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R-robustly measure expansive homoclinic classes are hyperbolic

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Abstract

Let $f : M \to M$ be a diffeomorphism on a closed smooth $n(n \geqslant 2)$-dimensional manifold M and let p be a hyperbolic periodic point of f. We show that if the homoclinic class $H_f(p)$ is R-robustly measure expansive then it is hyperbolic.

Keywords: Expansive, measure expansive, local product structure, shadowing, hyperbolic, homoclinic class, generic.

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1. Introduction

Roughly speaking, definition of expansiveness is, if two orbits are closed then they are one orbit which was introduced by Utz [22]. A main research is to study structure of the orbits in differentiable dynamical systems, and so a goal of differentiable dynamical system is to study stability properties (Anosov, Axiom A, hyperbolic, structurally stable, etc.). Therefore, expansiveness is an important notion to study stability properties. For instance, Mañé [11] proved that if a diffeomorphism is C^1 robustly expansive then it is quasi-Anosov. Arbieto [1] proved that for C^1 generic an expansive diffeomorphism is Axiom A without cycles. For expansivity, we can find various generalization notations, that is, continuum-wise expansive [5], n-expansive [13], and measure expansive [14]. Among that, we study measure expansiveness in the paper. Let M be a closed smooth n $(n \geqslant 2)$-dimensional Riemmanian manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let Λ be a closed f-invariant set. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leqslant C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leqslant C\lambda^n$$

for all $x \in \Lambda$ and $n \geqslant 0$. If $\Lambda = M$, then we say that f is Anosov.

For any closed f-invariant set $\Lambda \subset M$, we say that Λ is *expansive* for f, if there is $e > 0$ such that for any $x, y \in \Lambda$ if $d(f^n(x), f^n(y)) \leqslant e$ then $x = y$. Equivalently, Λ is expansive for f if there is $e > 0$ such that

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$\Gamma_e^f(x) = \{x\}$ for all $x \in \Lambda$, where $\Gamma_e^f(x) = \{y \in \Lambda : d(f^i(x), f^i(y)) \leqslant e$ for all $i \in \mathbb{Z}\}$. Let $\mathcal{M}(M)$ be the set of all Borel probability measures on M endowed with the weak* topology, and let $\mathcal{M}^*(M)$ be the set of nonatomic measures $\mu \in \mathcal{M}(M)$. For any $\mu \in \mathcal{M}^*(M)$, we say that Λ is μ-*expansive* for f if $\mu(\Gamma_e^f(x)) = 0$. Λ is said to be *measure expansive* for f if Λ is μ-expansive for all $\mu \in \mathcal{M}^*(M)$; that is, there is a constant $e > 0$ such that for any $\mu \in \mathcal{M}^*(M)$ and $x \in \Lambda$, $\mu(\Gamma_e^f(x)) = 0$. Here e is called a *measure expansive constant* of $f|_\Lambda$. Clearly, the expansiveness implies the measure expansiveness, but the converse does not hold in general (see [14, Theorem 1.35]). We say that f is *quasi-Anosov* if for any $v \in TM \setminus \{0\}$, the set $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded. Sakai et al. [19] proved that if a diffeomorphism f is C^1 robustly measure expansive then it is quasi-Anosov. Lee [7] proved that for C^1 generic f, if f is measure expansive then it is Axiom A without cycles. It is well known that if p is a hyperbolic periodic point of f with period $\pi(p)$ then the sets

$$W^s(p) = \{x \in M : f^{\pi(p)n}(x) \to p \text{ as } n \to \infty\}$$

and

$$W^u(p) = \{x \in M : f^{-\pi(p)n}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M. A point $x \in W^s(p) \pitchfork W^u(p)$ is called a *homoclinic point* of f associated to p. The closure of the homoclinic points of f associated to p is called the *homoclinic class* of f associated to p, and it is denoted by $H_f(p)$. It is clear that $H_f(p)$ is compact, transitive, and invariant.

