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Natural Map in Feblin's Type Fuzzy Normed Linear Spaces

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Abstract

In this paper, we aim to present some properties of the space of all weakly fuzzy bounded linear operators, with the Bag and Samanta's operator norm on fuzzy normed linear spaces. We introduced the natural linear injection of a fuzzy normed linear space X in to its second dual spaceX″.

Key words and phrases: Fuzzy real number, Fuzzy norm linear space, Weakly fuzzy continuouse, Weakly fuzzy bounded.

1. Introduction

In 1992, Felbin [3], by the notion of a fuzzy metric space introduced the concept of a fuzzy normed linear space. In fact she did this by assigning a non-negative fuzzy real number to each element of a linear space. In [4], Felbin introduced an idea of fuzzy bounded linear operator over fuzzy normed linear spaces and defined "fuzzy norm" for such operators. A modified definition of a fuzzy bounded linear operator and a "fuzzy norm" for such an operator was introduced by Bag and Samanta [1]. In this work the dual of a fuzzy normed space and Hahn-Banach's theorem for fuzzy strongly bounded linear functional were established. In this paper we consider the isometric isomorphism of X into X″ in classical analysis of Bag and Samanta fuzzy bounded linear operators is established. In section 2, we state some preliminaries and concepts, whit are needed for our study. In section 3, we try to make an isometric linear map of X into its second dual X″.

2. Preliminaries

We denote the set of all real number by $\mathbb{R}$ and by $\mathbb{Z}^+$ the set of all positive integers. In this paper, we consider the concept of fuzzy real number in the sense Xiao and Zhu [8] which is defined below:

A mapping $\eta(t)\colon \mathbb{R} \longrightarrow [0,1]$, whose α- level set is denoted by $[\eta]_\alpha = \{t\colon \eta(t) \geq \alpha\}$, is called a fuzzy real number if it satisfies two axioms:

(N1) There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$;

(N2) For each $\alpha \in (0,1]$; $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, where $-\infty < \eta_\alpha^- < \eta_\alpha^+ < +\infty$.

The set of all fuzzy real number is denoted by $\mathbb{F}$. For each $r \in \mathbb{R}$, let $\bar{r} \in \mathbb{F}$ be defined by

$$\bar{r}(t) = \begin{cases} 1 & t = r, \\ 0 & t \neq r. \end{cases}$$

So $\bar{r}$ is a fuzzy real number and $\mathbb{R}$ can be embedded in $\mathbb{F}$.

Let $\eta \in \mathbb{F}$, η is called positive fuzzy real number if for all $t < 0, \eta(t) = 0$. The set of all positive fuzzy real numbers is denoted by $\mathbb{F}^+$.

A partial order "$\preccurlyeq$" in $\mathbb{F}$ is defined as follows, $\eta \leq \delta$ if and only if for all $\alpha \in (0,1]$; $\eta_\alpha^- \leq \delta_\alpha^-$ and $\eta_\alpha^+ \leq \delta_\alpha^+$ where, $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$ and $[\delta]_\alpha = [\delta_\alpha^-, \delta_\alpha^+]$. The strict inequality in $\mathbb{F}$ is defined by $\eta \prec \delta$ if and only if for all $\alpha \in (0,1]$; $\eta_\alpha^- < \delta_\alpha^-$ and $\eta_\alpha^+ < \delta_\alpha^+$.

Kaleva and Seikkala, in [5] proved a sufficient condition for a family of intervals to represent the α-level sets of a fuzzy real number. In fact, let $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, be a given family of nonempty intervals. If:

(i) for all $0 < \alpha_1 < \alpha_2, [a_{\alpha_1}, b_{\alpha_1}] \supseteq [a_{\alpha_2}, b_{\alpha_2}]$.

