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Third-order differential sandwich-type results involving the Liu-Owa integral operator

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Abstract

Some third-order differential subordination and superordination results are derived for multivalent analytic functions in the open unit disk, which are defined by using the Liu-Owa integral operator. In addition, we obtain new third-order differential sandwich-type results for this operator.

Keywords: Differential subordination and superordination, multivalent analytic functions, admissible functions, sandwich-type results, Liu-Owa integral operator.

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1. Introduction, definitions, and preliminaries

Let $\mathbb{C}$ be complex plane and $\mathcal{H}(\mathbb{U})$ be the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

For $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$ and $a \in \mathbb{C}$, we suppose that

$$\mathcal{H}[a, n] = \{f : f \in \mathcal{H}(\mathbb{U}) \quad \text{and} \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$$

and $\mathcal{H}_0 = \mathcal{H}[0, 1]$.

Let f and F be members of $\mathcal{H}(\mathbb{U})$. The function f is said to be subordinate to F, or F is superordinate to f, if there exists a Schwarz function $\mathfrak{w}(z)$, analytic in $\mathbb{U}$ with

$$\mathfrak{w}(0) = 0 \qquad \text{and} \qquad |\mathfrak{w}(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = F(\mathfrak{w}(z)) \qquad (z \in \mathbb{U}).$$

In this case, we write

$$f \prec F \qquad \text{or} \qquad f(z) \prec F(z) \qquad (z \in \mathbb{U}).$$

Furthermore, if the function F is univalent in $\mathbb{U}$, then we have the following equivalence (see, for details,

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[23, 24]; see also [20, 28]):

$$f(z) \prec F(z) \quad (z \in \mathbb{U}) \iff f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

Assume that $\mathcal{A}(p)$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N};\ z \in \mathbb{U}). \tag{1.1}$$

In 2004, Liu and Owa [22] (see also [8–13, 31]) introduced the integral operator $Q^{\alpha}_{\beta,p} : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ as below:

$$Q^{\alpha}_{\beta,p} f(z) = \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\,dt \quad (\alpha > 0;\ \beta > -1;\ p \in \mathbb{N}) \tag{1.2}$$

and

$$Q^{0}_{\beta,p} f(z) = f(z) \quad (\alpha = 0;\ \beta > -1).$$

If the function $f \in \mathcal{A}(p)$ shown by (1.1), then from (1.2), we deduce that

$$Q^{\alpha}_{\beta,p} f(z) = z^p + \frac{\Gamma(\alpha+\beta+p)}{\Gamma(\beta+p)} \sum_{n=1}^{\infty} \frac{\Gamma(\beta+p+n)}{\Gamma(\alpha+\beta+p+n)} a_{p+n} z^{p+n} \quad (\alpha \geqq 0;\ \beta > -1;\ p \in \mathbb{N}). \tag{1.3}$$

Also, we easily get the relationship, from (1.3), that (see [22])

$$z\left(Q^{\alpha}_{\beta,p} f(z)\right)' = (\alpha+\beta+p-1) Q^{\alpha-1}_{\beta,p} f(z) - (\alpha+\beta-1) Q^{\alpha}_{\beta,p} f(z). \tag{1.4}$$

We note that

(i) for $p = 1$, $Q^{\alpha}_{\beta,1} = Q^{\alpha}_{\beta}$, which was called Jung-Kim-Srivastava integral operator (see [19]; also see [7, 18]);

(ii) for $\alpha = 1$ and $\beta = \delta$, $Q^{1}_{\delta,p} = J_{\delta,p}$ $(\delta > -p)$, which was called the generalized Libera operator and presented as follows (see [14, 25]; see also [21])

$$Q^{1}_{\delta,p} f(z) = J_{\delta,p}(f)(z) = \frac{\delta+p}{z^{\delta}} \int_0^z t^{\delta-1} f(t)\,dt \quad (\delta > -p;\ \ p \in \mathbb{N}).$$

In our next investigation, we need the following definitions and lemmas.

Definition 1.1 ([6])**.** Let $\mathcal{Q}$ be the set of analytic and univalent functions q on the set $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \{\xi : \xi \in \partial\mathbb{U} \quad \text{and} \quad \lim_{z\to\xi} q(z) = \infty\},$$

and are such that

$$\min |q'(\xi)| = \rho > 0$$

for $\xi \in \partial\mathbb{U} \setminus E(q)$. Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be written by $\mathcal{Q}(a)$ and

$$\mathcal{Q}(0) = \mathcal{Q}_0.$$

Definition 1.2 ([6, 32, 34])**.** Let $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the following third-order differential subordination:

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) \prec h(z) \qquad (z \in \mathbb{U}), \tag{1.5}$$

then $p(z)$ is called a solution of the differential subordination. A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination or, more simply, a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.5). A dominant $\widetilde{q}(z)$ that satisfies $\widetilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.5) is said to be the best dominant.

Definition 1.3 ([33]). Let $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be analytic in $\mathbb{U}$. If the functions $p(z)$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$$

are univalent in $\mathbb{U}$ and satisfy the following third-order differential superordination:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z), \tag{1.6}$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination or, more simply, a subordinant if $q(z) \prec p(z)$ for $p(z)$ satisfying (1.6). A univalent subordinant $\widetilde{q}(z)$ that satisfies the following condition:

$$q(z) \prec \widetilde{q}(z) \qquad (z \in \mathbb{U})$$

for all subordinants $q(z)$ of (1.6) is said to be the best subordinant. We note that the best subordinant is unique up to a rotation of $\mathbb{U}$.

Definition 1.4 ([6, 32]). Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and $n \in \mathbb{N} \setminus \{1\}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, u; z) \notin \Omega$$

whenever

$$r = q(\xi), \qquad s = k\xi q'(\xi), \qquad \Re\left(\frac{t}{s} + 1\right) \geqq k\Re\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right),$$

and

$$\Re\left(\frac{u}{s}\right) \geqq k^2\Re\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(q)$ and $k \geqq n$.

If $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$, $q \in \mathcal{Q}$ and $n \in \mathbb{N} \setminus \{1\}$, then we obtain

$$\psi\left(q(\xi), k\xi q'(\xi); z\right) \notin \Omega \quad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ k \geqq n).$$

If $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$, $q \in \mathcal{Q}$ and $n \in \mathbb{N} \setminus \{1\}$, then we have

$$\psi(r, s, t; z) \notin \Omega,$$

whenever $r = q(\xi)$, $s = k\xi q'(\xi)$, and

$$\Re\left(\frac{t}{s} + 1\right) \geqq k\Re\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right) \quad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ k \geqq n).$$

Definition 1.5 ([33]). Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}[a, n]$, and $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, u; \xi) \in \Omega,$$

whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m}, \quad \Re\left(\frac{t}{s} + 1\right) \leqq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\Re\left(\frac{u}{s}\right) \leqq \frac{1}{m^2}\Re\left(\frac{z^2 q'''(z)}{q'(z)}\right),$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U}$, and $m \geqq n \geqq 2$.

