{-1} is increasing and continuous and $G^{-1}(0) = 0$.

Definition 1.1. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leqslant \psi(d(x, y)), \quad \forall x, y \in X.$$

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doi: 10.22436/jmcs.018.01.06

Received 2016-11-01

Clearly, any contractive mapping is an α-ψ contractive mapping with $\alpha(x,y) = 1$ for all $x, y \in X$ and $\psi(t) = kt,\ k \in (0,1)$.

Definition 1.2. Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. We say that T is an α-admissible mapping if for all $x, y \in X$ we have the following implication:

$$\alpha(x,y) \geqslant 1 \Rightarrow \alpha(Tx,Ty) \geqslant 1.$$

Definition 1.3. Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. We say that T is a triangular α-admissible mapping if for all $x, y, z \in X$ we have

$$\alpha(x,y) \geqslant 1 \Rightarrow \alpha(Tx,Ty) \geqslant 1,$$

and

$$\alpha(x,y) \geqslant 1, \alpha(y,z) \geqslant 1 \Rightarrow \alpha(x,z) \geqslant 1.$$

Various examples of the above mappings are presented in [16, 29] and [18].

Some results of fixed point in b-metric space have been obtained (see, e.g., [8, 9, 11, 12]). Now, we present some definitions in b-metric space.

Definition 1.4. Let X be a nonempty set and the mapping $b : X \times X \to R^+$ satisfies:

(b1) $b(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(b2) $b(x,y) = b(y,x)$ for all $x, y \in X$;

(b3) there exists a real number $s \geqslant 1$ such that $b(x,y) \leqslant s[b(x,z) + b(z,y)]$ for all $x, y, z \in X$.

Then b is called a b-metric on X and (X, b) is called a b-metric space with coefficient s.

Remark 1.5. It is clear that every metric space is a b-metric space with coefficient $s = 1$.

Definition 1.6. Let (X, b) be a b-metric space, then for $x \in X$ and $\epsilon > 0$, the b-ball with center x and radius ϵ is

$$B(x,\epsilon) = \{y \in X | b(x,y) < \varepsilon\}.$$

Definition 1.7. Let (X, b) be a b-metric space, $A \subset X$. A is said to be a closed if and only if $x \in X$ and for all $\epsilon > 0$, $B(x,\epsilon) \cap A \neq \phi$, then $x \in A$.

Definition 1.8. Let (X, b) be a b-metric space, $A \subset X$. The diameter of A is

$$\delta(A) = \sup_{x,y \in A} b(x,y).$$

Definition 1.9 ([32])**.** A sequence $\{x_n\}$ in a b-metric space (X, b) is said to be:

(i) a Cauchy sequence if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in N$ such that for each $n, m \geqslant n(\epsilon)$ we have $b(x_n, x_m) < \epsilon$;

(ii) a convergent sequence if and only if there exists $x \in X$ such that for all $\epsilon > 0$ there exists $n(\epsilon) \in N$ such that for each $n \geqslant n(\epsilon)$ we have $b(x_n, x) < \epsilon$.

Definition 1.10. A b-metric space (X, b) is said to be complete if every Cauchy sequence $\{x_n\} \subset X$ converges to some $x \in X$.

Definition 1.11. Let (X, b) be a b-metric space and $T : X \to X$ be a mapping. T is continuous at $x \in X$, if and only if whenever $\{x_n\}$ is convergent to x, then $\{Tx_n\}$ is convergent to Tx.

2. Main results

We introduce a new concept of generalized α-ψ contractive type mappings as follows.

Definition 2.1. Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a generalized α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$, $\psi \in \Psi$, for all $x, y \in X$ such that

$$\alpha(x, y)d(Tx, Ty) \leqslant \psi(\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}).$$

Remark 2.2. Since ψ is increasing, clearly every α-ψ contractive mapping is generalized α-ψ contractive mapping.

Our results are the following.

Theorem 2.3. *Let (X, b) be a complete b-metric space with coefficient $s \geqslant 1$ and $T : X \to X$ be a given mapping. If there exist a function $\psi \in \Psi$ and constant $\lambda \in (0, \frac{1}{s}]$, for all $x, y \in X$ such that*

$$\alpha(x, y)b(Tx, Ty) \leqslant \lambda\psi(\max\{b(x, y), b(x, Tx), b(y, Ty), b(x, Ty), b(y, Tx)\}), \tag{2.1}$$

and which satisfies:

(i) *T is triangular α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$;*

(iii) *T is continuous,*

then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$. Take $x_{n+1} = Tx_n = T^n x_0$ for all $n \in N$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then x_{n_0} is a fixed point of T. So, we can assume that $x_{n+1} \neq x_n$ for all n. Since T is triangular α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geqslant 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geqslant 1.$$

