Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci., 18 (2018), 87–93
Research Article
Online: ISSN 2008-949x
Journal of Mathematics and Computer Science
Journal Homepage: www.tjmcs.com - www.isr-publications.com/jmcs

A Grüss type inequality for two weighted functions

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Abstract

Since Grüss in 1935 presented the so-called Grüss type inequality, a variety of its variants and generalizations have been investigated. Among those things, Dragomir in 2000 established a Grüss type inequality for a functional with a weighted function. In this sequel, we aim to present a Grüss type inequality for a functional with two weighted functions. We also apply our main result to give some other inequalities.

Keywords: Grüss type inequality and its generalization, Chebyshev inequality, Grüss type inequality with a weighted function, Grüss type inequality with two weighted functions, synchronous functions.

2010 MSC: 26D10, 26D15, 26A33, 05A30.

1. Introduction and preliminaries

In 1935, Grüss [8] established an interesting inequality that reveals the difference of the integral of product of two functions and product of integrals of the two functions:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\,\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leqslant \frac{1}{4}(\Phi-\varphi)(\Psi-\psi), \tag{1.1}$$

where $f,\ g : [a,\ b] \to \mathbb{R}$ are integrable functions such that $\varphi < f(x) < \Phi$ and $\psi < g(x) < \Psi$ for all $x \in [a,\ b]$ and φ, Φ, ψ, Ψ are real constants. Here the constant $\frac{1}{4}$ is sharp, for example, $f(x) = g(x) = \operatorname{sgn}\left(x - \frac{a+b}{2}\right)$. Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, real and positive real numbers, and positive, and non-positive integers, respectively.

Since the inequality (1.1) appeared, a number of its generalizations have been presented (see, e.g., [11, Chapter X]; see also [13]). If $T(f, g)$ is defined by

$$T(f,g) := (b-a)\int_a^b f(x)g(x)\,dx - \int_a^b f(x)\,dx \int_a^b g(x)\,dx,$$

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doi: 10.22436/jmcs.018.01.09

Received 2017-04-02

then the inequality (1.1) can be rewritten as follows:

$$|T(f,g)| \leqslant \frac{1}{4}(\Phi-\varphi)(\Psi-\psi)\,(b-a)^2.$$

Let $p:[a,\,b]\to[0,\,\infty)$ be an integrable function such that $\int_a^b p(x)\,dx>0$. Let f, g be the functions in (1.1). Then, Dragomir [7] presented the following Grüss type inequality with a weight functional p:

$$|T(f,g;p)| \leqslant \frac{1}{4}(\Phi-\varphi)\left(\int_a^b p(x)\,dx\right)^2, \tag{1.2}$$

where

$$T(f,g;p) := \int_a^b p(x)\,dx \int_a^b p(x)\,f(x)g(x)\,dx - \int_a^b p(x)\,f(x))\,dx \int_a^b p(x)\,g(x)\,dx. \tag{1.3}$$

It is obvious that $T(f,g;1)=T(f,g)$.

Let $q:[a,b]\to[0,\infty)$ be another integrable function such that $\int_a^b q(x)dx>0$. Then $T(f,g;p,q)$ is a functional with two weight functions p and q defined as follows:

$$\begin{aligned} T(f,g;p,q) := &\int_a^b q(x)dx\int_a^b p(x)f(x)g(x)dx + \int_a^b p(x)dx\int_a^b q(x)f(x)g(x)dx \\ &- \int_a^b q(x)f(x)dx\int_a^b p(x)g(x)dx - \int_a^b p(x)f(x)dx\int_a^b q(x)g(x)dx. \end{aligned} \tag{1.4}$$

Two functions f and g are said to be *synchronous* on $[a,b]$ if

$$(f(x)-f(y))\,(g(x)-g(y)) \geqslant 0 \quad (x,y\in[a,b]).$$

Two functions f and g are said to be *asynchronous* on $[a,b]$ if

$$(f(x)-f(y))\,(g(x)-g(y)) \leqslant 0 \quad (x,y\in[a,b]). \tag{1.5}$$

If two functions f,g are synchronous on $[a,b]$ and p,q are positive integrable functions on $[a,b]$, then the following inequalities hold (see, e.g., [9, 10]):

$$T(f,g)\geqslant 0, \quad T(f,g;p)\geqslant 0, \quad \text{and} \quad T(f,g;p,q)\geqslant 0. \tag{1.6}$$

In particular, $T(f,g;p,p)\geqslant 0$ is called Chebyshev inequality (see [2]).