If $\psi : \mathbb{C}^2 \times \overline{\mathbb{U}} \to \mathbb{C}$ and $q \in \mathcal{H}[a, n]$, then we have

$$\psi\left(q(z), \frac{zq'(z)}{m}; \xi\right) \in \Omega \quad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ m \geqq n \geqq 2).$$

If $\psi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$, then we get

$$\psi(r, s, t; \xi) \in \Omega,$$

whenever $r = q(z)$, $s = \frac{zq'(z)}{m}$, and

$$\Re\left(\frac{t}{s} + 1\right) \leqq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right) \quad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ m \geqq n \geqq 2).$$

For the class of admissible functions $\Psi_n[\Omega, q]$, Antonino and Miller [6] proved the following result.

Lemma 1.6 ([6]). *Let $p \in \mathcal{H}[a, n]$ with $n \in \mathbb{N} \setminus \{1\}$. Also let $q \in \mathcal{Q}(a)$ satisfying*

$$\Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geqq 0 \qquad \text{and} \qquad \left|\frac{zp'(z)}{q'(\xi)}\right| \leqq k,$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(q)$ and $k \geqq n$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$, and

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) \in \Omega,$$

then

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}).$$

On the other word, Tang et al. [33] (see also [34]) obtained the following result for the class of admissible functions $\Psi'_n[\Omega, q]$.

Lemma 1.7 ([33, 34]). *Let $q \in \mathcal{H}[a, n]$ and $\psi \in \Psi'_n[\Omega, q]$. If $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ is univalent in $\mathbb{U}$ and $p \in \mathcal{Q}(a)$ satisfies*

$$\Re\left(\frac{zq''(z)}{q'(z)}\right) \geqq 0 \quad \text{and} \quad \left|\frac{zp'(z)}{q'(z)}\right| \leqq m \quad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ m \geqq n \geqq 2),$$

then

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in \mathbb{U}\right\}$$

implies that

$$q(z) \prec p(z) \quad (z \in \mathbb{U}).$$

In recent years, several authors studied first-order and second-order differential subordination and superordination problems associated with various linear and nonlinear operators and obtained many interesting results, the interested readers can see, for example, [1–5, 15–17, 26, 27, 29, 30, 32–35]. More recently, Antonino and Miller [6] introduced the notion of third-order differential subordination, and Tang and Deniz [32] studied the third-order differential subordination results for analytic functions involving the generalized Bessel functions. Later on, Tang et al. [33] introduced the notion of third-order differential superordination and also studied the corresponding third-order differential superordination involving the generalized Bessel functions. Based on [6] and [33], Tang et al. [34] considered third-order differential subordination and superordination results for meromorphically multivalent functions associated with the Liu-Srivastava operator. Here, in the present paper, we aim to study third-order differential sandwich-type results of multivalent analytic functions involving the Liu-Owa integral operator $Q^{\alpha}_{\beta,p}$ defined by (1.2).

2. Third-order differential subordination of $Q^{\alpha}_{\beta,p}$

We define the following class of admissible functions that will be needed in proving our first result.

Definition 2.1. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_0 \cap \mathcal{H}[0, p]$. The class of admissible functions $\Phi_Q[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(b, c, d, e; z) \notin \Omega,$$

whenever

$$\begin{aligned}
&b = q(\xi),\\
&c = \frac{k\xi q'(\xi) + (\alpha+\beta-1)q(\xi)}{\alpha+\beta+p-1},\\
&\Re\left(\frac{(\alpha+\beta+p-1)[c(2-\alpha-\beta)+d(\alpha+\beta+p-2)]}{(\alpha+\beta+p-1)c-(\alpha+\beta-1)b} - (\alpha+\beta-1)\right) \geqq k\Re\left(\frac{\xi q''(\xi)}{q'(\xi)}+1\right)
\end{aligned}$$

and

$$\begin{aligned}
\Re\Bigg(&\frac{(\alpha+\beta+p-1)(\alpha+\beta+p-2)[(\alpha+\beta+p-3)e-3(\alpha+\beta-1)d]}{(\alpha+\beta+p-1)c-(\alpha+\beta-1)b}\\
&+\frac{2(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)b}{(\alpha+\beta+p-1)c-(\alpha+\beta-1)b} + 3(\alpha+\beta)(\alpha+\beta-1)\Bigg) \geqq k^2\Re\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),
\end{aligned}$$

where $z \in \mathbb{U}$, $p \in \mathbb{N}$, $\alpha \geqq 3$, $\beta > -1$, $\xi \in \partial\mathbb{U} \setminus E(q)$, and $k \in \mathbb{N} \setminus \{1\}$.

Theorem 2.2. *Suppose that* $\phi \in \Phi_Q[\Omega, q]$. *If* $f \in \mathcal{A}(p)$ *and* $q \in \mathcal{Q}_0$ *satisfy*

$$\Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geqq 0, \qquad \left|\frac{Q_{\beta,p}^{\alpha-1}f(z)}{q'(\xi)}\right| \leqq k, \tag{2.1}$$

and

$$\left\{\phi\left(Q_{\beta,p}^{\alpha}f(z), Q_{\beta,p}^{\alpha-1}f(z), Q_{\beta,p}^{\alpha-2}f(z), Q_{\beta,p}^{\alpha-3}f(z); z\right) : z \in \mathbb{U}\right\} \subset \Omega, \tag{2.2}$$

then

$$Q_{\beta,p}^{\alpha}f(z) \prec q(z) \qquad (z \in \mathbb{U}).$$

Proof. Let

$$\mathfrak{p}(z) = Q_{\beta,p}^{\alpha}f(z). \tag{2.3}$$

Then, differentiating (2.3) on z and from (1.4), we get

$$Q_{\beta,p}^{\alpha-1}f(z) = \frac{z\mathfrak{p}'(z) + (\alpha+\beta-1)\mathfrak{p}(z)}{\alpha+\beta+p-1}. \tag{2.4}$$

