Dynamical analysis on prey refuge in a predator-prey model with square root functional response

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Abstract

In this paper, we consider a predator-prey model with square root functional response and prey refuge. The study reveals that the dynamical behavior near the origin of the model is subtle and interesting. We also show that the model undergoes Transcritical bifurcation and Hopf bifurcation. Numerical simulations not only illustrate our results, but also exhibit richer dynamical behaviors of the model than those with Holling II type functional response. Taking prey refuge as control variable, it is feasible to decrease predation rate and then control predator density properly so as to avoid two of population extinction and promote coexistence.

Keywords: Square root functional response, prey refuges, limit cycle, global stability, transcritical bifurcation, Hopf bifurcation.

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1. Introduction

The effects of prey refuges on a predator-prey interaction with various classical functional responses have been studied during the recent decades, such as Holling type I-IV functional response, Rosenzweig functional response, etc. The refuges are considered as two types: a constant proportion of prey and a fixed number of prey using refuges. Prey refuges are widely believed to prevent prey extinction and damp predator-prey oscillations. For example, Chen et al. have shown that prey refuges having a stabilizing effect on predator-prey interactions incorporating a constant prey refuge and the dynamic behaviors deeply dependent on the prey refuge parameter m [3, 4]. Kar [10] and Huang et al. [7] have studied a Lotka-Volterra type predator-prey system with Holling type II response function and Holling type III response function, respectively, incorporating a constant proportion of prey using refuges mx, which protects mx of prey from predation. They have all derived that increasing the amount of refuge could increase prey densities and lead to population outbreaks. On the other hand, Ma et al. have obtained that prey refuges can destabilize the interior equilibrium point of a predator-prey model with a class of functional responses under a very restricted set of conditions [14]. In [15], McNair obtained that a

prey refuge with legitimate entry-exit dynamics was capable of amplifying predator-prey oscillations and several kinds of refuges could exert a locally destabilizing effect in [16]. For more biological backgrounds and results on the effects of a prey refuge, one could refer to [5, 6, 9, 11–13] and the references therein.

Recently, Ajraldi et al. [1] and Braza [2] modeled herd behavior in population system, in which the individuals of one population gather together in herds, while the other one shows a more individualistic behavior and propound a square root functional response. The authors [1] have discussed the following forms of symbiosis, competition, and the predator-prey interactions model with square root functional response:

$$\dot{x} = rx\left(1 - \frac{x}{K_x}\right) + a\sqrt{x}y,$$
$$\dot{y} = my\left(1 - \frac{y}{K_y}\right) + ae\sqrt{x}y,$$

where $a > 0, ae > 0$ represent the symbiosis case, while $a < 0, ae < 0$ represent the competition case, and

$$\dot{x} = rx\left(1 - \frac{x}{K_x}\right) - a\sqrt{x}y,$$
$$\dot{y} = -my + ae\sqrt{x}y,$$

where $a > 0, ae > 0$ represent the predator-prey case. By the method of qualitative analysis, they have shown that the socialized herd behavior prevents the competing individualistic population from becoming extinct and for the predator-prey case, they have derived sustained limit cycles. Braza [2] propounded the following predator-prey system with square root functional response, which appropriately models system in which the prey exhibits strong herd structure and the predator interacts with the prey along the outer corridor of the herd

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - \frac{\alpha\sqrt{x}y}{1 + h\alpha\sqrt{x}},$$
$$\dot{y} = -sy + \frac{c\alpha\sqrt{x}y}{1 + h\alpha\sqrt{x}},$$

(1.1)

where x, y denote prey and predator population at time t, respectively; r, k, s, α, c are positive constants and h is nonnegative. Here r represents the intrinsic growth rate and k the carrying capacity of the prey; h represents predator average handling time of prey; s is the death rate of the predator; and c is the conversion factor denoting the number of newly born predators for each captured prey. The term $\frac{\alpha\sqrt{x}}{1 + h\alpha\sqrt{x}}$ denotes the square root functional response of the predator. In [2], the author has studied the case of $h = 0$, which gives the modified Lotka-Volterra interaction term $\sqrt{x}y$ and is in line with the work [1], and it shows that if the prey is considerably smaller than the predator, then the prey first goes extinct and causes the predator to follow suit.

