



Multi-valued tripled fixed point results via CLR property in metric spaces with application

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Abstract

In this work, using CLR property, tripled coincidence and common fixed point theorems for hybrid pair of mappings are studied. As an application, existence of solution to the system of integral equation is also discussed.

Keywords: Hybrid maps, tripled fixed point, CLR property.

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1. Introduction and preliminaries

Alber and Gurre [2] presented weak contraction by generalizing the concept of contraction and showed the presence of fixed points for a self-map in Hilbert space. In [18], Rhoades proved that the result is still true for ϕ -weak contraction in metric space. Dutta and Choudhury [9] investigated fixed point results by using (ψ, ϕ) weak contraction. In [22], Zhang and Song proved fixed point theorem for generalized ϕ -weak contraction for two self-maps. Doric [8] generalized the result of Zhang and Song for the presence of common fixed point with (ψ, ϕ) weak contraction.

Bhaskar and Lakshmikantham [10] derived some coupled fixed point results using mixed monotone property. Furthermore, they applied their results on a first order differential equation with periodic boundary conditions [11]. Lakshmikantham and Ćirić [13] generalized the concept of mixed monotone mapping and established a coupled fixed point theorem for nonlinear contractions in partially ordered metric spaces.

Berinde and Borcut [5] introduced the concept of tripled fixed point for single-valued mappings in partially ordered metric spaces. Rao et al. [15] obtained a unique common triple fixed point theorem for hybrid pair of mappings in metric spaces. Aydi and Abbas [4] introduced the concept of w -compatibility for single-valued tripled hybrid mappings.

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Deshpande et al. [7] introduced the concept of w -compatibility and weakly commutativity for hybrid pair of mappings, $F : \Delta \times \Delta \times \Delta \rightarrow 2^\Delta$ and $g : \Delta \rightarrow \Delta$ and established a common tripled fixed point theorem with generalized nonlinear contraction. In [6] the authors introduced the concept of E.A property and occasionally w -compatibility for hybrid pair $F : \Delta \times \Delta \times \Delta \rightarrow 2^\Delta$ and $g : \Delta \rightarrow \Delta$. Khan and Sumitra [12, 17] defined E.A and CLR property for coupled mapping. Rao et al. [16] defined CLRg property for multi-valued coupled maps in fuzzy metric spaces. For detail see ([3, 20]).

In the present manuscript, by using CLR property, some common hybrid tripled fixed point results for (ψ, ϕ) weak contraction in the setting of metric space are established. Throughout the paper \mathbb{R}^+ , \mathbb{N} and \mathbb{N}_0 stand for the set of all non-negative real numbers, the set of positive integers and the set of non-negative, integers respectively.

Definition 1.1. Suppose Δ is nonempty set and let $d : \Delta \times \Delta \rightarrow \mathbb{R}^+$ be a function satisfying the conditions:

1. $d(\xi, \zeta) = 0$ implies that $\xi = \zeta$, for all $\xi, \zeta \in \Delta$;
2. $d(\xi, \zeta) > 0$, for all $\xi \neq \zeta, \xi, \zeta \in \Delta$;
3. $d(\xi, \zeta) = d(\zeta, \xi)$, where $\xi, \zeta \in \Delta$;
4. $d(\xi, \zeta) \leq d(\xi, \tilde{z}) + d(\zeta, \tilde{z})$, for all $\xi, \zeta, \tilde{z} \in \Delta$.

Then d is a metric on Δ and the pair (Δ, d) is called metric space.

Definition 1.2 ([19]). Let Δ be a nonempty set and $F : \Delta \times \Delta \times \Delta \rightarrow \Delta$ be a given mapping. An element $(x, y, z) \in \Delta \times \Delta \times \Delta$ is called a tripled fixed point of the mapping F if $F(x, y, z) = x$, $F(y, z, x) = y$ and $F(z, x, y) = z$.

We denote the set of tripled coincidence points of mappings F and g by $C(F, g)$. Note that if $(x, y, z) \in C(F, g)$, then (y, z, x) and (z, x, y) are also in $C(F, g)$.

Definition 1.3 ([7]). Let $F : \Delta \times \Delta \times \Delta \rightarrow 2^\Delta$ be a multi-valued mapping and g be a self-map on Δ . The hybrid pair (F, g) is called w -compatible if $g(F(x, y, z)) \in F(gx, gy, gz)$ whenever $(x, y, z) \in C(F, g)$.

Definition 1.4 ([7]). Let $F : \Delta \times \Delta \times \Delta \rightarrow 2^\Delta$ be a multi-valued mapping and g be a self-map on Δ . The mapping g is called F -weakly commuting at some point $(x, y, z) \in \Delta^3$, if $g^2x \in F(gx, gy, gz)$, $g^2y \in F(gy, gz, gx)$ and $g^2z \in F(gz, gx, gy)$.

Definition 1.5 ([7]). Let Δ be a nonempty set, $F : \Delta \times \Delta \times \Delta \rightarrow 2^\Delta$ (a collection of all nonempty subsets of Δ) and g be a self-map on Δ . An element $(x, y, z) \in \Delta \times \Delta \times \Delta$ is called

- (1) a tripled fixed point of F if $x \in F(x, y, z)$, $y \in F(y, z, x)$ and $z \in F(z, x, y)$;
- (2) a tripled coincidence point of hybrid pair $\{F, g\}$ if $g(x) \in F(x, y, z)$, $g(y) \in F(y, z, x)$ and $g(z) \in F(z, x, y)$;
- (3) a common tripled fixed point of hybrid pair F, g if $x = g(x) \in F(x, y, z)$, $y = g(y) \in F(y, z, x)$ and $z = g(z) \in F(z, x, y)$.

Definition 1.6 ([21]). Maps $f, g : \Delta \rightarrow \Delta$ are said to satisfy the common limit in the range of f with respect to g (shortly, the (CLR_f) -property with respect to g) if there exists a sequence ξ_n in Δ such that for some $u \in \Delta$, $\lim_{n \rightarrow \infty} f\xi_n = \lim_{n \rightarrow \infty} g\xi_n = fu$.