Denote by $P(f)$ the set of all periodic points of f. Let q be a hyperbolic periodic point of f. We say that p and q are *homoclinically related*, and write $p \sim q$ if

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \text{ and } W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

It is clear that if $p \sim q$ then $\text{index}(p) = \text{index}(q)$, that is, $\dim W^s(p) = \dim W^s(q)$. By the Smale's transverse homoclinic point theorem, $H_f(p) = \overline{\{q \in P_h(f) : q \sim p\}}$, where $\overline{A}$ is the closure of the set A and $P_h(f)$ is the set of all hyperbolic periodic points. Note that if p is a hyperbolic periodic point of f then there is a neighborhood U of p and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$ there exists a unique hyperbolic periodic point p_g of g in U with the same period as p and $\text{index}(p_g) = \text{index}(p)$. Such a point p_g is called the *continuation* of $p = p_f$. We say that Λ is *locally maximal* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$.

In differentiable dynamical systems, a main research topic is to study that for a given system, if the system has a property then we consider that a system which is C^1-nearby system has the same property. Then, we consider various type of C^1-perturbation property on a closed invariant set which are the following.

(a) We say that $H_f(p)$ is C^1 *robustly* $\mathfrak{P}$ *property* if there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $H_g(p_g)$ is $\mathfrak{P}$ property. If $\mathfrak{P}$ is expansive then the expansive constant is uniform, which means that the constant only depends on f (see [15, 16]).

(b) We say that $H_f(p)$ is C^1 *persistently* $\mathfrak{P}$ *property* if there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $H_g(p_g)$ is $\mathfrak{P}$ property. If $\mathfrak{P}$ is expansive then the expansive constant is not uniform which means that the constant depends on $g \in \mathcal{U}(f)$ (see [20]).

(c) We say that $H_f(p)$ is C^1 *stably* $\mathfrak{P}$ *property* if there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of $H_f(p)$ such that for any $g \in \mathcal{U}(f)$, $\Lambda_g(U)$ is $\mathfrak{P}$ property, where $\Lambda_g(U) = \bigcap_{n \in \mathbb{Z}} g^n(U)$ is the continuation of $H_f(p)$. If $\mathfrak{P}$ is expansive, then the expansive constant is not uniform which means that the constant depends on $g \in \mathcal{U}(f)$ (see [8]).

In the item (c), we can also consider a closed invariant set. We say that a subset $\mathcal{G} \subset \text{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of Diff(M); in this case $\mathcal{G}$ is dense in Diff(M). A property $\mathcal{P}$ is said to be *(C^1)-generic* if $\mathcal{P}$ holds for all diffeomorphisms which belong to some residual subset of Diff(M).

Li [10] introduced another C^1 robust property which is called *R-robustly* $\mathfrak{P}$ *property*. Using to the notion, we consider the following.

Definition 1.1. Let the homoclinic class $H_f(p)$ associated to a hyperbolic periodic point p. We say that $H_f(p)$ is *R-robustly measure expansive* if there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a residual set $\mathcal{G}$ of $\mathcal{U}(f)$ such that for any $g \in \mathcal{G}$, $H_g(p_g)$ is measure expansive, where p_g is the continuation of p.

Recently, Pacificao and Vieites [17] proved that a diffeomorphism f in a residual subset far from homoclinic tangencies are measure expansive. Lee and Lee [9] proved that if the homoclinic class $H_f(p)$ is C^1 stably measure expansive then it is hyperbolic. Koo et al. [6] proved that for C^1 generic f, if a locally maximal homoclinic class $H_f(p)$ is measure expansive, then it is hyperbolic. Owing to the result, we have the following which is a main theorem of the paper.