As we have seen, the considered models in [1, 2] reveals some interesting results because of the square root term. A natural question is: how do the prey refuges affect the dynamical behaviors of the model (1.1)? Stabilize or destabilize or no influence? That will be a popular and interesting issue.

To the best of the authors knowledge, to this day, still no scholars investigate the influence of prey refuges on a predator-prey interactions with square root functional response, this motivates us to propose the following model

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - \frac{\alpha\sqrt{(1-m)x}y}{1 + h\alpha\sqrt{(1-m)x}},$$
$$\dot{y} = -sy + \frac{c\alpha\sqrt{(1-m)x}y}{1 + h\alpha\sqrt{(1-m)x}},$$

(1.2)

where $m \in [0, 1)$ denotes a constant proportion of prey using refuges, which protects mx of prey from predation, and we will study the dynamical behavior of model (1.2).

The rest of the paper is organized as follows. Dynamical analysis of the model are given in Section 2 and the influence of prey refuges is analyzed in Section 3. Some numeric simulations which illustrate the feasibility of our finding are also given in Section 3. This paper ends by a brief conclusion.

2. Dynamical analysis of model (1.2)

Let $\Omega = \{(x, y)|x > 0, y > 0\}$, for practical biological meaning, we simply study model (1.2) in Ω or in $\overline{\Omega}$.

Similarly to the proof in [3, 4], we have the following.

Lemma 2.1. *The solution* $x(t), y(t)$ *of model* (1.2) *with initial values* $x(0) > 0, y(0) > 0$ *are positive and bounded for all* $t \geqslant 0$.

For simplicity, we take the following scaling: $t = \left(1 + h\alpha\sqrt{(1-m)x}\right)\tau$ and then system (1.2) takes the following form (still denotes τ as t)

$$
\begin{aligned}
\dot{x} &= rx\left(1 - \frac{x}{K}\right)\left[1 + h\alpha\sqrt{(1-m)x}\right] - \alpha\sqrt{(1-m)x}\,y \overset{\triangle}{=} P(x, y), \\
\dot{y} &= \left[-s + \alpha(c - hs)\sqrt{(1-m)x}\right]y \overset{\triangle}{=} Q(x, y).
\end{aligned}
\tag{2.1}
$$

System (2.1) has at most three equilibria $E_1(0, 0), E_2(K, 0), E_*(x_*, y_*)$ in $\overline{\Omega}$, where $x_* = \dfrac{s^2}{\alpha^2(c - hs)^2(1 - m)}$, $y_* = \dfrac{crs[K\alpha^2(c - hs)^2(1 - m) - s^2]}{K\alpha^4(c - hs)^4(1 - m)^2}$. $E_*(x_*, y_*)$ is a unique positive equilibrium if and only if $0 \leqslant m < 1 - \dfrac{s^2}{K\alpha^2(c - hs)^2}$. If $c \leqslant hs$ holds, then it follows from (2.1) that $\lim_{t \to +\infty} y(t) = 0$. Without loss of generality, in the following we always suppose that $c > hs$.

Let $p = s^2(3c + hs) - K\alpha^2(c - hs)^2(c + hs)(1 - m)$, $A = 1 - \dfrac{s^2}{K\alpha^2(c - hs)^2}$, and

$$
B = 1 - \frac{s^2(3c + hs)}{K\alpha^2(c - hs)^2(c + hs)}.
$$

Lemma 2.2.

(I) *If* $A < m < 1$ *holds, then* E_2 *is stable node point; if* $0 \leqslant m < A$ *holds, then* E_2 *is saddle point;*

(II) E_* *is locally asymptotically stable for* $B < m < A$ *and* E_* *is unstable for* $0 \leqslant m < B$.

Proof.