After some computations, we show that

$$Q_{\beta,p}^{\alpha-2}f(z) = \frac{z^2\mathfrak{p}''(z) + 2(\alpha+\beta-1)z\mathfrak{p}'(z) + (\alpha+\beta-1)(\alpha+\beta-2)\mathfrak{p}(z)}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)} \tag{2.5}$$

and

$$\begin{aligned}
Q_{\beta,p}^{\alpha-3}f(z) = &\frac{z^3\mathfrak{p}'''(z) + 3(\alpha+\beta-1)z^2\mathfrak{p}''(z) + 3(\alpha+\beta-1)(\alpha+\beta-2)z\mathfrak{p}'(z)}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)}\\
&+\frac{(\alpha+\beta-1)(\alpha+\beta-2)(\alpha+\beta-3)\mathfrak{p}(z)}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)}.
\end{aligned} \tag{2.6}$$

Now, we define the transformation $\mathbb{C}^4 \to \mathbb{C}$ as follows:

$$\begin{aligned}
&b(r,s,t,u) = r, \quad c(r,s,t,u) = \frac{s+(\alpha+\beta-1)r}{\alpha+\beta+p-1},\\
&d(r,s,t,u) = \frac{t+2(\alpha+\beta-1)s+(\alpha+\beta-1)(\alpha+\beta-2)r}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)},\\
&e(r,s,t,u) = \frac{u+3(\alpha+\beta-1)t+3(\alpha+\beta-1)(\alpha+\beta-2)s}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)}\\
&\qquad\qquad + \frac{(\alpha+\beta-1)(\alpha+\beta-2)(\alpha+\beta-3)r}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)}.
\end{aligned} \tag{2.7}$$

Assume that

$$\begin{aligned}\psi(r,s,t,u;z) &= \phi(b,c,d,e;z)\\ &= \phi\Bigg(r, \frac{s+(\alpha+\beta-1)r}{\alpha+\beta+p-1}, \frac{t+2(\alpha+\beta-1)s+(\alpha+\beta-1)(\alpha+\beta-2)r}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)},\\ &\quad \frac{u+3(\alpha+\beta-1)t+3(\alpha+\beta-1)(\alpha+\beta-2)s}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)}\\ &\quad + \frac{(\alpha+\beta-1)(\alpha+\beta-2)(\alpha+\beta-3)r}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)};z\Bigg).\end{aligned} \tag{2.8}$$

Using (2.3)-(2.6), we find from (2.8) that

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) = \phi\left(Q^{\alpha}_{\beta,p}f(z), Q^{\alpha-1}_{\beta,p}f(z), Q^{\alpha-2}_{\beta,p}f(z), Q^{\alpha-3}_{\beta,p}f(z); z\right). \tag{2.9}$$

So, clearly, (2.2) reduces to

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) \in \Omega.$$

We also notice that

$$\frac{t}{s}+1 = \frac{(\alpha+\beta+p-1)[c(2-\alpha-\beta)+d(\alpha+\beta+p-2)]}{(\alpha+\beta+p-1)c-(\alpha+\beta-1)b} - (\alpha+\beta-1)$$

and

$$\begin{aligned}\frac{u}{s} &= \frac{(\alpha+\beta+p-1)(\alpha+\beta+p-2)[(\alpha+\beta+p-3)e-3(\alpha+\beta-1)d]}{(\alpha+\beta+p-1)c-(\alpha+\beta-1)b}\\ &\quad + \frac{2(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)b}{(\alpha+\beta+p-1)c-(\alpha+\beta-1)b} + 3(\alpha+\beta)(\alpha+\beta-1).\end{aligned}$$

Thus, the admissibility condition for $\phi \in \Phi_Q[\Omega, q]$ in Definition 2.1 is equivalent to that for $\psi \in \Psi_2[\Omega, q]$ as shown in Definition 1.4 with $n = 2$. Hence, by applying (2.1) and Lemma 1.6, we obtain

$$p(z) \prec q(z) \qquad (z \in \mathbb{U})$$

or

$$Q^{\alpha}_{\beta,p}f(z) \prec q(z) \qquad (z \in \mathbb{U}),$$

which completes the proof of Theorem 2.2. □

When the behavior of $q(z)$ on $\partial\mathbb{U}$ is not known, we easily get the following result.

Corollary 2.3. *Let $\Omega \subset \mathbb{C}$ and q be univalent in $\mathbb{U}$ with $q(0) = 0$. Assume that $\phi \in \Phi_Q[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}(p)$ and q_ρ satisfy*

$$\Re\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geqq 0, \qquad \left|\frac{Q^{\alpha-1}_{\beta,p}f(z)}{q_\rho'(\xi)}\right| \leqq k \quad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q_\rho))$$

and

$$\phi\left(Q^{\alpha}_{\beta,p}f(z), Q^{\alpha-1}_{\beta,p}f(z), Q^{\alpha-2}_{\beta,p}f(z), Q^{\alpha-3}_{\beta,p}f(z); z\right) \in \Omega,$$

then

$$Q^{\alpha}_{\beta,p}f(z) \prec q(z) \qquad (z \in \mathbb{U}).$$

Proof. From Theorem 2.2, we have

$$Q^{\alpha}_{\beta,p}f(z) \prec q_\rho(z) \qquad (z \in \mathbb{U}).$$

Since

$$q_\rho(z) \prec q(z) \qquad (z \in \mathbb{U}),$$

then we deduce Corollary 2.3. □

If $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In such a case, we write $\Phi_Q[h(\mathbb{U}), q]$ as $\Phi_Q[h, q]$ and easily get the following two results.

Theorem 2.4. *Let $\phi \in \Phi_Q[h, q]$. If $f \in \mathcal{A}(p)$ and $q \in \mathcal{Q}_0$ satisfy*

$$\Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geqq 0, \qquad \left|\frac{Q^{\alpha-1}_{\beta,p} f(z)}{q'(\xi)}\right| \leqq k,$$

and

$$\phi\big(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z), Q^{\alpha-3}_{\beta,p} f(z); z\big) \prec h(z), \tag{2.10}$$

then

$$Q^{\alpha}_{\beta,p} f(z) \prec q(z) \qquad (z \in \mathbb{U}).$$

Corollary 2.5. *Let $\Omega \subset \mathbb{C}$ and q be univalent in $\mathbb{U}$ with $q(0) = 0$. Also let $\phi \in \Phi_Q[h, q_\rho]$ for some $\rho \in (0, 1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}(p)$ and q_ρ satisfy*

$$\Re\left(\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right) \geqq 0, \qquad \left|\frac{Q^{\alpha-1}_{\beta,p} f(z)}{q_\rho'(\xi)}\right| \leqq k \qquad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q_\rho))$$

and

$$\phi\big(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z), Q^{\alpha-3}_{\beta,p} f(z); z\big) \prec h(z),$$

then

$$Q^{\alpha}_{\beta,p} f(z) \prec q(z) \qquad (z \in \mathbb{U}).$$

Next, we yield the best dominant of the differential subordination (2.10).