Moreover

$$\alpha(x_0, x_1) \geqslant 1, \quad \alpha(x_1, x_2) \geqslant 1 \Rightarrow \alpha(x_1, x_3) \geqslant 1.$$

Inductively, for all $m, n \in N$, $n < m$, we easily obtain

$$\alpha(x_n, x_m) \geqslant 1. \tag{2.2}$$

Let us denote $O_T(x_0; n) = \{x_0, Tx_0, \cdots, T^n x_0\}$ and $\delta O_T(x_0; n)$ denotes the diameter of $O_T(x_0; n)$. From (2.1) and (2.2), for each $1 \leqslant i < j \leqslant n$, $i, j \in N$, we have

$$\begin{aligned} b(x_i, x_j) &= b(Tx_{i-1}, Tx_{j-1}) \\ &\leqslant \alpha(x_{i-1}, x_{j-1})b(Tx_{i-1}, Tx_{j-1}) \\ &\leqslant \lambda\psi(\max\{b(x_{i-1}, x_{j-1}), b(x_{i-1}, x_i), b(x_{j-1}, x_j), b(x_{i-1}, x_j), b(x_i, x_{j-1})\}) \\ &\leqslant \lambda\psi(\delta O_T(x_0; n)) \qquad (2.3) \\ &\leqslant \psi(\delta O_T(x_0; n)). \qquad (2.4) \end{aligned}$$

It is easy to see that there exists $k \leqslant n, k \in N$ such that

$$b(x_0, T^k x_0) = \delta O_T(x_0; n). \tag{2.5}$$

Indeed, if there exists $i, j \neq 0,\ i < j$ such that $\delta O_T(x_0; n) = b(x_i, x_j)$, from (2.4) we have

$$\delta O_T(x_0; n) = b(x_i, x_j) \leqslant \psi(\delta O_T(x_0; n)) < \delta O_T(x_0; n).$$

It is a contradiction. Hence, by applying (2.3), (2.5) and the triangular inequality, we have

$$\begin{aligned}\delta O_T(x_0; n) &= b(x_0, T^k x_0)\\ &\leqslant sb(x_0, Tx_0) + sb(Tx_0, T^k x_0)\\ &\leqslant sb(x_0, Tx_0) + s\lambda\psi(\delta O_T(x_0; n)),\end{aligned}$$

which leads to

$$\delta O_T(x_0; n) - s\lambda\psi(\delta O_T(x_0; n)) \leqslant sb(x_0, Tx_0).$$

For $G(t) = t - s\lambda\psi(t)$, since G^{-1} is increasing, then

$$\delta O_T(x_0; n) \leqslant G^{-1}(sb(x_0, Tx_0)). \tag{2.6}$$

Also, for all $m, n \in N$ and $m > n$, using (2.4), it results

$$b(x_n, x_m) \leqslant \psi(r_1), \tag{2.7}$$

where

$$r_1 = \delta O_T(x_{n-1}; m - n + 1).$$

Now, by (2.5), there exists $k_1 \in N,\ k_1 \leqslant m - n + 1$ such that

$$r_1 = \delta O_T(x_{n-1}; m - n + 1) = b(x_{n-1}, T^{k_1} x_{n-1}).$$

By using again (2.5) we have

$$r_1 = b(x_{n-1}, T^{k_1} x_{n-1}) = b(Tx_{n-2}, T^{k_1+1} x_{n-2}) \leqslant \psi(r_2), \tag{2.8}$$

where

$$r_2 = \delta O_T(x_{n-2}; k_1 + 1).$$

Since ψ is monotone increasing and $k_1 + 1 \leqslant m - n + 2$, from (2.7) and (2.8) we obtain

$$b(x_n, x_m) \leqslant \psi^2(\delta O_T(x_{n-2}; m - n + 2)).$$

So, for all $m, n \in N$, and $m > n$, by induction, we get

$$b(x_n, x_m) \leqslant \psi^n(\delta O_T(x_0; m)).$$

By (2.6), we get

$$b(x_n, x_m) \leqslant \psi^n(G^{-1}(sb(x_0, Tx_0))). \tag{2.9}$$

Letting $n \to \infty$ in (2.9), we get

$$b(x_n, x_m) \to 0. \tag{2.10}$$

It implies $\{x_n\}$ is a Cauchy sequence, hence it is convergent. So there exists $x^* \in X$ such that

$$\lim_{n\to\infty} b(x_n, x^*) = 0. \tag{2.11}$$

Next we will show that $x^* \in F(T)$. Since T is continuous, then $Tx_n \to Tx^*$ as $n \to \infty$. Using the triangular inequality, we have

$$b(x^*, Tx^*) \leqslant sb(x^*, x_{n+1}) + sb(Tx_n, Tx^*). \tag{2.12}$$

Letting $n \to \infty$ in (2.12), we get $b(x^*, Tx^*) = 0$, which means $x^* \in F(T)$. □