(I) The Jacobian matrix of system (2.1) for E_2 is given by

$$
J(E_2) = \begin{pmatrix} -r - rh\alpha\sqrt{(1-m)K} & -\alpha\sqrt{(1-m)K} \\ 0 & -s + \alpha(c - hs)\sqrt{(1-m)K} \end{pmatrix}.
$$

If $A < m < 1$ holds, the eigenvalues of matrix $-r - rh\alpha\sqrt{(1-m)K}$ and $-s + \alpha(c - hs)\sqrt{(1-m)K}$ are negative, hence, E_2 is locally asymptotically stable and furthermore E_2 is stable node point. If $0 \leqslant m < A$ holds, one of the eigenvalues of matrix $-s + \alpha(c - hs)\sqrt{(1-m)K}$ is positive, hence, E_2 is unstable and furthermore E_2 is saddle point.

(II) The Jacobian matrix of system (2.1) for E_* is given by

$$
J(E_*) = \begin{pmatrix} -\frac{rp}{2K\alpha^2(c-hs)^3(1-m)} & -\frac{s}{c-hs} \\ \frac{rc[K\alpha^2(c-hs)^2(1-m)-s^2]}{2K\alpha^2(c-hs)^2(1-m)} & 0 \end{pmatrix}.
$$

Note that $0 \leqslant m < A$ holds,

$$\det(J(E_*)) = \frac{rcs\left[K\alpha^2(c - hs)^2(1 - m) - s^2\right]}{2K\alpha^2(c - hs)^3(1 - m)} > 0.$$

If $B < m < A$ holds, then $p > 0$ and $\text{tr}(J(E_*)) = -\dfrac{rp}{2K\alpha^2(c - hs)^3(1 - m)} < 0$, E_* is locally asymptotically stable. If $0 \leqslant m < B$ holds, then $p < 0$ and $\text{tr}(J(E_*)) > 0$, E_* is unstable. This completes the proof. □

Remark 2.3. At the value $m = A$, E_2 coincides with E_*. As parameter m decreases from the value A, E_2 becomes unstable and E_* becomes feasible, while m passes this value, E_2 becomes stable and E_* becomes infeasible, which shows the transcritical bifurcation.

2.1. Dynamical behavior near the origin

In this subsection, we apply the research method in [2] to discuss the dynamical behavior near the origin of model (2.1). Similarly, we consider (2.1) for $x \ll 1$ and $y \ll 1$, then $1 - \dfrac{x}{K} \approx 1$, $1 + h\alpha\sqrt{(1 - m)x} \approx 1$ and $\sqrt{x}y \ll y$, so that near the origin, model (2.1) takes the following form

$$\dot{x} \approx rx - \alpha\sqrt{(1 - m)x}\,y,$$
$$\dot{y} \approx -sy. \tag{2.2}$$

From the second equation of (2.2), we have $y(t) = y_0 e^{-st}$ with $y_0 \ll 1$ and consider $x = O(y^\beta)$, where $\beta = 1, \beta = 2$, and $\beta > 2$. For $\beta = 1$, it is easy to derive that the origin is a saddle. For $\beta > 2$, since $x \ll \sqrt{x}y$, we can reduce (2.2) to the model $\dot{x} \approx -\alpha\sqrt{(1 - m)x}\,y$, $\dot{y} \approx -sy$, which gives the curve

$$y = y_0 + \frac{2s}{\alpha\sqrt{1 - m}}(\sqrt{x} - \sqrt{x_0 y_0^\beta}).$$

It is part of a parabola that starts at $(x_0 y_0^\beta, y_0)$ and terminates on the y−axis at $y = y_0 - \frac{2s}{\alpha\sqrt{1-m}}\sqrt{x_0 y_0^\beta} > 0$, after which y declines to zero since $\dot{y} \approx -sy$. This means that if the prey population is considerably smaller than the predator population, then the prey first goes extinct, causing the predator to follow suit. For $\beta = 2$, both x and $\sqrt{x}y$ are $O(y^2)$ so that equations (2.2) can be solved as

$$\sqrt{x} = \frac{\alpha\sqrt{1 - m}\,y^{1 + \frac{r}{2s}} + [(r + 2s)\sqrt{x_0} - \alpha\sqrt{1 - m}\,y_0]y_0^{\frac{r}{2s}}}{(r + 2s)y^{\frac{r}{2s}}}$$

in which $x(y_0) = x_0$. Similar to the discussion in [2], the trajectory goes into the origin along the parabola $\alpha\sqrt{1 - m}\,y = (r + 2s)\sqrt{x}$ with the initial conditions satisfying $\alpha\sqrt{1 - m}\,y_0 = (r + 2s)\sqrt{x_0}$. For $\alpha\sqrt{1 - m}\,y_0 > (r + 2s)\sqrt{x_0}$, the trajectory terminates at $x = 0$ at some positive value of y like in the $\beta > 2$ case above. For $\alpha\sqrt{1 - m}\,y_0 < (r + 2s)\sqrt{x_0}$, the origin acts like a saddle.