Theorem 2.6. *Let h be univalent in $\mathbb{U}$. Also let $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and ψ be given by (2.8). Assume that the differential equation:*

$$\psi\left(q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z\right) = h(z) \tag{2.11}$$

has a solution $q(z)$ with $q(0) = 0$ satisfying the condition (2.1). If $f \in \mathcal{A}(p)$ satisfies (2.10) and $\phi\big(Q^{\alpha}_{\beta,p} f(z), Q^{\alpha-1}_{\beta,p} f(z), Q^{\alpha-2}_{\beta,p} f(z), Q^{\alpha-3}_{\beta,p} f(z); z\big)$ is analytic in $\mathbb{U}$, then

$$Q^{\alpha}_{\beta,p} f(z) \prec q(z) \qquad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant.

Proof. By using Theorem 2.2, we deduce that q is a dominant of (2.10). Because q satisfies (2.11), so q is also a solution of (2.10). Hence, q will be dominated by all dominants and also q is the best dominant. □

In particular, when $q(z) = Mz \ \ (M > 0)$, we write $\Phi_Q[\Omega, q]$ as $\Phi_Q[\Omega, M]$ and give the corresponding definition.

Definition 2.7. Let $\Omega \subset \mathbb{C}$. Also let $p \in \mathbb{N}$, $\alpha \geqq 3$, $\beta > -1$, and $M > 0$. The class $\Phi_Q[\Omega, M]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ such that

$$\begin{aligned}\phi\Bigg(&Me^{i\theta}, \frac{k+\alpha+\beta-1}{\alpha+\beta+p-1} Me^{i\theta}, \frac{L + (\alpha+\beta-1)(2k+\alpha+\beta-2)Me^{i\theta}}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)},\\ &\frac{N + 3(\alpha+\beta-1)L + (\alpha+\beta-1)(\alpha+\beta-2)(3k+\alpha+\beta-3)Me^{i\theta}}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)}; z\Bigg) \notin \Omega,\end{aligned} \tag{2.12}$$

whenever $z \in \mathbb{U}$, $\Re(Le^{-i\theta}) \geqq (k-1)kM$, and $\Re(Ne^{-i\theta}) \geqq 0$ for all $\theta \in \mathbb{R}$ and $k \in \mathbb{N} \setminus \{1\}$.

Corollary 2.8. *Let* $\phi \in \Phi_Q[\Omega, M]$. *If* $f \in \mathcal{A}(p)$ *satisfies*

$$\left|Q^{\alpha-1}_{\beta,p}f(z)\right| \leqq kM \qquad (z \in \mathbb{U};\ k \in \mathbb{N}\setminus\{1\};\ M > 0)$$

and

$$\phi\left(Q^{\alpha}_{\beta,p}f(z), Q^{\alpha-1}_{\beta,p}f(z), Q^{\alpha-2}_{\beta,p}f(z), Q^{\alpha-3}_{\beta,p}f(z); z\right) \in \Omega,$$

then

$$\left|Q^{\alpha}_{\beta,p}f(z)\right| < M \qquad (z \in \mathbb{U};\ M > 0).$$

Specially, if

$$\Omega = q(\mathbb{U}) = \{\omega : |\omega| < M\ \ (M > 0)\},$$

then we denote $\Phi_Q[\Omega, M]$ by $\Phi_Q[M]$. Corollary 2.8 can now be rewritten as below.

Corollary 2.9. *Let* $\phi \in \Phi_Q[M]$. *If* $f \in \mathcal{A}(p)$ *satisfies*

$$\left|Q^{\alpha-1}_{\beta,p}f(z)\right| \leqq kM \qquad (z \in \mathbb{U};\ k \in \mathbb{N}\setminus\{1\};\ M > 0)$$

and

$$\left|\phi\left(Q^{\alpha}_{\beta,p}f(z), Q^{\alpha-1}_{\beta,p}f(z), Q^{\alpha-2}_{\beta,p}f(z), Q^{\alpha-3}_{\beta,p}f(z); z\right)\right| < M,$$

then

$$\left|Q^{\alpha}_{\beta,p}f(z)\right| < M \qquad (z \in \mathbb{U};\ M > 0).$$

Corollary 2.10. *Let* $p \in \mathbb{N}$, $k \in \mathbb{N}\setminus\{1\}$, $\alpha \geqq 3$, $\beta > -1$ *with* $\alpha + \beta \geqq \frac{2-p-k}{2}$ *and* $M > 0$. *If* $f \in \mathcal{A}(p)$ *satisfies*

$$\left|Q^{\alpha-1}_{\beta,p}f(z)\right| \leqq M \qquad (z \in \mathbb{U}),$$

then

$$\left|Q^{\alpha}_{\beta,p}f(z)\right| < M \qquad (z \in \mathbb{U}).$$

Proof. If we put

$$\phi\,(b, c, d, e; z) = c = \frac{k+\alpha+\beta-1}{\alpha+\beta+p-1}Me^{i\theta},$$

then from Corollary 2.9, we get Corollary 2.10. □

Corollary 2.11. *Let* $p \in \mathbb{N}$, $\alpha \geqq 3$, $\beta > -1$, *and* $M > 0$. *If* $f \in \mathcal{A}(p)$ *satisfies*

$$\left|Q^{\alpha-1}_{\beta,p}f(z)\right| \leqq kM \qquad (z \in \mathbb{U};\ k \in \mathbb{N}\setminus\{1\})$$

and

$$\left|Q^{\alpha-1}_{\beta,p}f(z) + \left(\frac{p}{\alpha+\beta+p-1} - 1\right) Q^{\alpha}_{\beta,p}f(z)\right| < \frac{2M}{|\alpha+\beta+p-1|} \qquad (z \in \mathbb{U}),$$

then

$$\left|Q^{\alpha}_{\beta,p}f(z)\right| < M \qquad (z \in \mathbb{U}).$$

Proof. Suppose that

$$\phi\,(b, c, d, e; z) = c + \left(\frac{p}{\alpha+\beta+p-1} - 1\right) b \qquad \text{and} \qquad \Omega = h(\mathbb{U}),$$

where

$$h(z) = \frac{2Mz}{|\alpha+\beta+p-1|} \qquad (M > 0).$$

To apply Corollary 2.8, we must show that $\phi \in \Phi_Q[\Omega, M]$, that is, that (2.12) is satisfied. In fact, it follows easily, because of

$$\begin{aligned}
&\left|\phi\left(Me^{i\theta}, \frac{k+\alpha+\beta-1}{\alpha+\beta+p-1}Me^{i\theta}, \frac{L+(\alpha+\beta-1)(2k+\alpha+\beta-2)Me^{i\theta}}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)},\right.\right.\\
&\qquad\left.\left.\frac{N+3(\alpha+\beta-1)L+(\alpha+\beta-1)(\alpha+\beta-2)(3k+\alpha+\beta-3)Me^{i\theta}}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)}; z\right)\right|\\
&\quad= \left|\frac{k+\alpha+\beta-1}{\alpha+\beta+p-1}Me^{i\theta} + \left(\frac{p}{\alpha+\beta+p-1}-1\right)Me^{i\theta}\right|\\
&\quad= \frac{kM}{|\alpha+\beta+p-1|} \geqq \frac{2M}{|\alpha+\beta+p-1|},
\end{aligned}$$

whenever $z \in \mathbb{U}$, $\theta \in \mathbb{R}$, and $k \in \mathbb{N}\setminus\{1\}$. The required result now follows from Corollary 2.8. □