Theorem 1.2. *Let the homoclinic class $H_f(p)$ associated to a hyperbolic periodic point p. If $H_f(p)$ is R-robustly measure expansive then it is hyperbolic.*

2. Dominated splitting and Hyperbolic periodic points in $H_f(p)$

Let M be as before, and let $f \in \text{Diff}(M)$. A *periodic point* for f is a point $p \in M$ such that $f^{\pi(p)}(p) = p$, where $\pi(p)$ is the minimum period of p. Denote by $P(f)$ the set of all periodic points of f. For given $x, y \in M$, we write $x \to y$ if for any $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=0}^{n} (n > 1)$ of f such that $x_0 = x$ and $x_n = y$. We write $x \leftrightarrow y$ if $x \to y$ and $y \to x$. The set of points $\{x \in M : x \leftrightarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. It is clear that $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$. Here $\Omega(f)$ is the non-wandering set of f. Let p be a hyperbolic periodic point of f. We say that the *chain component* if for any $x \in M$, $x \to p$ and $p \to x$ and denote it by $C_f(p)$. Note that the chain component $C_f(p)$ of f is a equivalent class, it is a closed set and f-invariant set. The following was proved by Bonatti and Crovisier [2].

Remark 2.1. There is a residual set $\mathcal{G}_1 \subset \text{Diff}(M)$ such that for any $f \in \mathcal{G}_1$, $H_g(p) = C_f(p)$ for some hyperbolic periodic point p.

Proposition 2.2. *Let the homoclinic class $H_f(p)$ be R-robustly measure expansive. If $x \in W^s(p) \cap W^u(p)$, then $x \in W^s(p) \pitchfork W^u(p)$.*

Proof. Since $H_f(p)$ is R-robustly measure expansive, there exists a C^1-neighborhood $\mathcal{U}(f)$ and a residual set $\mathcal{G} \subset \mathcal{U}(f)$ such that for any $g \in \mathcal{G}$, $H_g(p_g)$ is measure expansive. Let $\mathcal{G} = \mathcal{G}_1$. Since $x \in W^s(p) \cap W^u(p)$, by [17, Proposition 2.6], there is $g \in \mathcal{U}(f) \cap \mathcal{G}$ such that we can make a small arc $\mathcal{J} \subset W^s(p_g) \cap W^u(p_g)$. Since $H_g(p_g) = C_g(p_g)$, we know $\mathcal{J} \subset C_g(p_g)$. Let $\text{diam}(\mathcal{J}) = l$. We define a measure $\mu \in \mathcal{M}^*(M)$ by $\mu(C) = \nu(C \cap \mathcal{J})$ for any Borel set C of M, where ν is a normalized Lebesgue measure on $\mathcal{J}$. Let $e = l/4$ be a measure expansive constant. Since $\mathcal{J} \subset W^s(p_g) \cap W^u(p_g)$, there is $N > 0$ such that $\text{diam}(g^i(\mathcal{J})) \leqslant e/4$ for $-N \leqslant i \leqslant N$, and $g^i(\mathcal{J}) \subset W^s_{e/4}(p_g) \cap W^u_{e/4}(p_g)$ for $|i| > N$. Thus for all $i \in \mathbb{Z}$, we know that $\text{diam}(g^i(\mathcal{J})) \leqslant e$. Recall that

$$\Gamma_e(x) = \{y \in H_g(p_g) : d(g^i(x), g^i(y)) \leqslant e \text{ for } i \in \mathbb{Z}\}.$$

We can construct the set

$$A_e(x) = \{y \in \mathcal{J} : d(g^i(x), g^i(y)) \leqslant e \text{ for } i \in \mathbb{Z}\}.$$

Then we know $A_e(x) \subset \Gamma_e(x)$. Thus we have

$$0 < \mu(A_e(x)) \leqslant \mu(\Gamma_e(x)),$$

which is a contradiction to the measure expansivity of $H_g(p_g)$. □

For $f \in \text{Diff}(M)$, we say that a compact f-invariant set Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that for all $x \in \Lambda$ and $n \geqslant 0$, we have

$$\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \leqslant C\lambda^n.$$

Theorem 2.3. *Let $H_f(p)$ be the homoclinic class containing a hyperbolic periodic point p. Suppose that $H_f(p)$ is R-robustly measure expansive. Then there exist a C^1-neighborhood $\mathcal{U}(f)$ of f and a residual set $\mathcal{G} \subset \mathcal{U}(f)$ such that for any $g \in \mathcal{G}$, $H_g(p_g)$ admits a dominated splitting $T_{H_g(p_g)}M = E(g) \oplus F(g)$ with $\mathrm{index}(p_g) = \dim E(g)$.*