2.2. Global stability for E_2 and E_*

In this subsection, we assume that E_1 is a saddle point.

Theorem 2.4. *If $A < m < 1$ holds, $E_2(K, 0)$ is globally asymptotically stable.*

Proof. Notice that $A < m < 1$, model (2.1) does not have positive equilibrium and E_1 is unstable. If system (2.1) exists a closed orbit in Ω, then there must exist an equilibrium in the interior of the closed orbit, which is impossible. Hence, system (2.1) does not exist limit cycle, from the boundedness of system (2.1), the stable node point E_2 is globally asymptotically stable. □

Theorem 2.5. *If the positive equilibrium $E_*(x_*, y_*)$ of system (2.1) is locally asymptotically stable, then E_* is globally asymptotically stable.*

Proof. Define a Dulac function $B(x,y) = x^{-\frac{1}{2}}y^{\delta-1}$ and

$$\delta = \frac{r[3s^2(c+hs) - Khs\alpha^2(c-hs)^2(1-m)]}{Ks\alpha^2(c-hs)^3(1-m)}.$$

If the positive equilibrium E_* of system (2.1) is locally asymptotically stable, then $tr(J(E_*)) < 0$ for which $p > 0$, and hence $\delta > 0$. From system (2.1), we have

$$D = \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y}$$

$$= x^{-\frac{1}{2}}y^{\delta-1}[\frac{r}{2} + rh\alpha\sqrt{1-m}x^{\frac{1}{2}} - \frac{3r}{2K}x - \frac{2rh\alpha\sqrt{1-m}}{K}x^{\frac{3}{2}} - \delta s + \delta\alpha(c-hs)\sqrt{1-m}x^{\frac{1}{2}}]$$

$$\overset{\triangle}{=} x^{-\frac{1}{2}}y^{\delta-1}\phi(x).$$

Then $\phi'(x) = \frac{rh\alpha\sqrt{1-m}}{2}x^{-\frac{1}{2}} - \frac{3r}{2K} - \frac{3rh\alpha\sqrt{1-m}}{K}x^{\frac{1}{2}} + \frac{\delta\alpha(c-hs)\sqrt{1-m}}{2}x^{-\frac{1}{2}}$ and

$$\phi''(x) = -\frac{rh\alpha\sqrt{1-m}}{4}x^{-\frac{3}{2}} - \frac{3rh\alpha\sqrt{1-m}}{2K}x^{-\frac{1}{2}} - \frac{\delta\alpha(c-hs)\sqrt{1-m}}{4}x^{-\frac{3}{2}}.$$

It is easy to see that $\phi''(x) < 0$ for all $x > 0$, since $\delta > 0$, then, for all $x > 0$, $\phi'(x)$ decreases monotonously. Simple computation shows that $\phi'(x_*) = 0$, then $0 < x < x_*$ holds, $\phi'(x) > 0$ for which $\phi(x)$ increases monotonously; $x > x_*$ holds, $\phi'(x) < 0$ for which $\phi(x)$ decreases monotonously. Therefore, for all $x > 0$, $\phi(x) \leqslant \phi(x_*)$, and $\phi(x_*) = -\frac{rp}{2K\alpha^2(c-hs)^3(1-m)} < 0$, then we derive $\phi(x) < 0$, for all $x > 0$. That is for all $x > 0$, $D < 0$, system (2.1) does not exist limit cycle in Ω, from the boundedness of system (2.1), E_* is globally asymptotically stable. □

From Lemma 2.2 and Theorem 2.5, we have following corollary.

Corollary 2.6. *If* $B < m < A$ *holds, then* E_* *is globally asymptotically stable.*

2.3. Existence and uniqueness of limit cycle

In this subsection, we also assume that E_1 is a saddle point.