Definition 2.12. Let $\Omega \subset \mathbb{C}$ and $q \in \mathcal{Q}_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_{Q,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(b, c, d, e; z) \notin \Omega,$$

whenever

$$\begin{aligned}
&b = q(\xi), \quad c = \frac{k\xi q'(\xi) + (\alpha+\beta+p-2)q(\xi)}{\alpha+\beta+p-1},\\
&\Re\left(\frac{(\alpha+\beta+p-1)[c(3-\alpha-\beta-p)+(\alpha+\beta+p-2)d]}{(\alpha+\beta+p-1)c-(\alpha+\beta+p-2)b} - (\alpha+\beta+p-2)\right) \geqq k\Re\left(\frac{\xi q''(\xi)}{q'(\xi)}+1\right)
\end{aligned}$$

and

$$\begin{aligned}
&\Re\left(\frac{(\alpha+\beta+p-1)(\alpha+\beta+p-2)[(\alpha+\beta+p-3)e-3(\alpha+\beta+p-2)d]}{(\alpha+\beta+p-1)c-(\alpha+\beta+p-2)b}\right.\\
&\qquad\left.+\frac{2(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)b}{(\alpha+\beta+p-1)c-(\alpha+\beta+p-2)b} + 3(\alpha+\beta+p-1)(\alpha+\beta+p-2)\right)\\
&\quad\geqq k^2\Re\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right),
\end{aligned}$$

where $z \in \mathbb{U}$, $p \in \mathbb{N}$, $\alpha \geqq 3$, $\beta > -1$, $\xi \in \partial\mathbb{U}\setminus E(q)$, and $k \in \mathbb{N}\setminus\{1\}$.

Theorem 2.13. *Let $\phi \in \Phi_{Q,1}[\Omega, q]$. If $f \in \mathcal{A}(p)$ and $q \in \mathcal{Q}_0$ satisfy*

$$\Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geqq 0, \qquad \left|\frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}q'(\xi)}\right| \leqq k, \tag{2.13}$$

and

$$\left\{\phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-3}_{\beta,p}f(z)}{z^{p-1}}; z\right) : z \in \mathbb{U}\right\} \subset \Omega, \tag{2.14}$$

then

$$\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \prec q(z) \qquad (z \in \mathbb{U}).$$

Proof. Let

$$\mathfrak{p}(z) = \frac{Q^{\alpha}_{\beta,p} f(z)}{z^{p-1}}. \tag{2.15}$$

Then, from (1.4) and (2.15), we get

$$\frac{Q^{\alpha-1}_{\beta,p} f(z)}{z^{p-1}} = \frac{z\mathfrak{p}'(z) + (\alpha+\beta+p-2)\mathfrak{p}(z)}{\alpha+\beta+p-1}. \tag{2.16}$$

After a simper computation, we have

$$\frac{Q^{\alpha-2}_{\beta,p} f(z)}{z^{p-1}} = \frac{z^2\mathfrak{p}''(z) + 2(\alpha+\beta+p-2)z\mathfrak{p}'(z)}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)} + \frac{(\alpha+\beta+p-3)\mathfrak{p}(z)}{\alpha+\beta+p-1} \tag{2.17}$$

and

$$\begin{aligned}\frac{Q^{\alpha-3}_{\beta,p} f(z)}{z^{p-1}} &= \frac{z^3\mathfrak{p}'''(z) + 3(\alpha+\beta+p-2)z^2\mathfrak{p}''(z) + 3(\alpha+\beta+p-2)(\alpha+\beta+p-3)z\mathfrak{p}'(z)}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)} \\ &\quad + \frac{(\alpha+\beta+p-4)\mathfrak{p}(z)}{\alpha+\beta+p-1}.\end{aligned} \tag{2.18}$$

Next, we introduce the transformation $\mathbb{C}^4 \to \mathbb{C}$ as below:

$$\begin{aligned} b(r,s,t,u) &= r, \quad c(r,s,t,u) = \frac{s+(\alpha+\beta+p-2)r}{\alpha+\beta+p-1}, \\ d(r,s,t,u) &= \frac{t+2(\alpha+\beta+p-2)s}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)} + \frac{(\alpha+\beta+p-3)r}{\alpha+\beta+p-1}, \\ e(r,s,t,u) &= \frac{u+3(\alpha+\beta+p-2)t+3(\alpha+\beta+p-2)(\alpha+\beta+p-3)s}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)} + \frac{(\alpha+\beta+p-4)r}{\alpha+\beta+p-1}. \end{aligned} \tag{2.19}$$

Then, upon setting

$$\begin{aligned} \psi(r,s,t,u;z) &= \phi(b,c,d,e;z) \\ &= \phi\Bigg(r, \frac{s+(\alpha+\beta+p-2)r}{\alpha+\beta+p-1}, \frac{t+2(\alpha+\beta+p-2)s}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)} + \frac{(\alpha+\beta+p-3)r}{\alpha+\beta+p-1}, \\ &\qquad \frac{u+3(\alpha+\beta+p-2)t+3(\alpha+\beta+p-2)(\alpha+\beta+p-3)s}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)} + \frac{(\alpha+\beta+p-4)r}{\alpha+\beta+p-1}; z\Bigg). \end{aligned} \tag{2.20}$$

Using (2.15)-(2.18), we obtain from (2.20) that

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) = \phi\left(\frac{Q^{\alpha}_{\beta,p} f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p} f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p} f(z)}{z^{p-1}}, \frac{Q^{\alpha-3}_{\beta,p} f(z)}{z^{p-1}}; z\right), \tag{2.21}$$

so that (2.14) becomes

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) \in \Omega.$$

Again we notice that

$$\frac{t}{s} + 1 = \frac{(\alpha+\beta+p-1)[c(3-\alpha-\beta-p) + (\alpha+\beta+p-2)d]}{(\alpha+\beta+p-1)c - (\alpha+\beta+p-2)b} - (\alpha+\beta+p-2)$$

and

$$\frac{u}{s} = \frac{(\alpha+\beta+p-1)(\alpha+\beta+p-2)[(\alpha+\beta+p-3)e - 3(\alpha+\beta+p-2)d]}{(\alpha+\beta+p-1)c - (\alpha+\beta+p-2)b}$$

$$+\frac{2(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)b}{(\alpha+\beta+p-1)c-(\alpha+\beta+p-2)b}+3(\alpha+\beta+p-1)(\alpha+\beta+p-2).$$