(ii) $[\lim_{k\to\infty} a_{\alpha_k}, \lim_{k\to\infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$, whenever $\{\alpha_k\}$ is an increasing sequence in $(0,1]$ converging to α, then the family $[a_\alpha, b_\alpha]$ represents the α-level sets of a fuzzy real number. Conversely, if $[a_\alpha, b_\alpha], 0 < \alpha \leq 1$, are the α- level sets of a fuzzy real number then the conditions (i) and (ii) are satisfied. According to Dubois and Prade [2], the arithmetic operations $\oplus, \ominus, \odot, \oslash$ on $\mathbb{F} \times \mathbb{F}$ are defined by

$$(\eta \oplus \delta)(t) = sup_{t=r+s} min\{\eta(r), \delta(s)\}, \quad t \in \mathbb{R},$$

$$(\eta \ominus \delta)(t) = sup_{t=r-s} min\{\eta(r), \delta(s)\}, \quad t \in \mathbb{R},$$

$$(\eta \odot \delta)(t) = sup_{t=rs} min\{\eta(r), \delta(s)\}, \quad t \in \mathbb{R},$$

$$(\eta \oslash \delta)(t) = sup_{t=\frac{r}{s}} min\{\eta(r), \delta(s)\}, \quad t \in \mathbb{R},$$

Proposition 2.1 [1, proposition 2.2] Let $\{[a_\alpha, b_\alpha]: \alpha \in (0,1]\}$, be a family of nested bounded closed intervals and $\eta: \mathbb{R} \to (0,1]$ be a function defined by $\eta(t) = Sup\{\alpha \in (0,1]: t \in [a_\alpha, b_\alpha]\}$. Then η is a fuzzy real number.

Proposition 2.2 [1, Proposition 2.5] Let be a fuzzy real number with $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+], \forall \alpha \in (0,1]$, and η^* is the fuzzy real number generated by the family of nested bounded and closed intervals $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+], 0 < \alpha \leq 1$, then $\eta = \eta^*$.

Definition 2.3*[1, Definition 2.4] Let X be a linear space over* $\mathbb{R}$. *Suppose* $\|.\|: X \longrightarrow \mathbb{F}^+$ *is a mapping satisfying*

$$(i)\|x\| = \bar{0} \; if \; and \; only \; if \; x = 0,$$

$$(ii)\|rx\| = |r|\|x\|, \qquad x \in X, r \in R,$$

$$(iii) for \; all \; x, y \in X, \|x + y\| \preccurlyeq \|x\| \oplus \|y\|$$

And

$$(A'): x \neq 0 \implies \|x\|(t) = 0, \forall t \leq 0.$$

$(X, \|.\|)$ *is called a fuzzy normed linear space and* $\|.\|$ *is called a fuzzy norm on X. In the rest of this paper we use the previous definition of fuzzy norm. We note that (i) condition* (A') *in Definition 2.2 is equivalent to the condition*

(A'')*:For all* $x(\neq 0) \in X$ *and each* $\alpha \in (0,1], \|x\|_\alpha^- > 0$*, where* $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+]$ *and (ii)* $\|x\|_\alpha^i$*,* $i = 1,2$*, are crisp norms on X.*

Theorem 2.4 [6,Theorem 2.7] Let $(X, \|.\|)$ be an fuzzy normed linear space, $\alpha \in (0,1], \varepsilon > 0, N(\varepsilon, \alpha) = \{x: \|x\|_\alpha^+ < \varepsilon\}$. Then $(X, \|.\|)$ is a hausdorff topological vector space, whose neighborhood base of origin 0 is $\{N(\varepsilon, \alpha): \varepsilon > 0, \alpha \in (0,1]\}$.

Corollary 2.5 [6,Corollary 2.9] the two families of open sets $\{N(\varepsilon, \alpha): \varepsilon > 0, \alpha \in (0,1]\}$ and $\{N(\alpha, \alpha): \alpha \in (0,1]\}$ are equivalent 0-neighborhood bases.

According to J. Xiao and X. Zhu [8], let $(X, \|.\|)$ be an fuzzy normed linear space, $A \subseteq X$ and $x_0 \in X$. x_0 is called a point of closure of A if $\{x_0 + N(\alpha, \alpha)\} \cap A \neq \phi$ for every $\alpha \in (0,1]$; $\bar{A}$ denotes the set of all points of closure of A. A is called a fuzzy closed set if $A = \bar{A}$. A sequence $\{x_n\}$ in X is said converge to $x \in X$ if and only if for each $\alpha \in (0,1], \lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$.