Theorem 2.7. *If the positive equilibrium* $E_*(x_*, y_*)$ *of system* (2.1) *is unstable, then system* (2.1) *admits at least one limit cycle in* Ω.

Proof. For system (2.1), construct a Bendixson ring $\widehat{OABCO}$ including E_*. Define $\overline{OA}$, $\overline{AB}$, $\overline{BC}$, $\overline{CO}$ as a length of line $L_1 = y = 0$, $L_2 = x - K = 0$ $(y > 0)$, $L_3 = y + cx - n = 0$ $(0 < x < K$, n is large enough), and $L_4 = x = 0$, respectively. Since $\overline{OA}$, $\overline{CO}$ is a length of orbit of system (2.1), respectively and $\frac{dL_2}{dt}\big|_{(2.1)} = -\frac{\alpha\sqrt{(1-m)Ky}}{1+h\alpha\sqrt{(1-m)K}} < 0$ $(y > 0)$, $\frac{dL_3}{dt}\big|_{(2.1)} = -s(n-cx) + crx(1-\frac{x}{K}) < 0$ $(0 < x < K$, n is large enough), the orbits of system (2.1) go through into the interior of the Bendixson ring from the outer of $\overline{AB}$, $\overline{BC}$. By Poincaré-Bendixson Theorem, if $E_*(x_*, y_*)$ is an unstable equilibrium, then system (2.1) admits at least one limit cycle in region $\widehat{OABCO} \subset \Omega$. This completes the proof of Theorem 2.7. □

Theorem 2.8. *If the positive equilibrium* $E_*(x_*, y_*)$ *of system* (2.1) *is unstable, then system* (2.1) *admits at most one limit cycle which is globally asymptotically stable in* Ω.

Proof. In order to proof Theorem 2.8, we take the following change of variables $u = \alpha\sqrt{1-m}y$, then system (2.1) takes the following form

$$\dot{x} = \varphi(x)(h(x) - u),$$
$$\dot{u} = \psi(x)u,$$

where $\varphi(x) = \sqrt{x}$, $h(x) = r\sqrt{x}(1 - \frac{x}{K})(1 + h\alpha\sqrt{(1-m)x})$, $\psi(x) = -s + \alpha(c - hs)\sqrt{(1-m)x}$. According to Hwang [8], it is sufficient to show that

$$\frac{d}{dx}\big(\varphi(x)h'(x)/\psi(x)\big) < 0, \quad x \in (0, K) - \{x_*\}.$$

$$\frac{d}{dx}\big(\varphi(x)h'(x)/\psi(x)\big) = \frac{r\left[\frac{3s}{2K} - \frac{\alpha(c+hs)\sqrt{1-m}}{4}x^{-\frac{1}{2}} - \frac{3\alpha(c-5hs)\sqrt{1-m}}{4K}x^{\frac{1}{2}} - \frac{2h\alpha^2(c-hs)(1-m)}{K}x\right]}{\left(-s + \alpha(c-hs)\sqrt{(1-m)x}\right)^2},$$

$$\frac{d}{dx}(\varphi(x)h'(x)/\psi(x))\frac{(-s + \alpha(c-hs)\sqrt{(1-m)x})^2 x^{\frac{1}{2}}}{r} \triangleq H(x),$$

$$H(x) = \frac{3s}{2K}x^{\frac{1}{2}} - \frac{\alpha(c+hs)\sqrt{1-m}}{4} - \frac{3\alpha(c-5hs)\sqrt{1-m}}{4K}x - \frac{2h\alpha^2(c-hs)(1-m)}{K}x^{\frac{3}{2}}.$$

In the following, we will show that $H(x) < 0$ for all $x > 0$.

$$H'(x) = \frac{3s}{4K}x^{-\frac{1}{2}} - \frac{3\alpha(c-5hs)\sqrt{1-m}}{4K} - \frac{3h\alpha^2(c-hs)(1-m)}{K}x^{\frac{1}{2}},$$

$$H''(x) = -\frac{3s}{8K}x^{-\frac{3}{2}} - \frac{3h\alpha^2(c-hs)(1-m)}{2K}x^{-\frac{1}{2}}.$$