Therefore, we clearly know the admissibility condition for $\phi \in \Phi_{Q,1}[\Omega, q]$ in Definition 2.12 is equivalent to that for $\psi \in \Psi_2[\Omega, q]$ as presented in Definition 1.4 with $n = 2$. Thus, in view of (2.13) and Lemma 1.6, we have

$$\mathfrak{p}(z) \prec q(z) \qquad (z \in \mathbb{U})$$

or

$$\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \prec q(z) \qquad (z \in \mathbb{U}).$$

□

If $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, we write $\Phi_{Q,1}[h(\mathbb{U}), q]$ as $\Phi_{Q,1}[h, q]$ and get an immediate consequence of Theorem 2.13.

Theorem 2.14. *Let* $\phi \in \Phi_{Q,1}[h, q]$. *If* $f \in \mathcal{A}(p)$ *satisfies* (2.13), *then*

$$\phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-3}_{\beta,p}f(z)}{z^{p-1}}; z\right) \prec h(z)$$

implies that

$$\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \prec q(z) \qquad (z \in \mathbb{U}).$$

In particular, when $q(z) = 1 + Mz \quad (M > 0)$, we denote $\Phi_{Q,1}[\Omega, q]$ by $\Phi_{Q,1}[\Omega, M]$ and have the corresponding definition.

Definition 2.15. Let $\Omega \subset \mathbb{C}$. Also let $p \in \mathbb{N}$, $\alpha \geqq 3$, $\beta > -1$ and $M > 0$. The class $\Phi_{Q,1}[\Omega, M]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ such that

$$\begin{aligned}\phi\Bigg(&1+Me^{i\theta}, \frac{(k+\alpha+\beta+p-2)Me^{i\theta}+(\alpha+\beta+p-2)}{\alpha+\beta+p-1},\\ &\frac{L+(\alpha+\beta+p-2)(2k+\alpha+\beta+p-3)Me^{i\theta}}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)}+\frac{\alpha+\beta+p-3}{\alpha+\beta+p-1},\\ &\frac{N+3(\alpha+\beta+p-2)L+(\alpha+\beta+p-2)(\alpha+\beta+p-3)(3k+\alpha+\beta+p-4)Me^{i\theta}}{(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)}\\ &+\frac{\alpha+\beta+p-4}{\alpha+\beta+p-1}; z\Bigg) \notin \Omega,\end{aligned}$$

whenever $z \in \mathbb{U}$, $\Re(Le^{-i\theta}) \geqq (k-1)kM$, and $\Re(Ne^{-i\theta}) \geqq 0$ for all $\theta \in \mathbb{R}$ and $k \in \mathbb{N} \setminus \{1\}$.

Corollary 2.16. *Let* $\phi \in \Phi_{Q,1}[\Omega, M]$. *If* $f \in \mathcal{A}(p)$ *satisfies*

$$\left|\frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}\right| \leqq kM \qquad (z \in \mathbb{U};\ k \in \mathbb{N} \setminus \{1\};\ M > 0)$$

and

$$\phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-3}_{\beta,p}f(z)}{z^{p-1}}; z\right) \in \Omega,$$

then

$$\left|\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} - 1\right| < M \qquad (z \in \mathbb{U};\ M > 0).$$

Specially, if

$$\Omega = q(\mathbb{U}) = \{\omega : |\omega - 1| < M \quad (M > 0)\},$$

then we denote $\Phi_{Q,1}[\Omega, M]$ simply by $\Phi_{Q,1}[M]$. Thus, Corollary 2.16 can be restated as below.

Corollary 2.17. *Let* $\phi \in \Phi_{Q,1}[M]$*. If* $f \in \mathcal{A}(p)$ *satisfies*

$$\left|\frac{Q^{\alpha-1}_{\beta,p} f(z)}{z^{p-1}}\right| \leqq kM \qquad (z \in \mathbb{U};\ k \in \mathbb{N} \setminus \{1\};\ M > 0)$$

and

$$\left|\phi\left(\frac{Q^{\alpha}_{\beta,p} f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p} f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p} f(z)}{z^{p-1}}, \frac{Q^{\alpha-3}_{\beta,p} f(z)}{z^{p-1}}; z\right) - 1\right| < M,$$

then

$$\left|\frac{Q^{\alpha}_{\beta,p} f(z)}{z^{p-1}} - 1\right| < M \qquad (z \in \mathbb{U};\ M > 0).$$

Corollary 2.18. *Let* $p \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{1\}$, $\alpha \geqq 3$, $\beta > -1$ *with* $\alpha + \beta \geqq \frac{3-2p-k}{2}$ *and* $M > 0$*. If* $f \in \mathcal{A}(p)$ *satisfies*

$$\left|\frac{Q^{\alpha-1}_{\beta,p} f(z)}{z^{p-1}}\right| \leqq kM \qquad (z \in \mathbb{U};\ k \in \mathbb{N} \setminus \{1\};\ M > 0)$$

and

$$\left|\frac{Q^{\alpha-1}_{\beta,p} f(z)}{z^{p-1}} - 1\right| < M \qquad (z \in \mathbb{U};\ M > 0),$$

then

$$\left|\frac{Q^{\alpha}_{\beta,p} f(z)}{z^{p-1}} - 1\right| < M \qquad (z \in \mathbb{U};\ M > 0).$$

Proof. If we take

$$\phi\,(b, c, d, e; z) = c = \frac{(k + \alpha + \beta + p - 2)Me^{i\theta} + (\alpha + \beta + p - 2)}{\alpha + \beta + p - 1}$$

in Corollary 2.16, we easily get Corollary 2.18. □

3. Third-order differential superordination and sandwich-type results of $Q^{\alpha}_{\beta,p}$

In this section, we investigate the third-order differential superordination and sandwich-type results of the Liu-Owa operator $Q^{\alpha}_{\beta,p}$ defined by (1.2). For this, we first give the definition of the class of admissible functions.