Example 2.4. Let $X = [0, \infty)$, endow with the b-metric $b(x, y) = (x - y)^2$ with $s = 2$ for all $x, y \in X$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{4}, & x \in [0, 1], \\ 2x - \frac{7}{4}, & x \in (1, \infty). \end{cases}$$

We define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} e^{|x-y|}, & \text{if} \quad x, y \in (0, \frac{1}{4}], \\ e^{-|x-y|}, & \text{otherwise.} \end{cases}$$

Clearly, T is a triangular α-admissible and generalized α-ψ contractive mapping with $\psi(t) = \frac{t}{4}$ for all $t \in [0, \infty)$. In fact taking $\lambda = \frac{1}{4}$ for all $x, y \in X$, we have

$$\alpha(x, y)b(Tx, Ty) \leqslant \lambda\psi(\max\{b(x, y), b(x, Tx), b(y, Ty), b(x, Ty), b(y, Tx)\}).$$

Moreover, there exists $x_0 = \frac{1}{4} \in X$ such that

$$\alpha(x_0, Tx_0) = \alpha(\frac{1}{4}, \frac{1}{16}) = e^{\frac{3}{16}} \geqslant 1.$$

Obviously T is continuous.

Now, all the hypotheses of Theorem 2.3 are satisfied, T has a fixed point. In this example, 0 and $\frac{7}{4}$ are two fixed points of T.

Theorem 2.5. *Let (X, b) be a complete b-metric space with coefficient $s \geqslant 1$ and $T : X \to X$ be a given mapping. Suppose there exist a function $\psi \in \Psi$ and constant $\lambda \in (0, \frac{1}{s}]$, for all $x, y \in X$ such that*

$$\alpha(x, y)b(Tx, Ty) \leqslant \lambda\psi(\max\{b(x, y), b(x, Tx), b(y, Ty), b(x, Ty), b(y, Tx)\}), \tag{2.13}$$

and which satisfies:

(i) T *is triangular α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$;*

(iii) *if $\{x_n\}$ is a sequence in (X, b) such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in N$ and $x_n \to x^* \in X$ as $n \to \infty$, then $\alpha(x_n, x^*) \geqslant 1$.*

Then T has a fixed point.

Proof. Following the proof of Theorem 2.3, we know that the sequence x_n defined by $x_{n+1} = Tx_n$ for all $n \in N$, and converges to some $x^* \in X$. By applying (2.2) and condition (iii), we obtain $d(x_n, x^*) \geqslant 1$. So, by (2.1) and the triangular inequality, we have

$$\begin{aligned} b(x^*, Tx^*) &\leqslant sb(x^*, x_{n+1}) + sb(Tx_n, Tx^*) \\ &\leqslant sb(x^*, x_{n+1}) + s\alpha(x_n, x*)b(Tx_n, Tx^*) \\ &\leqslant sb(x^*, x_{n+1}) + s\lambda\psi(\max\{b(x_n, x^*), b(x_n, Tx^*), b(x_{n+1}, x^*), b(x^*, Tx^*), b(x_n, Tx_n)\}) \\ &= sb(x^*, x_{n+1}) + s\lambda\psi(M), \end{aligned} \tag{2.14}$$

where

$$M = \max\{b(x_n, x^*), b(x_n, Tx^*), b(x_{n+1}, x^*), b(x^*, Tx^*), b(x_n, Tx_n)\}.$$

There are three cases.

Case 1. If $M = \max\{b(x_n, x^*), b(x_{n+1}, x^*), b(x_n, x_{n+1})\}$.

Since ψ is continuous, let $n \to \infty$ in (2.14). By (2.10) and (2.11) we get $b(x^*, Tx^*) = 0$.

Case 2. If $M = b(x^*, Tx^*)$.

From (2.14), we have

$$b(x^*, Tx^*) - s\lambda\psi(b(x^*, Tx^*)) \leqslant sb(x_{n+1}, x^*),$$

this implies $b(x^*, Tx^*) \leqslant G^{-1}(sb(x_{n+1}, x^*))$, since G^{-1} is continuous and $G^{-1}(0) = 0$, let $n \to \infty$, by (2.11) we obtain $b(x^*, Tx^*) = 0$.