The following definition can be found in [1].

Definition 1.7. Mappings $f : \Delta \rightarrow \Delta$, $S : \Delta \rightarrow CB(\Delta)$ on metric space (Δ, d) are said to satisfy the common limit in the range of f with respect to S (shortly, the (CLR_f) -property with respect to S) if there exists a sequence ξ_n in Δ and $\Omega_1 \in CB(\Delta)$ such that for some $u \in \Delta$, $\lim_{n \rightarrow \infty} f\xi_n = fu \in \Omega_1 = \lim_{n \rightarrow \infty} S\xi_n = fu$.

Definition 1.8. Mappings $f, g : \Delta \rightarrow \Delta$ and $S, T : \Delta \rightarrow CB(\Delta)$ on metric space (Δ, d) are to satisfy the common limit in the range of f with respect to S (shortly, the (CLR_f) -property with respect to S) if there exist sequences ξ_n and ζ_n in Δ and $\Omega_1, \Omega_2 \in CB(\Delta)$ such that for some $u \in \Delta$, $\lim_{n \rightarrow \infty} S\xi_n = \Omega_1$, $\lim_{n \rightarrow \infty} T\zeta_n = \Omega_2$ and $\lim_{n \rightarrow \infty} f\xi_n = \lim_{n \rightarrow \infty} g\zeta_n = fu \in \Omega_1 \cap \Omega_2$.

Definition 1.9. Mappings $f : \Delta \rightarrow \Delta$ and $S : \Delta \times \Delta \rightarrow CB(\Delta)$ on metric space (Δ, d) are said to have E.A property if there exist sequences ξ_n and ζ_n in Δ and $\Omega_1, \Omega_2 \in CB(\Delta)$ such that for some $u \in \Delta$, $\lim_{n \rightarrow \infty} f\xi_n = u \in \Omega_1 = \lim_{n \rightarrow \infty} S(\xi_n, \zeta_n)$, $\lim_{n \rightarrow \infty} f\zeta_n = v \in \Omega_2 = \lim_{n \rightarrow \infty} S(\zeta_n, \xi_n)$.

Definition 1.10. Mappings $f : \Delta \rightarrow \Delta$ and $S : \Delta \times \Delta \rightarrow CB(\Delta)$ on metric space (Δ, d) are to satisfy the common limit in the range of f with respect to S (shortly, the (CLR_f) -property with respect to S) if there exist sequences ξ_n and ζ_n in Δ and $\Omega_1, \Omega_2 \in CB(\Delta)$ such that for some $u \in \Delta$

$$\lim_{n \rightarrow \infty} f\xi_n = u = f\xi \in \Omega_1 = \lim_{n \rightarrow \infty} S(\xi_n, \zeta_n),$$

$$\lim_{n \rightarrow \infty} f\zeta_n = v = f\zeta \in \Omega_2 = \lim_{n \rightarrow \infty} S(\zeta_n, \xi_n).$$

In the sequel we also need the following:

Assume that d is a metric on Δ , define the function $H : CB(\Delta) \times CB(\Delta) \rightarrow \mathbb{R}^+$ for $\Omega_1, \Omega_2 \in CB(\Delta)$ by

$$H(\Omega_1, \Omega_2) = \max\left\{\sup_{\sigma \in \Omega_1} d(\sigma, \Omega_2), \sup_{\zeta \in \Omega_1} d(\zeta, \Omega_1)\right\},$$

where

$$d(\xi, \Omega_1) = \inf\{d(\xi, \zeta) : \zeta \in \Omega_1\},$$

$$\delta(\Omega_1, \Omega_2) = \sup\{d(\sigma, \zeta) : \sigma \in \Omega_1, \zeta \in \Omega_2\},$$

and

$$D(\Omega_1, \Omega_2) = \inf\{d(\sigma, \zeta) : \sigma \in \Omega_1, \zeta \in \Omega_2\}.$$

The following can be deduced from the definition of δ

$$\delta(\Omega_1, \Omega_2) = \delta(\Omega_2, \Omega_1),$$

$$\delta(\Omega_1, \Omega_3) \leq \delta(\Omega_1, \Omega_2) + \delta(\Omega_2, \Omega_3),$$

$$\delta(\Omega_1, \Omega_2) = 0 \Leftrightarrow \Omega_1 = \Omega_2 = \{\sigma\},$$

$$\delta(\Omega_1, \Omega_1) = \text{diam}\Omega_1.$$

Lemma 1.11 ([14]). *Let (Δ, d) be a metric space. For any $\Omega_1, \Omega_2, \Omega_3 \in CB(\Delta)$ and $\xi, \zeta \in \Delta$, the following hold:*

1. $d(\xi, \Omega_2) \leq d(\xi, \zeta), \forall \zeta \in \Omega_2$;
2. $\delta(\Omega_1, \Omega_2) \leq H(\Omega_1, \Omega_2)$;
3. $d(\xi, \Omega_2) \leq H(\Omega_1, \Omega_2), \forall \xi \in \Omega_1$;
4. $H(\Omega_1, \Omega_1) = 0$;
5. $H(\Omega_1, \Omega_2) = H(\Omega_2, \Omega_1)$;
6. $H(\Omega_1, \Omega_3) \leq H(\Omega_1, \Omega_2) + H(\Omega_2, \Omega_3)$;
7. $H(\xi, \Omega_1) \leq d(\xi, \zeta) + d(\zeta, \Omega_1)$.

Lemma 1.12 ([14]). *Suppose (Δ, d) is metric space. Furthermore, let $\Omega_1, \Omega_2 \in CB(\Delta)$, then for every $h > 1$ and for each $\sigma \in \Omega_1$ there exists $\zeta(\sigma) \in \Omega_2$ such that $d(\sigma, \zeta) \leq hH(\Omega_1, \Omega_2)$.*

In [14] it is shown that the above lemma is also true for $h \geq 1$.