Definition 3.1. Let $\Omega \subset \mathbb{C}$ and $q \in \mathcal{H}[0, p]$ with $q'(z) \neq 0$. The class of admissible functions $\Phi'_Q[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(b, c, d, e; \xi) \in \Omega,$$

whenever

$$b = q(z), \quad c = \frac{zq'(z) + m(\alpha + \beta - 1)q(z)}{(\alpha + \beta + p - 1)m},$$

$$\Re\left(\frac{(\alpha + \beta + p - 1)[c(2 - \alpha - \beta) + d(\alpha + \beta + p - 2)]}{(\alpha + \beta + p - 1)c - (\alpha + \beta - 1)b} - (\alpha + \beta - 1)\right) \leqq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

and

$$\Re\left(\frac{(\alpha+\beta+p-1)(\alpha+\beta+p-2)[(\alpha+\beta+p-3)e-3(\alpha+\beta-1)d]}{(\alpha+\beta+p-1)c-(\alpha+\beta-1)b}\right.$$
$$\left.+\frac{2(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)b}{(\alpha+\beta+p-1)c-(\alpha+\beta-1)b}+3(\alpha+\beta)(\alpha+\beta-1)\right)\leqq\frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z\in\mathbb{U}$, $p\in\mathbb{N}$, $\alpha\geqq 3$, $\beta>-1$, $\xi\in\partial\mathbb{U}$, and $m\in\mathbb{N}\setminus\{1\}$.

Theorem 3.2. *Let $\phi\in\Phi'_Q[\Omega,q]$. If $f\in\mathcal{A}(p)$ and $Q^{\alpha}_{\beta,p}f(z)\in\mathcal{Q}_0$ satisfy*

$$\Re\left(\frac{zq''(z)}{q'(z)}\right)\geqq 0,\qquad\left|\frac{Q^{\alpha-1}_{\beta,p}f(z)}{q'(z)}\right|\leqq m,\tag{3.1}$$

and

$$\phi\left(Q^{\alpha}_{\beta,p}f(z),Q^{\alpha-1}_{\beta,p}f(z),Q^{\alpha-2}_{\beta,p}f(z),Q^{\alpha-3}_{\beta,p}f(z);z\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega\subset\left\{\phi\left(Q^{\alpha}_{\beta,p}f(z),Q^{\alpha-1}_{\beta,p}f(z),Q^{\alpha-2}_{\beta,p}f(z),Q^{\alpha-3}_{\beta,p}f(z);z\right):z\in\mathbb{U}\right\}\tag{3.2}$$

implies that

$$q(z)\prec Q^{\alpha}_{\beta,p}f(z)\quad(z\in\mathbb{U}).$$

Proof. Suppose that the functions $\mathfrak{p}(z)$ and ψ are given by (2.3) and (2.8). Because $\phi\in\Phi'_Q[\Omega,q]$, so (2.9) and (3.2) yield

$$\Omega\subset\left\{\psi(\mathfrak{p}(z),z\mathfrak{p}'(z),z^2\mathfrak{p}''(z),z^3\mathfrak{p}'''(z);z):z\in\mathbb{U}\right\}.$$

We see, from (2.7), that the admissible condition for $\phi\in\Phi'_Q[\Omega,q]$ in Definition 3.1 is equivalent to that for ψ as shown in Definition 1.5 with $n=2$. Hence $\psi\in\Psi'_2[\Omega,q]$, and from (3.1) and Lemma 1.7, we get

$$q(z)\prec\mathfrak{p}(z)\quad(z\in\mathbb{U})$$

or

$$q(z)\prec Q^{\alpha}_{\beta,p}f(z)\quad(z\in\mathbb{U}).$$

We complete the proof of Theorem 3.2. □

If $\Omega\neq\mathbb{C}$ is a simply-connected domain and $\Omega=h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω, then we write $\Phi'_Q[h(\mathbb{U}),q]$ as $\Phi'_Q[h,q]$. Proceedings similarly as in the previous section, the following theorem is an immediate consequence of Theorem 3.2.

Theorem 3.3. *Let $\phi\in\Phi'_Q[h,q]$. Also let h be analytic in $\mathbb{U}$. If $f\in\mathcal{A}(p)$ and $Q^{\alpha}_{\beta,p}f(z)\in\mathcal{Q}_0$ satisfy the condition (3.1) and*

$$\phi\left(Q^{\alpha}_{\beta,p}f(z),Q^{\alpha-1}_{\beta,p}f(z),Q^{\alpha-2}_{\beta,p}f(z),Q^{\alpha-3}_{\beta,p}f(z);z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z)\prec\phi\left(Q^{\alpha}_{\beta,p}f(z),Q^{\alpha-1}_{\beta,p}f(z),Q^{\alpha-2}_{\beta,p}f(z),Q^{\alpha-3}_{\beta,p}f(z);z\right)\tag{3.3}$$

implies that

$$q(z)\prec Q^{\alpha}_{\beta,p}f(z)\quad(z\in\mathbb{U}).$$

The following theorem proves the existence of the best subordinant of (3.3) for an appropriate ϕ.

Theorem 3.4. *Let h be analytic in $\mathbb{U}$, and assume that $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ and ψ are given by (2.8). Suppose that the differential equation*

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z)$$

has a solution $q(z) \in \mathcal{Q}_0$. If $f \in \mathcal{A}(p)$ and $Q^{\alpha}_{\beta,p}f(z) \in \mathcal{Q}_0$ satisfy the condition (3.1) and

$$\phi\left(Q^{\alpha}_{\beta,p}f(z), Q^{\alpha-1}_{\beta,p}f(z), Q^{\alpha-2}_{\beta,p}f(z), Q^{\alpha-3}_{\beta,p}f(z); z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(Q^{\alpha}_{\beta,p}f(z), Q^{\alpha-1}_{\beta,p}f(z), Q^{\alpha-2}_{\beta,p}f(z), Q^{\alpha-3}_{\beta,p}f(z); z\right)$$

implies that

$$q(z) \prec Q^{\alpha}_{\beta,p}f(z) \quad (z \in \mathbb{U})$$

and $q(z)$ is the best subordinant.

Proof. The proof of Theorem 3.4 is similar to that of Theorem 2.6 and now we choose to omit it. □

If we combine Theorems 2.4 and 3.3, then we obtain the following sandwich-type result.