In this case we write $\lim_{n\to\infty} x_n = x$. Also a sequence $\{x_n\}$ is called a Cauchy sequence if for each $\alpha \in (0,1], \lim_{m,n\to\infty} \|x_n - x_m\|_\alpha^+ = 0$. A fuzzy normed linear space $(X, \|.\|)$ is said to be complete if every cauchy sequence in X converges in X.

3. The main results

Let $(X, \|.\|)$ and $(Y, \|.\|)$ be two fuzzy normed linear spaces. A function $T: X \to Y$ is said to be weakly fuzzy continuous at $x_0 \in X$ if for a given $\varepsilon > 0, \exists \delta \in \mathbb{F}^+, \delta \succ \bar{0}$ such that

$$\|Tx - Tx_0\|_\alpha^- < \varepsilon \quad whenever \quad \|x - x_0\|_\alpha^+ < \delta_\alpha^+$$

$$\|Tx - Tx_0\|_\alpha^+ < \varepsilon \quad whenever \quad \|x - x_0\|_\alpha^- < \delta_\alpha^-$$

Where for $\alpha \in (0,1], [\delta]_\alpha = [\delta_\alpha^-, \delta_\alpha^+]$.

Also a linear mapping $T: X \to Y$ is called weakly fuzzy bounded, if there exists a fuzzy realNumber $\eta \in \mathrm{F}^+, \eta \succ 0$ such that for each $x(\neq 0) \in X, \|Tx\| \oslash \|x\| \preccurlyeq \eta$. In this case, the set of all weakly fuzzy bounded linear operators defined from $X\ to\ Y$is denoted by $B^{'}(X, Y)$ (see [1]).

From [7] we know that a linear mapping $T: X \to Y$ is weakly fuzzy continuous if and only if it is weakly fuzzy bounded. Also the set $B^{'}(X, Y)$ is a linear space with respect usual operations.

The following result of Bag Samanta [1] is essential in this paper. Let $(X, \|.\|)\ to (Y, \|.\|)$ be two fuzzy normed linear spaces $T \in B^{'}(X, Y)$. By definition there exists a fuzzy real number $\eta \in \mathrm{F}^+, \eta \succ 0$ such that for each $x(\neq 0) \in X$, $\|Tx\| \oslash \|x\| \preccurlyeq \eta$.

If $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+], 0 < \alpha < 1$, we get

$$\|Tx\|_\alpha^- \le \eta_\alpha^- \|x\|_\alpha^+ \quad and \quad \|Tx\|_\alpha^+ \le \eta_\alpha^+ \|x\|_\alpha^-$$

Define

$$\|T\|_\alpha^1 = sup_{x\in X,\ x\neq 0} \frac{\|Tx\|_\alpha^-}{\|x\|_\alpha^+}$$

$$\|T\|_\alpha^2 = sup_{x\in X,\ x\neq 0} \frac{\|Tx\|_\alpha^+}{\|x\|_\alpha^-}$$

Then $\{\|.\|_\alpha^1 : \alpha \in (0,1]\} and$ $\{\|.\|_\alpha^2 : \alpha \in (0,1]\}$ are, respectively, ascending and descending families of norms. Thus $\{[\|T\|_\alpha^1, \|T\|_\alpha^2] : \alpha \in (0,1]\}$ is a family of nested bounded closed intervals in $\mathbb{R}$. Define the function $\|T\|^{'} : \mathbb{R} \to [0,1]$ by

$$\|T\|^{'}(t) = \sup\{\alpha \in (0,1] : t \in [\|T\|_\alpha^1, \|T\|_\alpha^2]\}.$$