Proof. Suppose that $H_f(p)$ is R-robustly measure expansive. Then as in the proof of [20, Theorem 1], there is $m > 0$ such that for every $x \in W^s(p) \pitchfork W^u(p)$ there exists $m_1 \in [90, m]$ such that $\|Df^{m_1}|_{E(x)}\| \cdot \|Df^{-m_1}|_{F(f^{m_1}(x))}\| \leqslant 1/2$. Since the dominated splitting can be extended by continuity to the

$$\overline{W^s(p) \pitchfork W^u(p)} = H_f(p),$$

we have that $H_f(p)$ has a dominated splitting $E \oplus F$. □

Theorem 2.4. *Let the homoclinic class $H_f(p)$ be R-robustly measure expansive. Then there exist $C > 0, 0 < \lambda < 1$ and $m > 0$ such that q is a hyperbolic periodic point of period $\pi(q)$ and $q \sim p$, then*

$$\prod_{i=0}^{k-1} \|Df^m|_{E^s(f^{im}(q))}\| < C\lambda^k \text{ and } \prod_{i=0}^{k-1} \|Df^{-m}|_{E^u(f^{-im}(q))}\| < C\lambda^k,$$

where $k = [\pi(q)/m]$ ($[\cdot]$ represents the integer part).

Proof. Since $H_f(p)$ is R-robustly measure expansive, there are a C^1-neighborhood $\mathcal{U}(f)$ and a residual set $\mathcal{G} \subset \mathcal{U}(f)$ such that for any $g \in \mathcal{G}$, $H_g(p_g)$ is measure expansive. Let $\mathcal{G} = \mathcal{G}_1$. Since $q \in H_f(p)$ and $p \sim q$, as in the proof of [20], it is enough to show that the family of periodic sequences of linear isomorphisms of $\mathbb{R}^n$ generated by Df along the hyperbolic periodic points $q \in H_f(p), q \sim p$ and $\mathrm{index}(p) = \mathrm{index}(q)$ is uniformly hyperbolic. Suppose, by contradiction, that the assume does not hold. Then as in the proof of [18, Theorem B], we may assume that a hyperbolic periodic point $q \in H_f(p)$ such that the weakest normalized eigenvalue λ is close to 1. Then by Franks lemma, there is $g \in \mathcal{G}$ such that for any small $\gamma > 0$ we can construct a closed small curve $\mathcal{I}_q$ containing q or a closed small circle $\mathcal{C}_q$ centered at q such that $\mathcal{I}_q \subset C_g(p_g)$ and two endpoints are related to p_g and $\mathcal{C}_q \subset C_g(p_g)$. Note that $\mathcal{I}_q$ and $\mathcal{C}_q$ are $g^{\pi(q)}$-invariant, normally hyperbolic, and $g^{l\pi(q)}|_{\mathcal{I}_q}$ is the identity map for some $l > 0$ (see [18]). For $\mathcal{I}_q$, we define a measure $\mu \in \mathcal{M}^*(M)$ by

$$\mu(C) = \frac{1}{l\pi(q)} \sum_{i=0}^{l\pi(q)-1} \nu(g^{-i}(C \cap g^i(\mathcal{I}_q)))$$

for any Borel set C of M, where ν is a normalized Lebesgue measure on $\mathcal{I}_q$. Let $\gamma \leqslant e$ be a measure expansive constant of $g|_{H_g(p_g)}$. By [14, Proposition], g is measure expansive if and only if g^n is measure expansive for $n \in \mathbb{Z} \setminus \{0\}$. Let $\Gamma_e^g(x) = \{y \in H_g(p_g) : d(g^{l\pi(q)i}(x), g^{l\pi(q)i}(y)) \leqslant e, \text{ for all } i \in \mathbb{Z}\}$. Then we have

$$\left\{y \in \mathcal{I}_q : d(g^{l\pi(q)i}(x), g^{l\pi(q)i}(y)) \leqslant e \text{ for all } i \in \mathbb{Z}\right\} = \left\{y \in \mathcal{I}_q : d(g^i(x), g^i(y)) \leqslant e \text{ for all } i \in \mathbb{Z}\right\}.$$