Corollary 3.5. *Let h_1 and q_1 be analytic functions in $\mathbb{U}$. Also let h_2 be univalent in $\mathbb{U}$, $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$, and $\phi \in \Phi_Q[h_2, q_2] \cap \Phi'_Q[h_1, q_1]$. If $f \in \mathcal{A}(p)$, $Q^{\alpha}_{\beta,p}f(z) \in \mathcal{Q}_0 \cap \mathcal{H}[0, p]$ and*

$$\phi\left(Q^{\alpha}_{\beta,p}f(z), Q^{\alpha-1}_{\beta,p}f(z), Q^{\alpha-2}_{\beta,p}f(z), Q^{\alpha-3}_{\beta,p}f(z); z\right)$$

is univalent in $\mathbb{U}$, and (2.1) and (3.1) are satisfied, then

$$h_1(z) \prec \phi\left(Q^{\alpha}_{\beta,p}f(z), Q^{\alpha-1}_{\beta,p}f(z), Q^{\alpha-2}_{\beta,p}f(z), Q^{\alpha-3}_{\beta,p}f(z); z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec Q^{\alpha}_{\beta,p}f(z) \prec q_2(z) \quad (z \in \mathbb{U}).$$

Definition 3.6. Let $\Omega \subset \mathbb{C}$ and $q \in \mathcal{H}_0$ with $q'(z) \neq 0$. The class of admissible functions $\Phi'_{Q,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(b, c, d, e; \xi) \in \Omega,$$

whenever

$$b = q(z), \quad c = \frac{zq'(z) + m(\alpha+\beta+p-2)q(z)}{(\alpha+\beta+p-1)m},$$
$$\Re\left(\frac{(\alpha+\beta+p-1)[c(3-\alpha-\beta-p) + (\alpha+\beta+p-2)d]}{(\alpha+\beta+p-1)c - (\alpha+\beta+p-2)b} - (\alpha+\beta+p-2)\right) \leqq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right)$$

and

$$\begin{aligned}\Re\Bigg(&\frac{(\alpha+\beta+p-1)(\alpha+\beta+p-2)[(\alpha+\beta+p-3)e - 3(\alpha+\beta+p-2)d]}{(\alpha+\beta+p-1)c - (\alpha+\beta+p-2)b} \\ &+ \frac{2(\alpha+\beta+p-1)(\alpha+\beta+p-2)(\alpha+\beta+p-3)b}{(\alpha+\beta+p-1)c - (\alpha+\beta+p-2)b} + 3(\alpha+\beta+p-1)(\alpha+\beta+p-2)\Bigg) \\ &\leqq \frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right),\end{aligned}$$

where $z \in \mathbb{U}$, $p \in \mathbb{N}$, $\alpha \geqq 3$, $\beta > -1$, $\xi \in \partial\mathbb{U}$, and $m \in \mathbb{N} \setminus \{1\}$.

Theorem 3.7. *Let* $\phi \in \Phi'_{Q,1}[\Omega, q]$*. If* $f \in \mathcal{A}(p)$ *and* $\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \in \mathcal{Q}_0$ *satisfy*

$$\Re\left(\frac{zq''(z)}{q'(z)}\right) \geqq 0, \qquad \left|\frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}q'(z)}\right| \leqq m, \tag{3.4}$$

and

$$\phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-3}_{\beta,p}f(z)}{z^{p-1}}; z\right)$$

is univalent in $\mathbb{U}$*, then*

$$\Omega \subset \left\{\phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-3}_{\beta,p}f(z)}{z^{p-1}}; z\right) : z \in \mathbb{U}\right\} \tag{3.5}$$

implies that

$$q(z) \prec \frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

Proof. Let the functions $p(z)$ and ψ be given by (2.12) and (2.20). Because $\phi \in \Phi'_{Q}[\Omega, q]$, so we get from (2.21) and (3.5) that

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in \mathbb{U}\right\}.$$

We find, from (2.19), that the admissible condition for $\phi \in \Phi'_{Q,1}[\Omega, q]$ in Definition 3.6 is equivalent to that for ψ as shown in Definition 1.5 with $n = 2$. Therefore, $\psi \in \Psi'_2[\Omega, q]$, and by applying (3.4) and Lemma 1.7, we have

$$q(z) \prec p(z) \quad (z \in \mathbb{U})$$

or

$$q(z) \prec \frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

□

If $\Omega \neq \mathbb{C}$ is a simply-connected domain with $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω, then we write $\Phi'_{Q,1}[h(\mathbb{U}), q]$ as $\Phi'_{Q,1}[h, q]$. Proceedings similarly, we have an immediate consequence of Theorem 3.7.

Theorem 3.8. *Let* $\phi \in \Phi'_{Q,1}[h, q]$*. Also let* h *be analytic in* $\mathbb{U}$*. If* $f \in \mathcal{A}(p)$ *and* $\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \in \mathcal{Q}_0$ *satisfy* (3.4) *and*

$$\phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-3}_{\beta,p}f(z)}{z^{p-1}}; z\right)$$

is univalent in $\mathbb{U}$*, then*

$$h(z) \prec \phi\left(\frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p}f(z)}{z^{p-1}}, \frac{Q^{\alpha-3}_{\beta,p}f(z)}{z^{p-1}}; z\right)$$

implies that

$$q(z) \prec \frac{Q^{\alpha}_{\beta,p}f(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

If we combine Theorems 2.14 and 3.8, we obtain the following sandwich-type result.

Corollary 3.9. *Let* h_1 *and* q_1 *be analytic functions in* $\mathbb{U}$*. Also let* h_2 *be univalent in* $\mathbb{U}$*,* $q_2 \in \mathcal{Q}_0$ *with* $q_1(0) = q_2(0) = 0$*, and* $\phi \in \Phi_{Q,1}[h_2, q_2] \cap \Phi'_{Q,1}[h_1, q_1]$*. If* $f \in \mathcal{A}(p)$*,* $\frac{Q^{\alpha}_{\beta,p} f(z)}{z^{p-1}} \in \mathcal{Q}_0 \cap \mathcal{H}_0$*, and*

$$\phi\left(\frac{Q^{\alpha}_{\beta,p} f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p} f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p} f(z)}{z^{p-1}}, \frac{Q^{\alpha-3}_{\beta,p} f(z)}{z^{p-1}}; z\right)$$

is univalent in $\mathbb{U}$*, and* (2.13) *and* (3.4) *are satisfied, then*

$$h_1(z) \prec \phi\left(\frac{Q^{\alpha}_{\beta,p} f(z)}{z^{p-1}}, \frac{Q^{\alpha-1}_{\beta,p} f(z)}{z^{p-1}}, \frac{Q^{\alpha-2}_{\beta,p} f(z)}{z^{p-1}}, \frac{Q^{\alpha-3}_{\beta,p} f(z)}{z^{p-1}}; z\right) \prec h_2(z)$$

implies that

$$q_1(z) \prec \frac{Q^{\alpha}_{\beta,p} f(z)}{z^{p-1}} \prec q_2(z) \quad (z \in \mathbb{U}).$$

Remark 3.10. If we take $p = 1$ in all results of this paper, we can obtain the corresponding results for Jung-Kim-Srivastava operator Q^{α}_{β}.

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