It is easy to see that $H''(x) < 0$ for all $x > 0$, then, for all $x > 0$, $H'(x)$ decreases monotonously. Simple computation shows that $H'(x_*) = 0$, then $0 < x < x_*$ holds, $H'(x) > 0$ for which $H(x)$ increases monotonously; $x > x_*$ holds, $H'(x) < 0$ for which $H(x)$ decreases monotonously. Therefore, for all $x > 0$, $H(x) \leqslant H(x_*)$, and $H(x_*) = \dfrac{p}{4k\alpha(c-hs)^2\sqrt{1-m}} < 0$, since the positive equilibrium $E_*(x_*, y_*)$ of system (2.1) is unstable, then $\mathrm{tr}(J(E_*)) > 0$ for which $p < 0$. Then, we derive $H(x) < 0$ for all $x > 0$. That is, for all $x \in (0, K) - \{x_*\}$, $\frac{d}{dx}(\varphi(x)h'(x)/\psi(x)) < 0$. This completes the proof of Theorem 2.8. $\qquad\square$

From Lemma 2.2 and Theorems 2.7 and 2.8 we have the following corollary.

Corollary 2.9. *If $0 \leqslant m < B$ holds, then system (2.1) admits exactly one limit cycle which is globally asymptotically stable in Ω .*

Remark 2.10. At the value $m = B$, the eigenvalues of matrix $J(E_*)$ are a pair of purely imaginary, i.e., $\pm c\sqrt{\frac{rs}{(c-hs)(3c+hs)}}i$. With the parameter m passes through the value B, the stability of equilibrium E_* takes overturning, which meaning that model (2.1) undergoes Hopf bifurcation around E_* at $m = B$.

3. The influence of prey refuge and numerical simulations

It is easy to see that x_*, y_* are all continuous differential functions of parameter m and

$$\frac{dx_*}{dm} = \frac{s^2}{\alpha^2(c-hs)^2(1-m)^2} > 0, \quad \frac{dy_*}{dm} = \frac{rsc[K\alpha^2(c-hs)^2(1-m) - 2s^2]}{K\alpha^4(c-hs)^4(1-m)^3}.$$

Obviously, x_* is the strictly increasing function of parameter m and increasing the prey refuge leads to the increasing of the density of prey species. Let $m^* = 1 - \dfrac{2s^2}{K\alpha^2(c-hs)^2}$, then it follows that $\dfrac{dy_*}{dm} > 0$ for all $m \in (0, m^*)$ and $\dfrac{dy_*}{dm} < 0$ for all $m \in (m^*, 1)$. Thus, there exists a threshold $m = m^*$ such that y_* is the strictly increasing function of parameter m for all $m \in (0, m^*)$, while y_* is the strictly decreasing function of parameter m for all $m \in (m^*, 1)$. y_* attains its maximum value at $m = m^*$, the maximum value equals to $\frac{rcK}{4s}$.

Consider the following model

$$\dot{x} = 5x\left(1 - \frac{x}{60}\right) - \frac{1.2\sqrt{(1-m)x}y}{1 + 0.24\sqrt{(1-m)x}},$$

$$\dot{y} = -y + \frac{0.48\sqrt{(1-m)x}y}{1 + 0.24\sqrt{(1-m)x}}, \tag{3.1}$$

where $K = 60$, $r = 5$, $s = 1$, $\alpha = 1.2$, $h = 0.2$, $c = 0.4$. By simple computation, we derive that $m^* \approx 0.4213$, $A \approx 0.7106$, and $B \approx 0.3249$. Then, the densities of predator species increase as $m \in [0, 0.4213)$ while decrease as $m \in [0.4213, 1)$. Under the assumption that E_1 is saddle point, we have (1) $0 \leqslant m < 0.3249$, the condition of Corollary 2.9 holds, model (3.1) admits exactly one limit cycle; (2) $0.3249 < m < 0.7106$, the condition of Corollary 2.6 holds, E_* is globally stable; (3) $0.7106 < m < 1$, the condition of Theorem 2.4 holds , E_2 is globally stable. Fig. 1 shows above dynamic behavior of model (3.1). Under the assumption that E_1 is not saddle point, Fig. 2 (b) and Fig. 2 (c) show that three different orbits terminate on the predator axis and then decline to zero, another three different orbits tend to E_* or E_2, respectively, i.e. E_* or E_2 is locally asymptotically stable; Fig. 2 (a) shows that all trajectories go around the unstable equilibrium E_* and eventually terminate on the predator axis and then decline to zero. Dynamical analysis of model (1.2) can be summarized and classified into the Table 1.