Lemma 1.13. *Assume a metric space (Δ, d) . Let $\Omega_1, \Omega_2 \in CB(\Delta)$, then for every $h \geq 1$ and for each $\sigma \in \Omega_1$ there exists $\zeta(\sigma) \in \Omega_2$ such that $d(\sigma, \zeta) \leq hH(\Omega_1, \Omega_2)$.*

Definition 1.14. Let $f : \Delta \rightarrow \Delta$, $F : \Delta \times \Delta \times \Delta \rightarrow Cl(\Delta)$ on metric space (Δ, d) . Mappings (F, f) have (CLR_f) -property, if there exist sequences x_n, y_n, z_n in Δ for some $z_1, z_2, z_3 \in \Delta$ and $C_1, D_1, E_1 \in CB(\Delta)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} F(x_n, y_n, z_n) &= C_1, \lim_{n \rightarrow \infty} fx_n = fz_1 \in C_1, \\ \lim_{n \rightarrow \infty} F(y_n, x_n, z_n) &= D_1, \lim_{n \rightarrow \infty} fy_n = fz_2 \in D_1, \\ \lim_{n \rightarrow \infty} F(z_n, y_n, x_n) &= E_1, \lim_{n \rightarrow \infty} fz_n = fz_3 \in E_1. \end{aligned}$$

Definition 1.15. Let $f, g : \Delta \rightarrow \Delta$, $F, G : \Delta \times \Delta \times \Delta \rightarrow Cl(\Delta)$ on metric space (Δ, d) . Mappings (F, f) and (G, g) have (CLR_f) and (CLR_g) -property, respectively, if there exist sequences x_n, y_n, z_n and u_n, v_n, w_n in Δ for some $z_1, z_2, z_3 \in \Delta$ and $C_1, C_2, D_1, D_2, E_1, E_2 \in CB(\Delta)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} F(x_n, y_n, z_n) &= C_1, \lim_{n \rightarrow \infty} G(u_n, v_n, w_n) = C_2, \\ \lim_{n \rightarrow \infty} fx_n = fz_1 \in C_1, \lim_{n \rightarrow \infty} gu_n &= gz_1 \in C_2, \\ \lim_{n \rightarrow \infty} F(y_n, x_n, z_n) &= D_1, \lim_{n \rightarrow \infty} G(v_n, u_n, w_n) = D_2, \\ \lim_{n \rightarrow \infty} fy_n = fz_2 \in D_1, \lim_{n \rightarrow \infty} gv_n &= gz_2 \in D_2, \\ \lim_{n \rightarrow \infty} F(z_n, y_n, x_n) &= E_1, \lim_{n \rightarrow \infty} G(w_n, v_n, u_n) = E_2, \\ \lim_{n \rightarrow \infty} fz_n = fz_3 \in E_1, \lim_{n \rightarrow \infty} gw_n &= gz_3 \in E_2. \end{aligned}$$

2. Main results

In our main results we use the following two classes.

$$\begin{aligned} \Psi &= \left\{ \begin{array}{l} \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ is continuous and non-decreasing function such that} \\ \psi(\tau) = 0 \text{ if and only if } \tau = 0. \end{array} \right\}, \\ \Phi &= \left\{ \begin{array}{l} \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ is a lower semicontinuous and non-decreasing} \\ \text{function such that } \phi(\tau) = 0 \text{ if and only if } \tau = 0 \end{array} \right\}. \end{aligned}$$

Theorem 2.1. Let $f, g : \Delta \rightarrow \Delta$, $F, G : \Delta \times \Delta \times \Delta \rightarrow Cl(\Delta)$ on metric space (Δ, d) . Furthermore assume that (F, f) have CLR_f and (G, g) have CLR_g -property, also the following condition holds:

$$\psi(H^p(F(x, y, z), G(u, v, w))) \leq \psi(\Theta(x, y, z, u, v, w)) - \phi(\Theta(x, y, z, u, v, w)), \tag{2.1}$$

where

$$\Theta(x, y, z, u, v, w) = \max \left\{ \begin{array}{l} d^p(fx, gu), d^p(fx, F(x, y, z)), d^p(gu, G(u, v, w)), \\ d^p(fy, gv), d^p(fy, F(y, z, x)), d^p(gv, G(v, w, u)), \\ d^p(fz, gw), d^p(fz, F(z, x, y)), d^p(gw, G(w, u, v)), \\ \frac{d^p(fx, gu) + d^p(fx, F(x, y, z)) + d^p(gu, G(u, v, w))}{2}, \\ \frac{d^p(fy, gv) + d^p(fy, F(y, z, x)) + d^p(gv, G(v, w, u))}{2}, \\ \frac{d^p(fz, gw) + d^p(fz, F(z, x, y)) + d^p(gw, G(w, u, v))}{2}. \end{array} \right\}.$$

Here, $\psi \in \Psi$, $\phi \in \Phi$ and $p \geq 1$. Then

- (A₁) (f, F) have tripled coincidence point.
- (A₂) (g, G) have tripled coincidence point.
- (A₃) If f is F -weakly commuting for $(z_1, z_2, z_3) \in C(F, f)$ and $f^2z_1 = fz_1, f^2z_2 = fz_2, f^2z_3 = fz_3$, then F and f have a common tripled fixed point.
- (A₄) If g is G -weakly commuting for $(w_1, w_2, w_3) \in C(G, g)$ and $g^2w_1 = gw_1, g^2w_2 = gw_2, g^2w_3 = gw_3$, then G and g have a common tripled fixed point.

(A₅) F, G, f and g have common tripled fixed point if (A₃) and (A₄) hold.

Proof. Since (F, f) have (CLR_f) -property, there exist sequences x_n, y_n, z_n and $C_1, D_1, E_1 \in CB(\Delta)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} F(x_n, y_n, z_n) &= C_1, & \lim_{n \rightarrow \infty} fx_n &= fz_1 \in C_1, \\ \lim_{n \rightarrow \infty} F(y_n, x_n, z_n) &= D_1, & \lim_{n \rightarrow \infty} fy_n &= fz_2 \in D_1, \\ \lim_{n \rightarrow \infty} F(z_n, y_n, x_n) &= E_1, & \lim_{n \rightarrow \infty} fz_n &= fz_3 \in E_1, \end{aligned}$$

for some $z_1, z_2, z_3 \in \Delta$.