Thus we know

$$0 < \mu(\{y \in \mathcal{I}_q : d(g^i(x), g^i(y)) \leqslant e \text{ for all } i \in \mathbb{Z}\}) \leqslant \mu(\Gamma_e^g(x)).$$

Since $H_g(p_g)$ is measure expansive for g, we know $\mu(\Gamma_e^g(x)) = 0$. Thus we have

$$\mu(\{y \in \mathcal{I}_q : d(g^i(x), g^i(y)) \leqslant e \text{ for all } i \in \mathbb{Z}\}) = 0.$$

This is a contradiction.

For $\mathcal{C}_q$ case, if $\mathcal{C}_q$ is irrational rotation then using the Franks' lemma, there is $h \in \mathcal{U}(g) \cap \mathcal{G}$ such that $\mathcal{C}_{q_h}$ is rational rotation which is centered at q_h, where $\mathcal{U}(g)$ is a C^1-neighborhood of g, and q_h is the continuation of q for h. Then there is $k > 0$ such that $h^k : \mathcal{C}_{q_h} \to \mathcal{C}_{q_h}$ is the identity map. Then we define a measure $\mu \in \mathcal{M}^*(M)$ by

$$\mu(B) = \frac{1}{k}\sum_{i=0}^{k-1} \eta(h^i(B \cap h^{-i}(\mathcal{C}_{q_h}))$$

for any Borel set B of M, where η is a normalized Lebesgue measure on $\mathcal{C}_{q_h}$. Then as in the proof of previous argument, we can derive a contradiction. □

By [14, Proposition], g is measure expansive if and only if g^n is measure expansive for $n \in \mathbb{Z} \setminus \{0\}$. Theorem 2.4 can be rewritten as the following.

Theorem 2.5. *Let the homoclinic class $H_f(p)$ be R-robustly measure expansive. Then there exist $0 < \lambda < 1$ and $L \leqslant 1$ such that q is a hyperbolic periodic point of period $\pi(q)$ with $L > \pi(q)$ and $q \sim p$, then*

$$\prod_{i=0}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| < \lambda^{\pi(q)} \text{ and } \prod_{i=0}^{\pi(q)-1} \|Df^{-1}|_{E^u(f^{-i}(q))}\| < \lambda^{\pi(q)}.$$

3. Local product structure

Let Λ be a closed, f-invariant set. We say that Λ has a *local product structure* if for given $\epsilon > 0$ there exists a $\delta > 0$ such that if $d(x, y) < \delta$ and $x, y \in \Lambda$, then

$$\emptyset \neq W^s_\epsilon(x) \cap W^u_\epsilon(y) \subset \Lambda.$$

By the uniqueness of the dominated splitting, if $q \in H_f(p)$ is a periodic point with $q \sim p$ then we have $E(q) = E^s(q)$ and $F(q) = E^u(q)$. Let $\dim E = s$ and by $\dim F = u$, and put $D^j_r = \{x \in \mathbb{R}^j : \|x\| \leqslant r\}$ $(r > 0)$, for $j = s, u$. Let $\mathrm{Emb}_\Lambda(D^j_1, M)$ be the space of C^1 embeddings $\beta : D^j_1 \to M$ such that $\beta(0) \in \Lambda$ endowed with the C^1 topology. Then we have the following.

Proposition 3.1 ([4, 12])**.** *Let $H_f(p)$ be the homoclinic class of f associated to a hyperbolic periodic point p, and let $\Lambda = H_f(p)$. Suppose that Λ has a dominated splitting $E \oplus F$. Then there exist sections $\phi^s : \Lambda \to \mathrm{Emb}_\Lambda(D^s_1, M)$ and $\phi^u : \Lambda \to \mathrm{Emb}_\Lambda(D^u_1, M)$ such that by defining $W^{cs}_\epsilon(x) = \phi^s(x)D^s_\epsilon$ and $W^{cu}_\epsilon(x) = \phi^u(x)D^u_\epsilon$, for each $x \in \Lambda$, we have*