Ostrowski in [12] established the following generalization of the Chebyshev inequality: if f and g are two differentiable and synchronous functions on $[a,b]$, p is a positive integrable function on $[a,b]$, and $|f'(x)|\geqslant \mathbf{m}$ and $|g'(x)|\geqslant \mathbf{r}$ $(x\in[a,b])$ for some real numbers $\mathbf{m}$ and $\mathbf{r}$, then

$$T(f,g;p) := T(f,g;p,p) \geqslant \mathbf{mr}T(x-a,x-a,p)\geqslant 0. \tag{1.7}$$

If f and g are asynchronous on $[a,b]$, then the inequality in (1.7) is reversed:

$$T(f,g,p)\leqslant \mathbf{mr}T(x-a,x-a,p)\leqslant 0.$$

If f and g are differentiable functions on $[a,b]$, p is a positive integrable function on $[a,b]$, and $|f'(x)|\leqslant \mathbf{M}$, $|g'(x)|\leqslant \mathbf{R}$ $(x\in[a,b])$ for some real numbers $\mathbf{M}$ and $\mathbf{R}$, then

$$|T(f,g,p)|\leqslant \mathbf{MR}T(x-a,x-a;p)\leqslant 0.$$

Here, in this paper, we aim at establishing a Grüss type inequality for the two weighted functional $T(f,g,p,q)$ in (1.4) like (1.1) and (1.2). We also apply the main result to give some other inequalities.

2. A Grüss type inequality for the two weighted functional $T(f, g; p, q)$

Here, a Grüss type inequality for the two weighted functional $T(f, g, p, q)$ in (1.4) has been tried by modifying the methods of proof given in [7, 8, 12]. The resulting inequality which is not so simple as those in (1.1) and (1.2) is given in the following theorem.

Theorem 2.1. *Let* $f, g : [a, b] \to \mathbb{R}$ *be integrable functions such that* $\varphi < f(x) < \Phi$ *and* $\psi < g(x) < \Psi$ *for all* $x \in [a, b]$, *where* φ, Φ, ψ, Ψ *are constants. Also let* $p, q : [a, b] \to [0, \infty)$ *be integrable functions on* $[a, b]$ *such that*

$$\min\left\{\int_a^b p(x)dx, \int_a^b q(x)dx\right\} > 0.$$

Then the two weighted functional $T(f, g; p, q)$ *in* (1.4) *satisfies the following inequality:*

$$|T(f, g; p, q)| \leqslant \sqrt{\left\{\frac{(\Phi-\varphi)^2}{2} + 2\|f\|_\infty^2\right\}\left\{\frac{(\Psi-\psi)^2}{2} + 2\|g\|_\infty^2\right\}} \times \int_a^b p(x)dx \int_a^b q(x)dx, \tag{2.1}$$

where

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)| \quad \text{and} \quad \|g\|_\infty = \sup_{x\in[a,b]} |g(x)|.$$

Proof. We find the following relation:

$$T(f, g; p, q) = \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y))p(x)q(y)dxdy. \tag{2.2}$$

Let

$$\mathcal{I} := \frac{1}{\int_a^b p(x)dx \int_a^b q(x)dx} \int_a^b \int_a^b (f(x) - f(y))(g(x) - g(y))p(x)q(y)dxdy. \tag{2.3}$$

Applying Cauchy-Buniakowski-Schwarz's inequality for double integrals, we have

$$\begin{aligned}\mathcal{I}^2 \leqslant &\frac{1}{\int_a^b p(x)dx \int_a^b q(x)dx} \int_a^b \int_a^b (f(x) - f(y))^2 p(x)q(y)dxdy\\ &\times \frac{1}{\int_a^b p(x)dx \int_a^b q(x)dx} \int_a^b \int_a^b (g(x) - g(y))^2 p(x)q(y)dxdy = \mathcal{A}\mathcal{B},\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\mathcal{A} =& \frac{1}{\int_a^b p(x)dx} \int_a^b f^2(x)p(x)dx + \frac{1}{\int_a^b q(x)dx} \int_a^b f^2(x)q(x)dx\\ &- \frac{2}{\int_a^b p(x)dx \int_a^b q(x)dx} \int_a^b f(x)p(x)dx \int_a^b f(x)q(x)dx\end{aligned}$$

and

$$\begin{aligned}\mathcal{B} =& \frac{1}{\int_a^b p(x)dx} \int_a^b g^2(x)p(x)dx + \frac{1}{\int_a^b q(x)dx} \int_a^b g^2(x)q(x)dx\\ &- \frac{2}{\int_a^b p(x)dx \int_a^b q(x)dx} \int_a^b g(x)p(x)dx \int_a^b g(x)q(x)dx.\end{aligned}$$