Similarly, since (G, g) have (CLR_g) -property, there exist sequences u_n, v_n, w_n and $C_2, D_2, E_2 \in CB(\Delta)$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} G(u_n, v_n, w_n) &= C_2, & \lim_{n \rightarrow \infty} gu_n &= gz_1 \in C_2, \\ \lim_{n \rightarrow \infty} G(v_n, u_n, w_n) &= D_2, & \lim_{n \rightarrow \infty} gv_n &= gz_2 \in D_2, \\ \lim_{n \rightarrow \infty} G(w_n, v_n, u_n) &= E_2, & \lim_{n \rightarrow \infty} gw_n &= gz_3 \in E_2, \end{aligned}$$

for some $z_1, z_2, z_3 \in \Delta$.

Putting $x = x_n, y = y_n, z = z_n$ and $u = u_n, v = v_n, w = w_n$ in inequality (2.1) we get

$$\psi(H^P(F(x_n, y_n, z_n), G(u_n, v_n, w_n))) \leq \psi(\Theta(x_n, y_n, z_n, u_n, v_n, w_n)) - \phi(\Theta(x_n, y_n, z_n, u_n, v_n, w_n)), \tag{2.2}$$

where

$$\Theta(x_n, y_n, z_n, u_n, v_n, w_n) = \max \left\{ \begin{aligned} & d^P(fx_n, gu_n), d^P(fx_n, F(x_n, y_n, z_n)), d^P(gu_n, G(u_n, v_n, w_n)), \\ & d^P(fy_n, gv_n), d^P(fy_n, F(y_n, z_n, x_n)), d^P(gv_n, G(v_n, w_n, u_n)), \\ & d^P(fz_n, gw_n), d^P(fz_n, F(z_n, x_n, y_n)), d^P(gw_n, G(w_n, u_n, v_n)), \\ & \frac{d^P(fx_n, gu_n) + d^P(fx_n, F(x_n, y_n, z_n)) + d^P(gu_n, G(u_n, v_n, w_n))}{2}, \\ & \frac{d^P(fy_n, gv_n) + d^P(fy_n, F(y_n, z_n, x_n)) + d^P(gv_n, G(v_n, w_n, u_n))}{2}, \\ & \frac{d^P(fz_n, gw_n) + d^P(fz_n, F(z_n, x_n, y_n)) + d^P(gw_n, G(w_n, u_n, v_n))}{2}. \end{aligned} \right\}.$$

Applying limit to (2.2) we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi(H^P(F(x_n, y_n, z_n), G(u_n, v_n, w_n))) &\leq \lim_{n \rightarrow \infty} \psi(\Theta(x_n, y_n, z_n, u_n, v_n, w_n)) \\ &\quad - \lim_{n \rightarrow \infty} \phi(\Theta(x_n, y_n, z_n, u_n, v_n, w_n)), \end{aligned}$$

where

$$\lim_{n \rightarrow \infty} \Theta(x_n, y_n, z_n, u_n, v_n, w_n) = \max \{ d^P(fz_1, gz_1), d^P(fz_2, gz_2), d^P(fz_3, gz_3) \},$$

which implies

$$\psi(d^P(fz_1, gz_1)) \leq \psi \left(\max \left\{ \begin{aligned} & d^P(fz_1, gz_1), \\ & d^P(fz_2, gz_2), \\ & d^P(fz_3, gz_3) \end{aligned} \right\} \right) - \phi \left(\max \left\{ \begin{aligned} & d^P(fz_1, gz_1), \\ & d^P(fz_2, gz_2), \\ & d^P(fz_3, gz_3) \end{aligned} \right\} \right). \tag{2.3}$$

By putting $x = y_n, y = z_n, z = x_n$ and $u = v_n, v = w_n, w = u_n$ in inequality (2.1) we get

$$\psi(H^P(F(x_n, y_n, z_n), G(u_n, v_n, w_n))) \leq \psi(\Theta(x_n, y_n, z_n, u_n, v_n, w_n)) - \phi(\Theta(x_n, y_n, z_n, u_n, v_n, w_n)). \tag{2.4}$$

By applying limit to (2.4) we conclude that

$$\psi(d^P(fz_2, gz_2)) \leq \psi \left(\max \left\{ \begin{aligned} & d^P(fz_1, gz_1), \\ & d^P(fz_2, gz_2), \\ & d^P(fz_3, gz_3) \end{aligned} \right\} \right) - \phi \left(\max \left\{ \begin{aligned} & d^P(fz_1, gz_1), \\ & d^P(fz_2, gz_2), \\ & d^P(fz_3, gz_3) \end{aligned} \right\} \right). \tag{2.5}$$

Next, putting $x = z_n, y = y_n, z = x_n$ and $u = w_n, v = v_n, w = u_n$ in inequality (2.1) we get

$$\psi(H^P(F(z_n, y_n, x_n), G(w_n, v_n, u_n))) \leq \psi(\Theta(z_n, y_n, x_n, w_n, v_n, u_n)) - \phi(\Theta(z_n, y_n, x_n, w_n, v_n, u_n)). \tag{2.6}$$

Again, applying limit to (2.6), by simplification one can write

$$\psi(d^P(fz_3, gz_3)) \leq \psi\left(\max\left\{\begin{matrix} d^P(fz_1, gz_1), \\ d^P(fz_2, gz_2), \\ d^P(fz_3, gz_3) \end{matrix}\right\}\right) - \phi\left(\max\left\{\begin{matrix} d^P(fz_1, gz_1), \\ d^P(fz_2, gz_2), \\ d^P(fz_3, gz_3) \end{matrix}\right\}\right). \tag{2.7}$$