(1) $T_xW^{cs}_\epsilon(x) = E(x)$ *and* $T_xW^{cu}_\epsilon(x) = F(x)$;
(2) *for every* $0 < \epsilon_1 < 1$ *there exists* $0 < \epsilon_2 < 1$ *such that* $f(W^{cs}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f(x))$ *and* $f^{-1}(W^{cu}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f^{-1}(x))$;
(3) *for every* $0 < \epsilon_1 < 1$ *there exists* $0 < \delta < 1$ *such that if* $d(x, y) < \delta$ $(x, y \in \Lambda)$ *then* $W^{cs}_{\epsilon_1}(x) \cap W^{cu}_{\epsilon_1}(y) \neq \emptyset$, *and this intersection is transverse.*

The sets $W^{cs}_\epsilon(x)$ and $W^{cu}_\epsilon(x)$ are called the *local center stable* and *local unstable manifolds* of x, respectively. The following lemma can be proved similarly to that of Lemma 4 in [20].

Lemma 3.2. *Let $H_f(p)$ be the homoclinic class of f associated to a hyperbolic periodic point p, and suppose that $H_f(p)$ is R-robustly measure expansive. Then for C, λ as in Theorem 3.1 and $\delta > 0$ satisfying $\lambda' = \lambda(1+\delta) < 1$ and $q \sim p$, there exists $0 < \epsilon_1 < \epsilon$ such that if for all $0 \leqslant n \leqslant \pi(q)$ it holds that for some $\epsilon_2 > 0$, $f^n(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q))$, then*

$$f^{\pi(q)}(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{C\lambda'^{\pi(q)}\epsilon_2}(q).$$

Similarly, if $f^{-n}(W^{cu}_{\epsilon_2}(q)) \subset W^{cu}_{\epsilon_1}(f^{-n}(q))$, then

$$f^{-\pi(q)}(W^{cu}_{\epsilon_2}(q)) \subset W^{cu}_{C\lambda'^{\pi(q)}\epsilon_2}(q).$$

Recall that by using the Smale's transverse theorem, we have $H_f(p) = \overline{\text{homo}_p}$, where $\text{homo}_p = \{q \in P_h(f) : q \sim p\}$.

Lemma 3.3. *Let $H_f(p)$ be the homoclinic class of f associated to a hyperbolic periodic point p, and let $e > 0$ be a measure expansive constant. Suppose that $H_f(p)$ is R-robustly measure expansive. Then*

(a) *for any hyperbolic periodic point $q \in \text{homo}_p$ and $0 < \epsilon_1 < e$, there is $\epsilon_2 > 0$ such that*

$$f^n(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q)) \text{ and } f^{-n}(W^{cu}_{\epsilon_2}(q)) \subset W^{cu}_{\epsilon_1}(f^{-n}(q)) \text{ for all } n \geqslant 0.$$

(b) *for any $y \in W^{cs}_{\epsilon_2}(q)$ and $q \in \text{homo}_p$ we have*

$$\lim_{n\to\infty} d(f^n(q), f^n(y)) = 0.$$

Proof. Let $f \in \mathcal{G} = \mathcal{G}_1$ and let $H_f(p)$ is R-robustly measure expansive. To prove (a), it is enough to show that $f^n(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q))$. Let $\sup\{\dim W^{cs}_{\epsilon_1}(q) : q \in \text{homo}_p\} < e$. Since $q \in \text{homo}_p$, we define

$$\epsilon(q) = \sup\{\epsilon > 0 : f^n(W^{cs}_{\epsilon}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q)) \text{ for all } n \geqslant 0\}.$$