Then $\|T\|^{'}$ is called the fuzzy norme of $T \in B^{'}(X,Y)$. By Propostition 2.2 we can prove cleary

$$[\|T\|_\alpha^1, \|T\|_\alpha^2] = [[\|T\|_\alpha^{'-}, \|T\|_\alpha^{'+}]$$

Definition 3.1 [1, Definition 6.2] A weakly fuzzy bounded operator defined from a fuzzy normed linear space $(X, \|.\|)\ to\ (\mathbb{R}, \|.\|)$, is called a weakly fuzzy bounded linear functional, where the functional $\|r\| : \mathbb{R} \to [0,1]$, is defined by

$$\|r\|(t) = \begin{cases} 1 & t = |r|, \\ 0 & t \neq |r|. \end{cases}$$

Then $\|r\|$ is a fuzzy norm on $\mathbb{R}$ and $\alpha -$ level sets of $\|r\|$ are given by $[\|r\|]_\alpha = [|r|, |r|], 0 < \alpha \leq 1$. Denoted by $X^{'}$ the set of all weal fuzzy bounded linear functional over $(X, \|.\|)$. We call $X^{'}$ the first fuzzy dual space of X.

Theorem 3.2 Let $(X, \|.\|)$ be fuzzy normed linear space and $x_0 \in X$. Then for all $\alpha \in (0,1]$, there exists $F_\alpha \in X^{'}$ such that $|F_\alpha(x_0)| = \|x_0\|_\alpha^-$ and for all $x \in X, |F(x)| \leq \|x\|_\alpha^-$.

Proof. The proof is routine.

Theorem 3.3 Let $(X, \|.\|)$ be fuzzy normed linear space and $x_0 \in X$. Then for all $\alpha \in (0,1]$, there exists $F_\alpha \in X^{'}$ such that $|F_\alpha(x_0)| = \|x_0\|_\alpha^+$ and for all $x \in X, |F(x)| \leq \|x\|_\alpha^+$.

Proof. The proof is routine.

Definition 3.4 [7, Definition 3.1] let $(X, \|.\|)\ and\ (Y, \|.\|)$ are two fuzzy normed linear spaces and $T \in B^{'}(X,Y)$. Operator $T^{'} : Y^{'} \to X^{'}$ is defined $(T^{'} \wedge)(x) = \wedge (Tx)$, $\wedge \in Y^{'}$, $x \in X$, operator $T^{'}$ is called adjoint operator T.

By theorem 6.2 in [1] we know $(X^{'}, \|.\|^{'})$ is a complete fuzzy normed linear space.

Theorem 3.5 Let $(X, \|.\|)$ be fuzzy normed linear space, then

$$\varphi : X \to X^{''}$$

$$\varphi(x)(x^{'}) = x^{'}(x)$$

Is an isometric linear map, where $x \in X,\ x^{'} \in X^{'}$.

Proof. Since

$$\|\varphi\|_\alpha^{''-} = sup_{x^{'} \in X^{'}} \frac{|x^{'}(x)|}{\|x^{'}\|_\alpha^2} = sup_{x^{'} \in X^{'}} \frac{|x^{'}(x)|}{sup_{x \in X} \frac{|x^{'}(x)|}{\|x\|_\alpha^-}} \leq sup_{x^{'} \in X^{'}} \frac{|x^{'}(x)|}{\frac{|x^{'}(x)|}{\|x^{'}\|_\alpha^-}} = \|x\|_\alpha^-$$

Similary, one can show that $\|\varphi\|_\alpha^{''+} \leq \|x\|_\alpha^+$. Using Theorem 3.3 for all $\alpha \in (0,1]$ there exist $x_1' \in X'$ such that.

$$\|x\|_\alpha^+ = |x_1'(x)|.$$

We have

$$\|x\|_\alpha^+ = |x_1'(x)| = \frac{|x_1'(x)|}{\|x_1'\|_\alpha^1} \leq sup_{x' \in X'} \frac{|x'(x)|}{\|x'\|_\alpha^1} = \|\varphi\|_\alpha^{''+}.$$

By theorem 3.2 for all $\alpha \in (0,1]$ there exist $x_2' \in X'$ such that

$$\|x\|_\alpha^- = |x_2'(x)|.$$

So we have

$$\|x\|_\alpha^+ = |x_2'(x)| = \frac{|x_2'(x)|}{\|x_2'\|_\alpha^2} \leq sup_{x' \in X'} \frac{|x'(x)|}{\|x'\|_\alpha^2} = \|\varphi\|_\alpha^{''-}.$$

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