Case 3. If $M = b(x_n, Tx^*)$.

Since ψ is continuous, let $n \to \infty$ in (2.14), by (2.11) we get

$$b(x^*, Tx^*) \leqslant s\lambda\psi(b(x^*, Tx^*)).$$

This implies $b(x^*, Tx^*) = 0$, or

$$b(x^*, Tx^*) \leqslant \psi(b(x^*, Tx^*)) < b(x^*, Tx^*).$$

It is a contradiction.

From the above three cases, we all obtain $b(x^*, Tx^*) = 0$, hence x^* is a fixed point of T. □

Example 2.6. Let $X = R$, endow with the b-metric $b(x, y) = (x - y)^2$ with $s = 2$ for all $x, y \in X$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{4}, & x \in Q, \\ x^2 - 1, & x \in R - Q. \end{cases}$$

We define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if} \quad x, y \in Q, \\ 0, & x \in R - Q. \end{cases}$$

Clearly, T is a triangular α-admissible and generalized α-ψ contractive mapping with $\psi(t) = \frac{t}{4}$ for all $t \in [0, \infty)$. In fact, taking $\lambda = \frac{1}{4}$ for all $x, y \in X$, we have

$$\alpha(x, y)b(Tx, Ty) \leqslant \lambda\psi(\max\{b(x, y), b(x, Tx), b(y, Ty), b(x, Ty), b(y, Tx)\}).$$

Moreover, there exists $x_0 = \frac{1}{4} \in X$ such that

$$\alpha(x_0, Tx_0) = \alpha(\frac{1}{4}, \frac{1}{16}) = 1.$$

Take $x_n = T^n x_0$. We easily obtain

$$\alpha(x_n, x_{n+1}) = \alpha(\frac{1}{4^n}, \frac{1}{4^{n+1}}) = 1,$$

and as $n \to \infty$, we have

$$x_n = \frac{1}{4^n} \to x = 0 \in X.$$

So

$$\alpha(x_n, x) = \alpha(\frac{1}{4^n}, 0) = 1.$$

Now, all the hypotheses of Theorem 2.5 are satisfied, T has a fixed point. In this example, 0, $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ are three fixed points of T.

(H) For all $x, y \in F(T)$, there exists $z \in X$ such that $\alpha(x, z) \geqslant 1$ and $\alpha(y, z) \geqslant 1$.

Theorem 2.7. *Adding condition* (H) *to Theorem* 2.3 *(resp., Theorem* 2.5*), then that* x^* *is the unique fixed point of* T.

Proof. Let that $x^*, y^* \in F(T)$. By condition (H), there exists $z \in X$ such that

$$\alpha(x^*, z) \geqslant 1, \quad \alpha(y^*, z) \geqslant 1.$$

Since T is α-admissible, from the above inequalities, for all $n \in N$, we obtain

$$\alpha(x^*, T^n z) \geqslant 1, \quad \alpha(y^*, T^n z) \geqslant 1.$$

So

$$\begin{aligned} b(x^*, T^n z) &\leqslant \alpha(x^*, T^{n-1}z) b(T^n x^*, T^n z) \\ &\leqslant \lambda\psi(\max\{b(x^*, T^{n-1}z), b(T^{n-1}z, T^n z), d(x^*, Tx^*), b(x^*, T^n z), b(x^*, T^{n-1}z)\}) \\ &= \lambda\psi(N) \\ &\leqslant \psi(N), \end{aligned} \tag{2.15}$$

where

$$N = \max\{b(x^*, T^{n-1}z), b(T^{n-1}z, T^n z), b(x^*, Tx^*), b(x^*, T^n z), b(x^*, T^{n-1}z)\}.$$

There are four cases.

Case 1. If $N = b(x^*, Tx^*)$.

It implies for all $n \in N$, we have $b(x^*, T^n z) = 0$.

Case 2. If $N = b(x^*, T^{n-1}z)$.

It results

$$b(x^*, T^n z) \leqslant \psi(b(x^*, T^{n-1}z)),$$

recursively, we obtain

$$b(x^*, T^n z) \leqslant \psi^n(b(x^*, z)).$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} b(x^*, T^n z) = 0.$$

Case 3. If $N = b(x^*, T^n z)$.

We get

$$b(x^*, T^n z) \leqslant \psi(b(x^*, T^n z)).$$

It implies for all $n \in N$, we have $b(x^*, T^n z) = 0$.

Case 4. If $N = b(T^{n-1}z, T^n z)$.