By Proposition 3.1 and Lemma 3.2, $\epsilon(q) > 0$. Let $\epsilon_0 = \inf\{\epsilon(q) : q \in \text{homo}_p\}$. If $\epsilon_0 > 0$ then it is a proof of (a). Suppose, by contradiction, that there is a sequence $\{q_n\} \subset \text{homo}_p$ such that $\epsilon(q_n) \to 0$ as $n \to \infty$. Then we have $0 < m_n < \pi(q_n)$ and $y_n \in W^{cs}_{\epsilon(q_n)}(q_n)$ such that $d(f^{m_n}(q_n), f^{m_n}(y_n)) = \epsilon_1$ for $f^{m_n}(q_n), f^{m_n}(y_n) \in W^{cs}_{\epsilon(q_n)}(q_n)$. Let I_n be a closed connected arc joining $f^{m_n}(q_n)$ with $f^{m_n}(y_n)$. Then we know that

(i) $I_n \subset W^{cs}_{\epsilon(q_n)}(q_n)$;
(ii) $f^i(I_n) \subset W^{cs}_{\epsilon_1}(f^i(q_n))$ for $0 \leqslant i \leqslant \pi(q_n)$;
(iii) $\text{diam}(I_n) = \epsilon_1$.

By Lemma 3.2, we know $f^{\pi(q_n)}(W^{cs}_{\epsilon(q_n)}(q_n)) \subset W^{cs}_{C\lambda'^{\pi(q_n)}\epsilon(q_n)}(q_n)$. Observe that if $n \to \infty$ then $m_n \to \infty$ and $\pi(q_n) - m_n \to \infty$. Suppose that $f^{m_n}(q_n) \to x$ and $f^{m_n}(y_n) \to y$ as $n \to \infty$. Then $I_n \to I$, where I is a close connected arc joining x with y. It means that $\text{diam}(f^j(I)) \leqslant \epsilon_1$ for all $j \in \mathbb{Z}$, and $x \in \overline{\text{homo}_p} = H_f(p)$. We show that the closed connected arc $I \subset H_f(p)$. Since $f \in \mathcal{G}$, $H_f(p) = C_f(p)$. For any $a \in I$, take $a_n \in W^{cs}_{\epsilon(q_n)}(q_n)$ such that $f^{m_n}(a_n) \to a$ as $n \to \infty$. As in the proof of [21, Lemma 2.6], let $\epsilon > 0$ be arbitrary. Let $n \in \mathbb{N}$ be such that $\epsilon(q_n) < \epsilon$. Then for n sufficiently large, $\{q_n, f(a_n), \ldots, f^{m_n-1}(a_n), a, f^{m_n+1}(a_n), \ldots, f^{\pi(q_n)-1}(a_n), q_n\}$ is a periodic ϵ-chain through a and having a point in $H_f(p)$. Since $q_n \in \text{homo}_p$, $H_f(q_n) = H_f(p) = C_f(q_n) = C_f(p)$ and so the closed connected arc $I \subset H_f(p)$. We define a measure $\mu \in \mathcal{M}^*(M)$ by $\mu(C) = \mu_I(C \cap I)$ for any Borel set C of M, where μ_I is a normalized Lebesgue measure on I. Let

$$\Gamma_e(x) = \{y \in H_f(p) : d(f^i(x), f^i(y)) \leqslant e \text{ for } i \in \mathbb{Z}\}.$$

Since for all $i \in \mathbb{Z}$, $\text{diam}(f^i(I)) \leqslant e$, we can construct the set

$$\{y \in I : d(f^i(x), f^i(y)) \leqslant e \text{ for } i \in \mathbb{Z}\}.$$

Then we know $\{y \in I : d(f^i(x), f^i(y)) \leqslant e \text{ for } i \in \mathbb{Z}\} \subset \Gamma_e(x)$. Thus we have

$$0 < \mu(\{y \in I : d(f^i(x), f^i(y)) \leqslant e \text{ for } i \in \mathbb{Z}\}) \leqslant \mu(\Gamma_e(x)).$$

Since $H_f(p)$ is measure expansive, $\mu(\Gamma_e(x)) = 0$. Thus $\mu(\{y \in I : d(f^i(x), f^i(y)) \leqslant e \text{ for } i \in \mathbb{Z}\}) = 0$ which is a contradiction.