Combining (2.3), (2.5) and (2.7) we have

$$\psi\left(\max\left\{\begin{matrix} d^P(fz_1, gz_1), \\ d^P(fz_2, gz_2), \\ d^P(fz_3, gz_3) \end{matrix}\right\}\right) \leq \psi\left(\max\left\{\begin{matrix} d^P(fz_1, gz_1), \\ d^P(fz_2, gz_2), \\ d^P(fz_3, gz_3) \end{matrix}\right\}\right) - \phi\left(\max\left\{\begin{matrix} d^P(fz_1, gz_1), \\ d^P(fz_2, gz_2), \\ d^P(fz_3, gz_3) \end{matrix}\right\}\right).$$

By using the definition of ϕ and ψ we conclude that

$$\max\{d^P(fz_1, gz_1), d^P(fz_2, gz_2), d^P(fz_3, gz_3)\} = 0,$$

which implies that

$$d^P(fz_1, gz_1) = d^P(fz_2, gz_2) = d^P(fz_3, gz_3) = 0.$$

Thus

$$fz_1 = gz_1, \quad fz_2 = gz_2, \quad fz_3 = gz_3. \tag{2.8}$$

By putting $x = z_1, y = z_2, z = z_3$ and $u = u_n, v = v_n, w = w_n$ in inequality (2.1) we get

$$\psi(H^P(F(z_1, z_2, z_3), G(u_n, v_n, w_n))) \leq \psi(\Theta(z_1, z_2, z_3, u_n, v_n, w_n)) - \phi(\Theta(z_1, z_2, z_3, u_n, v_n, w_n)), \tag{2.9}$$

where,

$$\Theta(z_1, z_2, z_3, u_n, v_n, w_n) = \max\left\{\begin{matrix} d^P(fz_1, gu_n), d^P(fz_1, F(z_1, z_2, z_3)), d^P(gu_n, G(u_n, v_n, w_n)), \\ d^P(fz_2, gv_n), d^P(fz_2, F(z_2, z_3, z_1)), d^P(gv_n, G(v_n, w_n, u_n)), \\ d^P(fz_3, gw_n), d^P(fz_3, F(z_3, z_1, z_2)), d^P(gw_n, G(w_n, u_n, v_n)), \\ \frac{d^P(fz_1, gu_n) + d^P(fz_1, F(z_1, z_2, z_3)) + d^P(gu_n, G(u_n, v_n, w_n))}{2}, \\ \frac{d^P(fz_2, gv_n) + d^P(fz_2, F(z_2, z_3, z_1)) + d^P(gv_n, G(v_n, w_n, u_n))}{2}, \\ \frac{d^P(fz_3, gw_n) + d^P(fz_3, F(z_3, z_1, z_2)) + d^P(gw_n, G(w_n, u_n, v_n))}{2} \end{matrix}\right\}.$$

Applying limit to (2.9) and using (2.8) we have

$$\begin{aligned} \psi(d^P(fz_1, F(z_1, z_2, z_3))) &\leq \psi\left(\max\left\{\begin{matrix} d^P(fz_1, F(z_1, z_2, z_3)), \\ d^P(fz_2, F(z_2, z_3, z_1)), \\ d^P(fz_3, F(z_3, z_1, z_2)) \end{matrix}\right\}\right) \\ &\quad - \phi\left(\max\left\{\begin{matrix} d^P(fz_1, F(z_1, z_2, z_3)), \\ d^P(fz_2, F(z_2, z_3, z_1)), \\ d^P(fz_3, F(z_3, z_1, z_2)) \end{matrix}\right\}\right). \end{aligned} \tag{2.10}$$

By putting $x = z_2, y = z_3, z = z_1$ and $u = u_n, v = v_n, w = w_n$ in inequality (2.1), by following similar step we get

$$\begin{aligned} \psi(d^P(fz_2, F(z_2, z_3, z_1))) &\leq \psi\left(\max\left\{\begin{matrix} d^P(fz_1, F(z_1, z_2, z_3)), \\ d^P(fz_2, F(z_2, z_3, z_1)), \\ d^P(fz_3, F(z_3, z_1, z_2)) \end{matrix}\right\}\right) \\ &\quad - \phi\left(\max\left\{\begin{matrix} d^P(fz_1, F(z_1, z_2, z_3)), \\ d^P(fz_2, F(z_2, z_3, z_1)), \\ d^P(fz_3, F(z_3, z_1, z_2)) \end{matrix}\right\}\right). \end{aligned} \tag{2.11}$$

By placing $x = z_3$, $y = z_1$, $z = z_2$ and $u = u_n$, $v = v_n$, $w = w_n$ in inequality (2.1), by following same line we have

$$\begin{aligned} \psi(d^p(fz_3, F(z_3, z_1, z_2))) &\leq \psi\left(\max\left\{\begin{array}{l} d^p(fz_1, F(z_1, z_2, z_3)), \\ d^p(fz_2, F(z_2, z_3, z_1)), \\ d^p(fz_3, F(z_3, z_1, z_2)) \end{array}\right\}\right) \\ &\quad - \phi\left(\max\left\{\begin{array}{l} d^p(fz_1, F(z_1, z_2, z_3)), \\ d^p(fz_2, F(z_2, z_3, z_1)), \\ d^p(fz_3, F(z_3, z_1, z_2)) \end{array}\right\}\right). \end{aligned} \tag{2.12}$$

Combining (2.10), (2.11) and (2.12) we have

$$\begin{aligned} \psi\left(\max\left\{\begin{array}{l} (d^p(fz_1, F(z_1, z_2, z_3))), \\ d^p(fz_2, F(z_2, z_3, z_1)), \\ d^p(fz_3, F(z_3, z_1, z_2)) \end{array}\right\}\right) &\leq \psi\left(\max\left\{\begin{array}{l} d^p(fz_1, F(z_1, z_2, z_3)), \\ d^p(fz_2, F(z_2, z_3, z_1)), \\ d^p(fz_3, F(z_3, z_1, z_2)) \end{array}\right\}\right) \\ &\quad - \phi\left(\max\left\{\begin{array}{l} [d^p(fz_1, F(z_1, z_2, z_3)), \\ d^p(fz_2, F(z_2, z_3, z_1)), \\ d^p(fz_3, F(z_3, z_1, z_2))] \end{array}\right\}\right), \end{aligned}$$

using the definition of ψ and ϕ one can get

$$\max\left\{\begin{array}{l} d^p(fz_1, F(z_1, z_2, z_3)), \\ d^p(fz_2, F(z_2, z_3, z_1)), \\ d^p(fz_3, F(z_3, z_1, z_2)) \end{array}\right\} = 0,$$

which implies $d^p(fz_1, F(z_1, z_2, z_3)) = 0$, $d^p(fz_2, F(z_2, z_3, z_1)) = 0$ and $d^p(fz_3, F(z_3, z_1, z_2)) = 0$. Thus $fz_1 \in F(z_1, z_2, z_3)$, $fz_2 \in F(z_2, z_3, z_1)$ and $fz_3 \in F(z_3, z_1, z_2)$.