Let $n \to \infty$ in (2.15). From (2.10) we obtain

$$\lim_{n\to\infty} b(x^*, T^n z) = 0.$$

From the above four cases, we all obtain

$$\lim_{n\to\infty} b(x^*, T^n z) = 0.$$

Similarly, we can get

$$\lim_{n\to\infty} b(y^*, T^n z) = 0.$$

Using the triangular inequality, we have

$$b(x^*, y^*) \leqslant s b(x^*, T^n z) + s b(y^*, T^n z).$$

Letting $n \to \infty$, we get $b(x^*, y^*) = 0$, i.e., $x^* = y*$. Hence T has the unique fixed point. □

3. Applications

Next, we will show that some results can be deduced easily from our Theorem 2.7.

3.1. Standard fixed point theorems

Letting $s = 1$ in Theorem 2.7, we may get the following fixed point theorem.

Corollary 3.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If there exists a function $\psi \in \Psi$ for all $x, y \in X$ such that*

$$\alpha(x, y)d(Tx, Ty) \leqslant \psi(\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}),$$

and which satisfies:

(i) *T is triangular α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$;*

(iii) *T is continuous or if $\{x_n\}$ is a sequence in (X, d) such that $\alpha(x_n, x_{n+1}) \geqslant 1$ for all $n \in N$ and $x_n \to x^* \in X$ as $n \to \infty$, then $\alpha(x_n, x^*) \geqslant 1$,*

then

(1) *T has a fixed point;*

(2) *if the condition (H) is satisfied, T has a unique fixed point.*

Letting $\alpha(x, y) = 1$ in Theorem 2.7, for all $x, y \in X$, we get the following fixed point theorem.

Corollary 3.2. *Let (X, b) be a complete b-metric space with coefficient $s \geqslant 1$ and $T : X \to X$ be a mapping. If there exist a function $\psi \in \Psi$ and constant $\lambda \in (0, \frac{1}{s}]$, for all $x, y \in X$ such that*

$$b(Tx, Ty) \leqslant \lambda\psi(\max\{b(x, y), b(x, Tx), b(y, Ty), b(x, Ty), b(y, Tx)\}),$$

then T has a unique fixed point.

Corollary 3.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If there exists a function $\psi \in \Psi$, for all $x, y \in X$ such that*

$$d(Tx, Ty) \leqslant \psi(\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}),$$

then T has a unique fixed point.

Corollary 3.4. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If there exist a function $\psi \in \Psi$ and constant $k \in (0, 1)$, for all $x, y \in X$ such that*

$$d(Tx, Ty) \leqslant k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

then T has a unique fixed point.

3.2. Fixed point theorem on b-metric spaces endowed with a partial order

Many exciting fixed point theorems on metric space with a partial have been obtained (see, e.g., [1, 13, 20, 24, 25, 28]). According to our Theorem 2.7, we will deduce fixed point theorems on metric space with a partial, and know that those exciting theorems will be obtained easily by our result. At first, we present some concepts.

Definition 3.5. Let $(X, \preceq)$ be a partially ordered set, $T : X \to X$ be a mapping. We say that T is increasing with respect to $\preceq$, if for all $x, y \in X$

$$x \preceq y \Rightarrow Tx \preceq Ty.$$

Definition 3.6. Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be increasing with respect to $\preceq$, if $x_n \preceq x_{n+1}$ for all n.

Definition 3.7. Let $(X, \preceq, d)$ be partially ordered metric space. We say that $(X, \preceq, d)$ is regular if for every increasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

We obtain the following result.

Corollary 3.8. *Let $(X, \preceq, b)$ be complete partially ordered b-metric space with coefficient $s \geqslant 1$ and $T : X \to X$ be an increasing mapping with respect to $\preceq$. Suppose there exist a function $\psi \in \Psi$ and constant $\lambda \in (0, \frac{1}{s}]$ such that*

$$d(Tx, Ty) \leqslant \lambda\psi(\max\{b(x, y), b(x, Tx), b(y, Ty), b(x, Ty), b(y, Tx)\})$$

for all $x, y \in X$ with $x \succeq y$ and suppose the following conditions are satisfied:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.*

(ii) *T is continuous or $(X, \preceq, b)$ is regular.*

Then T has a fixed point. And, suppose for all $x, y \in X$ there exists $z \in X$ such that $x \preceq y$ and $y \preceq z$, therefore the fixed point is unique.