Consider

$$U := \frac{1}{2\int_a^b p(x)dx \int_a^b q(x)dx} \left[\int_a^b \int_a^b (\Phi - f(x))(f(x) - \varphi)p(x)q(y)dxdy\right.$$

$$+\int_a^b\int_a^b(\Phi-f(x))(f(x)-\varphi)p(y)q(x)dxdy\Big]$$

$$=-\Phi\varphi+\frac{\Phi V}{2}+\frac{\varphi V}{2}-\frac{\int_a^b f^2(x)p(x)dx}{2\int_a^b p(x)dx}-\frac{\int_a^b f^2(x)q(x)dx}{2\int_a^b q(x)dx},$$

where

$$V:=\frac{\int_a^b f(x)p(x)dx}{\int_a^b p(x)dx}+\frac{\int_a^b f(x)q(x)dx}{\int_a^b q(x)dx}. \tag{2.5}$$

Now

$$W:=\left(\Phi-\frac{V}{2}\right)\left(\frac{V}{2}-\varphi\right)=\frac{\Phi V}{2}-\Phi\varphi-\frac{V^2}{4}+\frac{\varphi V}{2}.$$

Then we have

$$W-U=-\frac{V^2}{4}+\frac{\int_a^b f^2(x)p(x)dx}{2\int_a^b p(x)dx}+\frac{\int_a^b f^2(x)q(x)dx}{2\int_a^b q(x)dx}. \tag{2.6}$$

Here we find

$$\mathcal{A}=2\cdot\frac{\mathcal{A}}{2}=2\left\{\frac{\int_a^b f^2(x)p(x)dx}{2\int_a^b p(x)dx}+\frac{\int_a^b f^2(x)q(x)dx}{2\int_a^b q(x)dx}-\frac{\int_a^b f(x)p(x)dx\int_a^b f(x)q(x)dx}{\int_a^b p(x)dx\int_a^b q(x)dx}\right\}. \tag{2.7}$$

Combining (2.6) and (2.7), we obtain

$$\mathcal{A}=2\left\{W-U+\frac{V^2}{4}-\frac{\int_a^b f(x)p(x)dx\int_a^b f(x)q(x)dx}{\int_a^b p(x)dx\int_a^b q(x)dx}\right\}\leqslant 2\left\{W-U+\frac{V^2}{2}-\frac{\int_a^b f(x)p(x)dx\int_a^b f(x)q(x)dx}{\int_a^b p(x)dx\int_a^b q(x)dx}\right\}.$$

Using the definition of V in (2.5), we have

$$\mathcal{A}\leqslant 2\left\{W-U+\frac{1}{2}\left(\frac{\int_a^b f(x)p(x)dx}{\int_a^b p(x)dx}\right)^2+\frac{1}{2}\left(\frac{\int_a^b f(x)q(x)dx}{\int_a^b q(x)dx}\right)^2\right\}.$$

Since $(\Phi-f(x))(f(x)-\varphi)$ for all $x\in[a,b]$, $U\geqslant 0$. Also note that

$$\left(\int_a^b f(x)p(x)dx\right)^2=\left(\left|\int_a^b f(x)p(x)dx\right|\right)^2\leqslant\left(\int_a^b|f(x)|p(x)dx\right)^2\leqslant\left(\|f\|_\infty\int_a^b p(x)dx\right)^2.$$

So we get

$$\mathcal{A}\leqslant 2\left(W+\|f\|_\infty^2\right).$$

Using an elementary inequality: $4uv\leqslant(u+v)^2$ for all $u,v\in\mathbb{R}$, we find

$$4W=4\left(\Phi-\frac{V}{2}\right)\left(\frac{V}{2}-\varphi\right)\leqslant(\Phi-\varphi)^2.$$

Therefore, we obtain

$$\mathcal{A}\leqslant\frac{(\Phi-\varphi)^2}{2}+2\|f\|_\infty^2. \tag{2.8}$$

Similarly we have

$$\mathcal{B}\leqslant\frac{(\Psi-\psi)^2}{2}+2\|g\|_\infty^2. \tag{2.9}$$

Considering (2.2), (2.3), and (2.4), we obtain

$$\left(\frac{T(f,g;p,q)}{\int_a^b p(x)dx\int_a^b q(x)dx}\right)^2 \leqslant \mathcal{A}\mathcal{B}. \tag{2.10}$$

Finally applying the inequalities (2.8) and (2.9) to (2.10), the desired inequality (2.1) can be easily obtained. □

We find from (2.2) that

$$|T(f,g;p,q)| \leqslant (\Phi-\varphi)(\Psi-\psi)\int_a^b p(x)dx\int_a^b q(x)dx.$$

For the inequality in Theorem 2.1, there exists a corresponding discrete analogue as in the following corollary.