Similarly, by the same procedure one can obtain

$$gw_1 \in G(w_1, w_2, w_3), \quad gw_2 \in G(w_2, w_3, w_1), \quad gw_3 \in G(w_3, w_1, w_2).$$

Since, f is F -weakly commuting so $f^2z_1 \in F(fz_1, fz_2, fz_3)$, $f^2z_2 \in F(fz_2, fz_3, fz_1)$ and $f^2z_3 \in F(fz_3, fz_1, fz_2)$. Since, $f^2z_1 = fz_1$, $f^2z_2 = fz_2$ and $f^2z_3 = fz_3$, thus (fz_1, fz_2, fz_3) is a common fixed point. A similar argument proves (A_4) . Then using (2.8) (A_5) holds immediately. \square

By taking $f(x) = g(x) = x$ in Theorem 2.1 we deduce the following corollary.

Corollary 2.2. *Let $F, G : \Delta \times \Delta \times \Delta \rightarrow Cl(\Delta)$ on metric space (Δ, d) , and satisfying the following*

$$\psi(H^p(F(x, y, z), G(u, v, w))) \leq \psi(\Theta(x, y, z, u, v, w)) - \phi(\Theta(x, y, z, u, v, w)),$$

where

$$\Theta(x, y, z, u, v, w) = \max\left\{\begin{array}{l} d^p(x, u), d^p(x, F(x, y, z)), d^p(u, G(u, v, w)), \\ d^p(y, v), d^p(y, F(y, z, x)), d^p(v, G(v, w, u)), \\ d^p(z, w), d^p(z, F(z, x, y)), d^p(w, G(w, u, v)), \\ \frac{d^p(x, u) + d^p(x, F(x, y, z)) + d^p(u, G(u, v, w))}{2}, \\ \frac{d^p(y, v) + d^p(y, F(y, z, x)) + d^p(v, G(v, w, u))}{2}, \\ \frac{d^p(z, w) + d^p(z, F(z, x, y)) + d^p(w, G(w, u, v))}{2}. \end{array}\right\}.$$

Here, $\psi \in \Psi$, $\phi \in \Phi$ and $p \geq 1$. Then F and G have common tripled fixed point.

Similarly, we can prove the following result.

Theorem 2.3. Let $f : \Delta \rightarrow \Delta, F : \Delta \times \Delta \times \Delta \rightarrow Cl(\Delta)$ on metric space (Δ, d) . Furthermore assume that (F, f) have CLR_f -property and satisfy the following

$$\psi(H^p(F(x, y, z), F(u, v, w))) \leq \psi(\Theta(x, y, z, u, v, w)) - \phi(\Theta(x, y, z, u, v, w)),$$

where

$$\Theta(x, y, z, u, v, w) = \max \left\{ \begin{array}{l} d^p(fx, fu), d^p(fx, F(x, y, z)), d^p(fu, F(u, v, w)), \\ d^p(fy, fv), d^p(fy, F(y, z, x)), d^p(fv, F(v, w, u)), \\ d^p(fz, fw), d^p(fz, F(z, x, y)), d^p(fw, F(w, u, v)), \\ \frac{d^p(fx, fu) + d^p(fx, F(x, y, z)) + d^p(fu, F(u, v, w))}{2}, \\ \frac{d^p(fy, fv) + d^p(fy, F(y, z, x)) + d^p(fv, F(v, w, u))}{2}, \\ \frac{d^p(fz, fw) + d^p(fz, F(z, x, y)) + d^p(fw, F(w, u, v))}{2}. \end{array} \right\}.$$

Here, $\psi \in \Psi, \phi \in \Phi$ and $p \geq 1$. There exists a common fixed point of f and F if one of the following holds:

- (A₁) (F, f) are weakly compatible $\lim_{n \rightarrow \infty} f^n z_1 = u, \lim_{n \rightarrow \infty} f^n z_2 = v, \lim_{n \rightarrow \infty} f^n z_3 = w$, for $(z_1, z_2, z_3) \in C(f, F)$ where f is Continuous at u, v, w .
- (A₂) f is F weakly commuting $f^2 z_1 = f z_1, f^2 z_2 = f z_2, f^2 z_3 = f z_3$, for $(z_1, z_2, z_3) \in C(f, F)$.
- (A₃) $\lim_{n \rightarrow \infty} f^n u = z_1, \lim_{n \rightarrow \infty} f^n u = z_2, \lim_{n \rightarrow \infty} f^n w = z_3$, for $(z_1, z_2, z_3) \in C(f, F)$ where f is Continuous at z_1, z_2 and z_3 .
- (A₄) $f(C(f, F))$ is singleton set of $C(f, F)$.