The proof of (b) is similar as in the proof of item (b) of [21, Lemma 2.6]. □

Remark 3.4. In the Lemma 3.3, we consider $q \in \text{homo}_p$. Then we can extend $x \in H_f(p)$, that is, for any $x \in H_f(p)$ and $\epsilon_1 > 0$ there exists $\epsilon_2 > 0$ such that $f^n(W^{cs}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f^n(x))$ for all $n \geqslant 0$. And if $z \in W^{cs}_{\epsilon_2}(x)$ and $z \in H_f(p)$, then $d(f^i(z), f^i(x)) \to 0$ as $n \to \infty$.

Proposition 3.5. *Suppose that the homoclinic class $H_f(p)$ is R-robustly measure expansive. Then $H_f(p)$ has a local product structure.*

Proof. By Lemma 3.3, there is $\epsilon_2 > 0$ such that for any $q \in \text{homo}_p$

$$W^{cs}_{\epsilon_2}(q) = W^{s}_{\epsilon_2}(q) \text{ and } W^{cu}_{\epsilon_2}(q) = W^{u}_{\epsilon_2}(q).$$

By Proposition 3.1 (3), there is $\delta > 0$ such that for any $q, r \in \text{homo}_p$,

$$W^{s}_{\epsilon_2}(q) \cap W^{u}_{\epsilon_2}(r) \neq \emptyset.$$

By λ-lemma, $W^{s}_{\epsilon_2}(q) \subset \overline{W^s(p)}$ and $W^{u}_{\epsilon_2}(r) \subset \overline{W^u(p)}$. Thus we know that $W^{s}_{\epsilon_2}(q) \cap W^{u}_{\epsilon_2}(r) \subset H_f(p)$. This means that $H_f(p)$ has a local product structure. □

Corollary 3.6. *Suppose that the homoclinic class $H_f(p)$ is R-robustly measure expansive. Then for any hyperbolic periodic point $q \in H_f(p)$, $\text{index}(p) = \text{index}(q)$.*

Proof. The proof is directly obtained by Proposition 3.1 (3), Lemma 3.3, and Proposition 3.5. Thus for any hyperbolic periodic point $q \in H_f(p)$,

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \text{ and } W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

Thus we have $\text{index}(p) = \text{index}(q)$. □

4. Proof of Theorem 1.2

For any $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. Let Λ be a closed f-invariant set. We say that f has the *shadowing property* on Λ such that for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ there is $z \in M$ such that $d(f^i(z), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. The following proposition is a very useful result for proving of Theorem 1.2.

Proposition 4.1 ([23, Proposition 3.3])**.** *Let p be a hyperbolic periodic point, and let $H_f(p)$ be the homoclinic class of f containing p. Let $0 < \lambda < 1$ and $L \geqslant 1$ be given. Assume that $H_f(p)$ satisfies the following properties.*

(1) *There is a continuous Df-invariant splitting $T_{H_f(p)}M = E \oplus F$ with $\dim E = \text{index}(p)$ such that for any $x \in H_f(p)$,*

$$\|Df|_{E(x)}\|/m(Df|_{F(x)}) < \lambda^2,$$

where $m(A) = \inf\{\|A\| : \|v\| = 1\}$ denotes the mininorm of a linear map A.

(2) *For any $q \in H_f(p) \cap P(f)$, if q is hyperbolic and $\pi(q) > L$, then $\text{index}(p) = \text{index}(q)$ and*

$$\prod_{i=0}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| < \lambda^{\pi(q)}, \quad \prod_{i=0}^{\pi(q)-1} \|Df^{-1}|_{E^u(f^{-i}(q))}\| < \lambda^{\pi(q)}.$$

(3) *f has the shadowing property on $H_f(p)$.*

Then $H_f(p)$ is hyperbolic.

End of the Proof of Theorem 1.2. Since $H_f(p)$ is R-robustly measure expansive, by Theorems 2.3 and 2.5, propositions (1) and (2) hold. By Proposition 3.5 and Bowen's result [3, Proposition 3.6], if the homoclinic class $H_f(p)$ is R-robustly measure expansive then f has the shadowing property on $H_f(p)$, and so, proposition (3) also holds. Thus if $H_f(p)$ is R-robustly measure expansive then it is hyperbolic. □

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