Corollary 2.2. *Let $f=(f_1,\dots,f_n)$ and $g=(g_1,\dots,g_n)$ be two real sequences with $\varphi\leqslant f_i\leqslant\Phi$ and $\psi\leqslant g_i\leqslant\Psi$ $(1\leqslant i\leqslant n)$, where φ, Φ, ψ, Ψ are real constants, and let $p=(p_1,\dots,p_n)$ and $q=(q_1,\dots,q_n)$ be two non-negative sequences with*

$$\min\left\{\sum_{i=1}^n p_i, \sum_{i=1}^n q_i\right\} > 0.$$

Also let

$$T_s(f,g;p,q)=\sum_{i=1}^n q_i\sum_{i=1}^n p_if_ig_i+\sum_{i=1}^n p_i\sum_{i=1}^n q_if_ig_i-\sum_{i=1}^n q_if_i\sum_{i=1}^n p_ig_i-\sum_{i=1}^n p_if_i\sum_{i=1}^n q_ig_i.$$

Then the $T_s(f,g;p,q)$ satisfies the following inequality:

$$|T_s(f,g;p,q)| \leqslant \sqrt{\left\{\frac{(\Phi-\varphi)^2}{2}+2M_f^2\right\}\left\{\frac{(\Psi-\psi)^2}{2}+2M_g^2\right\}}\times\sum_{i=1}^n p_i\sum_{i=1}^n q_i,$$

where

$$M_f := \max_{1\leqslant i\leqslant n}|f_i| \quad \textit{and} \quad M_g := \max_{1\leqslant i\leqslant n}|g_i|.$$

3. f' and g' belong to some L_p-spaces

Here we give some inequalities of Grüss type for differentiable functions whose derivatives belong to $L_p(a,b)$ when $p=\infty$, $1<p<\infty$, and $p=1$. Here, theorems and their proofs would run in parallel with those in [7, Section 6]. So their proofs are omitted.

3.1. $L_\infty(a,b)$ case

Theorem 3.1. *Let $f,g:[a,b]\to\mathbb{R}$ be two differentiable functions on (a,b) with $f',g'\in L_\infty(a,b)$ and $p,q:[a,b]\to[0,\infty)$. Then $T(f,g;p,q)$ in (1.4) satisfies the following inequality:*

$$\begin{aligned}|T(f,g;p,q)| &\leqslant \int_a^b\int_a^b\left|\int_x^y|f'(u)|du\right|\left|\int_x^y|g'(v)|dv\right|p(x)q(y)dxdy\\ &\leqslant \|f'\|_\infty\|g'\|_\infty\left(\int_a^b q(x)dx\int_a^b x^2p(x)dx\right.\\ &\quad\left.-2\int_a^b xp(x)dx\int_a^b xq(x)dx+\int_a^b p(x)dx\int_a^b x^2q(x)dx\right).\end{aligned} \tag{3.1}$$

Moreover, the inequality in (3.1) is sharp.

Corollary 3.2. *Under the same assumptions as in Theorem 3.1 with $p = q$, we have the following inequality:*

$$\begin{aligned}\frac{1}{2}|T(f,g;p,p)| &= \left|\int_a^b p(x)dx\int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx\int_a^b p(x)g(x)dx\right| \\ &\leqslant \frac{1}{2}\int_a^b\int_a^b\left|\int_x^y |f'(u)|du\right|\left|\int_x^y |g'(v)|dv\right| p(x)p(y)dxdy \\ &\leqslant \|f'\|_\infty\|g'\|_\infty\left\{\int_a^b p(x)dx\int_a^b x^2p(x)dx - \left(\int_a^b xp(x)dx\right)^2\right\}.\end{aligned} \tag{3.2}$$

Moreover, the inequality in (3.2) is sharp.

Corollary 3.3. *Let $f, g : [a, b] \to \mathbb{R}$ be two differentiable functions on (a, b) with $f', g' \in L_\infty(a, b)$. Then the following inequality holds true:*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right| \\ &\quad\leqslant \frac{1}{2(b-a)^2}\int_a^b\int_a^b\left|\int_x^y |f'(u)|du\right|\left|\int_x^y |g'(v)|dv\right| dxdy \leqslant \frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty.\end{aligned} \tag{3.3}$$

Proof. Setting $p = 1$ in the inequality (3.2) and dividing each side of the resulting inequalities by $(b-a)^2$ yields the desired inequalities in (3.3). □

Remark 3.4. The results in Corollaries 3.2 and 3.3 are recorded in Theorem 6.1 and Corollary 6.2 in [7], respectively, with a correction of the first inequality in (3.3). The second inequality in (3.3) is due to Čebyšev (see, e.g., [11, p. 297, (2.1)]).

4. Results, discussion, and concluding remarks

Dragomir's Grüss type inequality with a weight function p in (1.2) is interesting and useful in dealing with some bound of the functional with a weighted function (1.3). The functional $T(f, g; p, q)$ with two weighted functions p and q in (1.4) are naturally arising in connection with the definitions of two or more functions being *synchronous* and *asynchronous*. The main result presented here in Theorem 2.1 can be applied to some other integral operators (see, e.g., [1, 3–6, 14] and the references cited therein) and motivated to give some inequalities for functionals with three or more weighted functions as in (1.2) and (2.1).

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