Proof. Following the line of the proof of Theorem 2.1, we conclude that (A₂) holds. Further, we have to check that (A₁), (A₃) and (A₄) hold respectively. Suppose condition (A₁) holds i.e., $\lim_{n \rightarrow \infty} f^n z_1 = u, \lim_{n \rightarrow \infty} f^n z_2 = v, \lim_{n \rightarrow \infty} f^n z_3 = w$. Since f is Continuous at u, v and w we have

$$fu = u, \quad fv = v, \quad fw = w. \tag{2.13}$$

As (F, f) are weakly compatible, therefore for $n \geq 1$

$$\left\{ \begin{array}{l} f^n z_1 \in F(f^{n-1} z_1, f^{n-1} z_2, f^{n-1} z_3), \\ f^n z_2 \in F(f^{n-1} z_2, f^{n-1} z_1, f^{n-1} z_3), \\ f^n z_3 \in F(f^{n-1} z_3, f^{n-1} z_2, f^{n-1} z_1). \end{array} \right\}. \tag{2.14}$$

Next,

$$\begin{aligned} \psi(D^p(f^n z_1, F(u, v, w))) &\leq \psi(H^p(F(f^{n-1} z_1, f^{n-1} z_2, f^{n-1} z_3), F(u, v, w))) \\ &\leq \psi(\Theta(f^{n-1} z_1, f^{n-1} z_2, f^{n-1} z_3, u, v, w)) \\ &\quad - \phi(\Theta(f^{n-1} z_1, f^{n-1} z_2, f^{n-1} z_3, u, v, w)), \end{aligned} \tag{2.15}$$

where

$$\Theta((f^{n-1} z_1, f^{n-1} z_2, f^{n-1} z_3, u, v, w)) = \max \left\{ \begin{array}{l} d^p(f^n z_1, fu), d^p(f^n z_1, F(f^{n-1} z_1, f^{n-1} z_2, f^{n-1} z_3)), \\ d^p(fu, F(u, v, w)), d^p(f^n z_2, fv), \\ d^p(f^n z_2, F(f^{n-1} z_1, f^{n-1} z_2, f^{n-1} z_3)), d^p(fv, F(v, w, u)), \\ d^p(f^n z_3, fw), d^p(f^n z_1, F(f^{n-1} z_1, f^{n-1} z_2, f^{n-1} z_3)), \\ d^p(fw, F(w, u, v)), \\ \frac{d^p(f^n z_1, fu) + d^p(f^n z_1, F(f^{n-1} z_1, f^{n-1} z_2, f^{n-1} z_3)) + d^p(fu, F(u, v, w))}{2}, \\ \frac{d^p(f^n z_2, fv) + d^p(f^n z_2, F(f^{n-1} z_1, f^{n-1} z_2, f^{n-1} z_3)) + d^p(fv, F(v, w, u))}{2}, \\ \frac{d^p(f^n z_3, fw) + d^p(f^n z_3, F(f^{n-1} z_1, f^{n-1} z_2, f^{n-1} z_3)) + d^p(fw, F(w, u, v))}{2}. \end{array} \right\}.$$

By taking limit in (2.15) and using (2.14) we get

$$\begin{aligned} \psi(D^p(fu, F(u, v, w))) &\leq \psi \left(\max \left\{ \begin{array}{l} d^p(fu, F(u, v, w)), \\ d^p(fv, F(v, w, u)), \\ d^p(fw, F(w, u, v)) \end{array} \right\} \right) \\ &- \phi \left(\max \left\{ \begin{array}{l} d^p(fu, F(u, v, w)), \\ d^p(fv, F(v, w, u)), \\ d^p(fw, F(w, u, v)) \end{array} \right\} \right). \end{aligned} \quad (2.16)$$

Similarly we have

$$\begin{aligned} \psi(D^p(fv, F(v, w, u))) &\leq \psi \left(\max \left\{ \begin{array}{l} d^p(fu, F(u, v, w)), \\ d^p(fv, F(v, w, u)), \\ d^p(fw, F(w, u, v)) \end{array} \right\} \right) \\ &- \phi \left(\max \left\{ \begin{array}{l} d^p(fu, F(u, v, w)), \\ d^p(fv, F(v, w, u)), \\ d^p(fw, F(w, u, v)) \end{array} \right\} \right), \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} \psi(D^p(fw, F(w, u, v))) &\leq \psi \left(\max \left\{ \begin{array}{l} d^p(fu, F(u, v, w)), \\ d^p(fv, F(v, w, u)), \\ d^p(fw, F(w, u, v)) \end{array} \right\} \right) \\ &- \phi \left(\max \left\{ \begin{array}{l} d^p(fu, F(u, v, w)), \\ d^p(fv, F(v, w, u)), \\ d^p(fw, F(w, u, v)) \end{array} \right\} \right). \end{aligned} \quad (2.18)$$

Combining (2.16), (2.17), (2.18) in last we get

$$D^p(fu, F(u, v, w)) = 0, \quad D^p(fv, F(v, w, u)) = 0, \quad D^p(fw, F(w, v, w)) = 0,$$

which implies $fu \in F(u, v, w)$, $fv \in F(v, w, u)$ and $fw \in F(w, v, w)$. Hence using (2.13) (u, v, w) is a common fixed point of F and f .

Suppose (A_3) holds. Assume $(z_1, z_2, z_3) \in C(f, F)$, then $\lim_{n \rightarrow \infty} f^n u = z_1$, $\lim_{n \rightarrow \infty} f^n v = z_2$, $\lim_{n \rightarrow \infty} f^n w = z_3$. Since f is Continuous at z_1, z_2 and z_3 , therefore

$$fz_1 = z_1, \quad fz_2 = z_2, \quad fz_3 = z_3. \quad (2.19)$$

Since $(z_1, z_2, z_3) \in C(f, F)$, therefore $fz_1 \in F(z_1, z_2, z_3)$, $fz_2 \in F(z_2, z_1, z_3)$ and $fz_3 \in F(z_3, z_2, z_1)$. Hence from (2.19), (z_1, z_2, z_3) is a common tripled fixed point of F and f .