Table 1: Schematic representation of our analytical findings: LAS=locally asymptotically stable, GAS=globally asymptotically stable.

Condition	E_1	E_2	E_*	Bifurcation	Phase portrait
$0 \leqslant m < B$	saddle point	Unstable	Unstable		Fig. 1 (a)
	not saddle point	Unstable	Unstable		Fig. 2 (a)
$m = B$				Hopf	
$B < m < A$	saddle point	Unstable	GAS		Fig. 1 (b)
	not saddle point	Unstable	LAS		Fig. 2 (b)
$m = A$				Transcritical at $E_2 = E_*$	
$A < m < 1$	saddle point	GAS	Infeasible		Fig. 1 (c)
	not saddle point	LAS	Infeasible		Fig. 2 (c)

4. Conclusions

In this paper, we have considered the dynamical behavior of a predator-prey model with square root functional response and prey refuge. Depending on the technique of qualitative analysis, we have given the complete qualitative analysis of the instability and global stability properties of the equilibria and the existence and uniqueness of limit cycles for the considered model. Taking prey refuge m as a bifurcation parameter, it is shown that the model undergoes Transcritical bifurcation and Hopf bifurcation. Numerical simulations indicate the more complex dynamical behaviors of the model than those with Holling II type functional response in which the origin is unstable (e.g. [2, 4, 7, 8]). Fig. 1 and Fig. 2 show that the origin of the model acts not only a saddle point, more specifically, some trajectories eventually terminate on the predator axis and then decline to zero. In particular, when prey refuge m is less than the Hopf bifurcation value, as the value of prey refuge m used in Fig. 2 (a), the origin can be the stable global attractor. From the perspective of ecological significance, if the value of prey refuge m is too small, then more prey population is captured by predator population and its own fecundity leads to the demise of the prey and then itself. Hence, taking prey refuge m as control variable, it is feasible to decrease predation rate and then control predator density properly so as to avoid two of population extinction and promote coexistence.
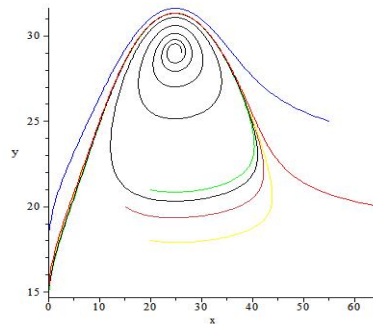
(a) there is a stable limit cycle surrounding $E_*(25.3447, 29.2775)$ with $m = 0.315$.

(b) $E_*(26.7094, 29.6391)$ is globally asymptotically stable with $m = 0.35$.
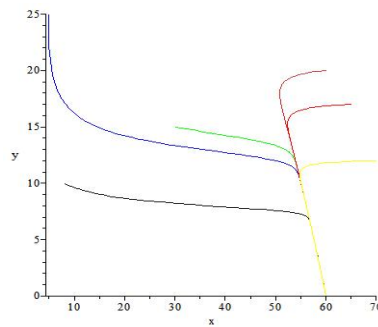
(c) $E_2(60, 0)$ is globally asymptotically stable with $m = 0.8$.

Figure 1: E_1 is saddle point. The phase portrait of system (1.2) for the different values of prey refuge m.

(a) All trajectories go around $E_*(24.8016, 29.0993)$ and eventually terminate on the predator axis and then decline to zero with $m = 0.3$.

(b) $E_*(28.9352, 29.9620)$ is locally asymptotically stable with $m = 0.4$.

(c) $E_2(60, 0)$ is locally asymptotically stable with $m = 0.75$.

Figure 2: E_1 is not saddle point. The phase portrait of system (1.2) for the different values of prey refuge m.

Acknowledgment

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