Proof. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if} \quad x \preceq y \quad \text{or} \quad x \succeq y, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that T is a generalized α-ψ contractive mapping, that is,

$$\alpha(x, y)b(Tx, Ty) \leqslant \lambda\psi(\max\{b(x, y), b(x, Tx), b(y, Ty), b(x, Ty), b(y, Tx)\})$$

for all $x, y \in X$. From condition (i), we have $\alpha(x_0, Tx_0) \geqslant 1$. Moreover, for all $x, y \in X$, from the monotone property of T, we have

$$\alpha(x, y) \geqslant 1 \Rightarrow x \preceq y \quad \text{or} \quad x \succeq y \Rightarrow Tx \succeq Ty \quad \text{or} \quad Tx \preceq Ty \Rightarrow \alpha(Tx, Ty) \geqslant 1,$$

and

$$\alpha(x, y) \geqslant 1, \alpha(y, z) \geqslant 1 \Rightarrow x \preceq y \preceq z \quad \text{or} \quad x \succeq y \succeq z \Rightarrow \alpha(y, z) \geqslant 1.$$

Thus T is triangular α-admissible. One the case that if T is continuous, then all the hypotheses of Theorem 2.3 are satisfied, so T has a fixed point. The other case if that $(X, \preceq, b)$ is regular. Take $Tx_n = x_n$, we may obtain $\alpha(x_n, x_{n+1}) \geqslant 1$, that is, $x_n \preceq x_{n+1}$ for all n and $x_n \to x \in X$. Then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. This implies that $\alpha(x_{n(k)}, x) \geqslant 1$ for all k. Then all the hypotheses of Theorem 2.5 are satisfied. So T has a fixed point. Next, we show the uniqueness. By hypothesis for $x, y \in X$, there exists $z \in X$ such that $x \preceq y$ and $y \preceq z$. So we get $\alpha(x, z) \geqslant 1$ and $\alpha(y, z) \geqslant 1$. Hence the uniqueness of the fixed point is obtained from Theorem 2.7. □

Corollary 3.9. *Let $(X, \preceq, d)$ be complete partially ordered metric space. Let $T : X \to X$ be an increasing mapping with respect to $\preceq$. Suppose there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leqslant \psi(\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\})$$

for all $x, y \in X$ with $x \succeq y$. Suppose the following conditions are satisfied:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.*

(ii) T *is continuous or* $(X, \preceq, d)$ *is regular.*

Then T *has a fixed point. And, suppose for all* $x, y \in X$ *there exists* $z \in X$ *such that* $x \preceq y$ *and* $y \preceq z$*, so the fixed point is unique.*

Corollary 3.10. *Let* $(X, \preceq, d)$ *be complete partially ordered metric space. Let* $T : X \to X$ *be an increasing mapping with respect to* $\preceq$*. Suppose there exists a constant* $k \in (0, 1)$ *such that*

$$d(Tx, Ty) \leqslant k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$ *with* $x \succeq y$*. Suppose the following conditions are satisfied:*

(i) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$*.*

(ii) T *is continuous or* $(X, \preceq, d)$ *is regular.*

Then T *has a fixed point. And, suppose for all* $x, y \in X$ *there exists* $z \in X$ *such that* $x \preceq y$ *and* $y \preceq z$*, so the fixed point is unique.*

3.3. *Fixed point theorems for cyclic contractive mappings*

Some fixed point theorems for cyclic contractive mappings are obtained (see, e.g., [15, 17, 22, 23, 26, 32]). Next, we will show that some fixed point theorems for cyclic contractive mappings are obtained by our Corollary 3.2.

Corollary 3.11. *Let* $\{A_i\}_{i=1}^2$ *be nonempty closed subsets of complete b-metric space* (X, b) *with coefficient* $s \geqslant 1$ *and* $T : Y \to Y$ *be a given mapping, where* $Y = A_1 \cup A_2$*. Suppose that the following conditions hold:*

(i) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(ii) *there exist a function* $\psi \in \Psi$ *and constant* $\lambda \in (0, \frac{1}{s}]$*, for all* $(x, y) \in A_1 \times A_2$ *such that*

$$b(Tx, Ty) \leqslant \lambda\psi(\max\{b(x, y), b(x, Tx), b(y, Ty), b(x, Ty), b(y, Tx)\}).$$

Then T *has a unique fixed point that belongs to* $A_1 \cap A_2$*.*

Proof. Since A_1 and A_2 are closed subsets in the complete b-metric space (X, b), then (Y, b) is complete. So, all the conditions of Corollary 3.2 are satisfied. Thus we may get that T has a unique fixed point, and it belongs to $A_1 \cap A_2$ (from (i)). □

Corollary 3.12. *Let* $\{A_i\}_{i=1}^2$ *be nonempty closed subsets of complete metric space* (X, d)*,* $T : Y \to Y$ *be a mapping, where* $Y = A_1 \cup A_2$*. Suppose that the following conditions hold:*