Finally, let $f(C(f, F)) = (z_1, z_2, z_3)$ then $\{z_1\} = \{fz_1\} = F(z_1, z_2, z_3)$. Hence (z_1, z_2, z_3) is tripled fixed point of f and F . \square

3. Applications to systems of integral equations

On ward, we develop sufficient conditions for the solutions to the following general nonlinear systems of Fredholm integral equations of 2nd kind given by

$$\begin{cases} \eta_i(t) = \phi(t) + \int_{\zeta}^{\sigma} K_1(f_i(t), f_{i+1}(t), f_{i+2}(t))dt, & t \in [\zeta, \sigma], \\ \zeta_i(t) = \phi(t) + \int_{\zeta}^{\sigma} K_2(f_i(t), f_{i+1}(t), f_{i+2}(t))dt, & t \in [\zeta, \sigma], \quad i = 1, 2, 3. \end{cases} \quad (3.1)$$

Let $\Lambda = L^1[\zeta, \sigma]$ be the set of all Lebesgue integrable functions on $[\zeta, \sigma]$. Define $d : \Lambda \times \Lambda \rightarrow \mathbb{R}^+$ by

$$d(\eta, \zeta) = \int_{\zeta}^{\sigma} |\eta(x) - \zeta(x)| dx.$$

Then (Λ, d) is a metric space on Λ . For the derivation of aforesaid condition, we give the following theorem.

Theorem 3.1. *Assume that the following assumptions holds:*

(A₁) $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous.

(A₂) There exist $K_1, K_2 : \mathbb{R}^3 \rightarrow \mathbb{R}$ which are continuous functions such that

$$\left| K_1(x_{i+1}, x_{i+2}, x_{i+3}) - K_2(y_{i+1}, y_{i+2}, y_{i+3}) \right| \leq \max \left\{ \begin{array}{l} |x_{i+1} - y_{i+2}|, \\ |x_{i+2} - y_{i+2}|, \\ |x_{i+3} - y_{i+3}|. \end{array} \right\},$$

for $i = 1, 2, 3$.

Then, the systems of integral equations (3.1) has a common solution in $L^1[\zeta, \sigma]$.

Proof. Define $F, G : L^1[\zeta, \sigma]^3 \rightarrow L^1[\zeta, \sigma]$ by

$$\begin{cases} F_{f_{i+1}f_{i+2}f_{i+3}}(t) = \phi(t) + \int_{\zeta}^{\sigma} K_1(f_i(t), f_{i+1}(t), f_{i+2}(t)) dt, & t \in [\zeta, \sigma], \\ G_{g_{i+1}g_{i+2}g_{i+3}}(t) = \phi(t) + \int_{\zeta}^{\sigma} K_2(g_i(t), g_{i+1}(t), g_{i+2}(t)) dt, & t \in [\zeta, \sigma], \quad i = 1, 2, 3. \end{cases}$$

Now, we have

$$\begin{aligned} & d(F_{f_{i+1}f_{i+2}f_{i+3}}, G_{g_{i+1}g_{i+2}g_{i+3}}) \\ &= \int_{\zeta}^{\sigma} |F_{f_{i+1}f_{i+2}f_{i+3}}(t) - G_{g_{i+1}g_{i+2}g_{i+3}}(t)| dx \\ &= \int_{\zeta}^{\sigma} \left(\int_{\zeta}^{\sigma} |k_1(f_{i+1}(t), f_{i+2}(t), f_{i+3}(t)) - k_2(g_{i+1}(t), g_{i+2}(t), g_{i+3}(t))| dt \right) dx \\ &\leq \int_{\zeta}^{\sigma} \left(\int_{\zeta}^{\sigma} \max\{|f_{i+1}(t) - g_{i+1}(t)|, |f_{i+2}(t) - g_{i+2}(t)|, |f_{i+3}(t) - g_{i+3}(t)|\} dt \right) dx. \end{aligned} \tag{3.2}$$

Now, there are three cases for $i = 1, 2, 3$.

If $i = 1$ again there are three cases.

Case 1. If

$$\max\{|f_2(t) - g_2(t)|, |f_3(t) - g_3(t)|, |f_4(t) - g_4(t)|\} = |f_2(t) - g_2(t)|,$$

then inequality (3.2) implies that

$$\begin{aligned} d(F_{f_2f_3f_4}, G_{g_2g_3g_4}) &\leq \int_{\zeta}^{\sigma} \left(\int_{\zeta}^{\sigma} |f_2(t) - g_2(t)| dt \right) dx \\ &= \int_{\zeta}^{\sigma} (d(f_2, g_2)) dx \\ &= (\sigma - \zeta) (d(f_2, g_2)) \\ &\leq (\sigma - \zeta) \Theta(f_2, f_3f_4, g_2, g_3, g_4). \end{aligned}$$

Case 2. If

$$\max\{|f_2(t) - g_2(t)|, |f_3(t) - g_3(t)|, |f_4(t) - g_4(t)|\} = |f_3(t) - g_3(t)|,$$

then from inequality (3.2) we have

$$\begin{aligned} d(F_{f_2 f_3 f_4}, G_{g_2 g_3 g_4}) &\leq \int_{\zeta}^{\sigma} \left(\int_{\zeta}^{\sigma} |f_3(t) - g_3(t)| dt \right) dx \\ &= \int_{\zeta}^{\sigma} \left(d(f_3, g_3) \right) dx \\ &= (\sigma - \zeta) \left(d(f_3, g_3) \right) \\ &\leq (\sigma - \zeta) \Theta(f_2, f_3 f_4, g_2, g_3, g_4). \end{aligned}$$

Case 3. If

$$\max\{|f_2(t) - g_2(t)|, |f_3(t) - g_3(t)|, |f_4(t) - g_4(t)|\} = |f_4(t) - g_4(t)|,$$

then from inequality (3.2) we have

$$\begin{aligned} d(F_{f_2 f_3 f_4}, G_{g_2 g_3 g_4}) &\leq \int_{\zeta}^{\sigma} \left(\int_{\zeta}^{\sigma} |f_4(t) - g_4(t)| dt \right) dx \\ &= \int_{\zeta}^{\sigma} \left(d(f_4, g_4) \right) dx \\ &= (\sigma - \zeta) \left(d(f_4, g_4) \right) \\ &\leq (\sigma - \zeta) \Theta(f_2, f_3 f_4, g_2, g_3, g_4). \end{aligned}$$

Similar result can be held for other two cases for $i = 2, 3$. By taking $\psi(\eta) = (\sigma - \zeta)\eta$, $\phi(\eta) = 0$, $p = 1$ from Corollary 2.2 the system of integral equations (3.1) has common solution in the form of (f_0, f_0, f_0) . \square

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