(i) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(ii) *there exists a function* $\psi \in \Psi$*, for all* $(x, y) \in A_1 \times A_2$ *such that*

$$d(Tx, Ty) \leqslant \psi(\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}).$$

Then T *has a unique fixed point that belongs to* $A_1 \cap A_2$*.*

Corollary 3.13. *Let* $\{A_i\}_{i=1}^2$ *be nonempty closed subsets of complete metric space* (X, d)*,* $T : Y \to Y$ *be a mapping, where* $Y = A_1 \cup A_2$*. Suppose that the following conditions hold:*

(i) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(ii) *there exists constant* $k \in (0, 1)$*, for all* $(x, y) \in A_1 \times A_2$ *such that*

$$d(Tx, Ty) \leqslant k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then T *has a unique fixed point that belongs to* $A_1 \cap A_2$*.*

3.4. *Fixed point theorem on metric spaces endowed with a graph*

Recently, Jachymski [14] obtained fixed point theorems on a metric space with a graph. Following the paper [14], some fixed point theorems on a metric space with a graph have appeared (see, e.g., [10, 21, 30]). At first, we need to introduce some concepts.

Let (X, d) be a metric space and Δ be the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, $E(G)$ being the set of the edges of the graph. Assuming that G has no parallel edges, we will suppose that G can be identified with the $(V(G), E(G))$.

If x and y are vertices of G, then a path in G from x to y of length $k \in N$ is a finite sequence $(x_i)_0^k$ of vertices such that $x_0 = x, x_k = y$ and $(x_{i-1}, x_i) \in E(G)$, for $i \in \{1, 2, \cdots, k\}$.

Let us denote by $\tilde{G}$ the undirected graph obtained from G by ignoring the direction of edges. Notice that a graph G is connected if there is a path between any two vertices and it is weakly connected if $\tilde{G}$ is connected.

The following results are obtained by Corollary 3.1.

Corollary 3.14. *Let* (X, d) *be a metric space and* G *be a directed graph and* $T : X \to X$ *be a given mapping. Suppose there exists a function* $\psi \in \Psi$, *for all* $x, y \in E(G)$ *such that*

$$d(Tx, Ty) \leqslant \psi(\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}),$$

and which satisfies:

(i) $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$, *and* $(x, y) \in E(G), (y, z) \in E(G) \Rightarrow (x, z) \in E(G)$;

(ii) *there exists* $x_0 \in X$ *such that* $(x_0, Tx_0) \in E(G)$;

(iii) T *is continuous or if* $\{x_n\}$ *is a sequence in* (X, d) *such that* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in N$ *and* $x_n \to x^* \in X$ *as* $n \to \infty$, *then* $(x_n, x^*) \in E(G)$.

Then

(1) T *has a fixed point;*

(2) *if* $x, y \in F(T)$, *there exists* $z \in X$ *such that* $(x, y) \in E(G), (y, z) \in E(G)$, T *has a unique fixed point.*

Proof. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in E(G), \\ 0, & \text{otherwise,} \end{cases}$$

which means all the hypotheses of Corollary 3.1 are satisfied. So we can deduce that T has a unique fixed point. □

Corollary 3.15. *Let* (X, d) *be a metric space and* G *be a directed graph and* $T : X \to X$ *be a given mapping. Suppose there exists a constant* $k \in (0, 1)$ *for all* $x, y \in E(G)$ *such that*

$$d(Tx, Ty) \leqslant k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

and which satisfies:

(i) $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$, *and* $(x, y) \in E(G), (y, z) \in E(G) \Rightarrow (x, z) \in E(G)$;

(ii) *there exists* $x_0 \in X$ *such that* $(x_0, Tx_0) \in E(G)$;

(iii) T *is continuous or if* $\{x_n\}$ *is a sequence in* (X, d) *such that* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in N$ *and* $x_n \to x^* \in X$ *as* $n \to \infty$, *then* $(x_n, x^*) \in E(G)$.

Then

(1) T *has a fixed point;*

(2) *if* $x, y \in F(T)$, *there exists* $z \in X$ *such that* $(x, y) \in E(G), (y, z) \in E(G)$, T *has a unique fixed point.*

3.5. *Application to integral equations*

Here, we are concerned with the nonlinear quadratic integral equation

$$x(t) = h(t) + \theta \int_0^t k(t,s)f(s,x(s))ds, \quad t \in [0,T], \ T > 0. \tag{3.1}$$

Let $X = C([0,T])$ be the set of continuous functions in $[0,T]$ and

$$b(x,y) = \sup_{t\in[0,T]} |x(t) - y(t)|^p, \quad x,y \in C([0,T]).$$

It is easy to see that (X,b) is the complete b-metric space with $s = 2^{p-1}$, $p \geqslant 1$ [2].

We consider (3.1) under the following assumptions:

(i) $h : [0,T] \to R$ is continuous;

(ii) $f : [0,T] \to R$ is continuous and for all $t \in [0,T]$, if $x \leqslant y$, we have

$$f(t,x) \leqslant f(t,y), \quad |f(t,x) - f(t,y)| \leqslant L|x-y|,$$

where $L > 0$ is a constant;

(iii) $k : [0,T] \times [0,T] \to [0,\infty)$ is continuous and there exists a constant $K > 0$ such that

$$\int_0^t k(t,s)|x(s) - y(s)|ds \leqslant K, \quad t \in [0,T];$$

(iv) there exists $x_0 \in X$ such that

$$x_0(t) = h(t) + \theta \int_0^t k(t,s)f(s,x_0(s))ds, \quad t \in [0,T], \ T > 0.$$

We have the following theorem.

Theorem 3.16. *Suppose the above conditions* (i)–(iv) *are satisfied. If* $\theta LKT < \frac{1}{2^{p-1}}$, *then the integral equation* (3.1) *has a unique continuous solution* $x^* \in C[0,T]$.

Proof. We consider the operator $T : X \to X$ defined by

$$Tx(t) = h(t) + \theta \int_0^t k(t,s)f(s,x(s))ds, \quad t \in [0,T], \ T > 0. \tag{3.2}$$

We show that T is an α-ψ generalized contractive mapping in b-metric spaces, that is,

$$\alpha(x,y)b(Tx(t),Ty(t)) \leqslant \lambda\psi(M(x,y)), \tag{3.3}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$.

Now we let the function $\alpha : X \times X \to R$ defined by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x(t) \leqslant y(t), \ t \in [0,T], \\ 0, & \text{otherwise,} \end{cases}$$

and the function $\psi : [0,\infty) \to [0,\infty)$ defined by

$$\psi(t) = (\theta KLT)^{p-1}t, \quad t \in [0,\infty).$$

Obviously, $\psi \in \Psi$.

$$b(Tx(t), Ty(t)) = \sup_{t\in[0,T]} |Tx(t) - Ty(t)|^p. \tag{3.4}$$

Also, if $x(t) \leqslant y(t)$ is not satisfied, then the inequality (3.3) holds immediately. So we may suppose $x(t) \leqslant y(t), t \in [0, T]$. From conditions (ii), (iii) and (3.2), we have

$$\begin{aligned}|Tx(t) - Ty(t)| &= |h(t) + \theta \int_0^t k(t,s)f(s,x(s))ds - h(t) - \theta \int_0^t k(t,s)f(s,y(s))ds| \\ &\leqslant \theta \int_0^t k(t,s)|f(s,x(s)) - f(s,y(s))|ds \\ &\leqslant \theta \int_0^t k(t,s)L|x(s)) - y(s)|ds \\ &\leqslant \theta KLT|x(s)) - y(s)|.\end{aligned}$$

So, from (3.4), we get

$$b(Tx(t), Ty(t)) \leqslant (\theta KLT)^p b(x, y) \leqslant (\theta KLT)^p M(x, y). \tag{3.5}$$

Taking $\lambda = \theta KLT$ and by (3.5) we obtain

$$\alpha(x, y)b(Tx(t), Ty(t)) \leqslant \lambda\psi(M(x, y)).$$

So, T is an α-ψ generalized contractive mapping in b-metric spaces.

Take $x_n = T^n x_0$, $n \in N$. From condition (iv), we get $\alpha(x_0, Tx_0) = 1$. And from condition (ii) we may obtain

$$\alpha(x, y) = 1 \Rightarrow \alpha(Tx, Ty) = 1.$$

So by induction, we get easily $\alpha(x_n, x_{n+1}) = 1$. Also from the proof of Theorem 2.3, we know that $x_n \to x^* \in X$, then $\alpha(x_n, x^*) = 1$. Hence all assumptions of Theorem 2.5 are satisfied. So, according to Theorem 2.5 we can deduce that x^* is a fixed point of T, that is, x^* is a solution to the integral equation (3.1).

Also, take $z(t) = \max\{x(t), y(t)\}$, $t \in [0, T]$. Then for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) = \alpha(y, z) = 1$. From Theorem 2.7, we know that x^* is the unique solution to the integral equation (3.1).

□

Acknowledgment

The first author is supported by the Educational Science Foundation of Chongqing, Chongqing of China (KG111309). The second author is supported by Chongqing City Board of Education (KJ1705136)

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