Numerical Solutions of MRLW Equation by a Fully Implicit Finite-Difference Scheme

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Abstract

In the present paper, a fully implicit finite difference method is introduced for the numerical solution of the modified regularized long wave (MRLW) equation. The accuracy of the method is examined by different problems of the MRLW equation. The results and comparisons with analytical and other numerical invariants clearly show that results obtained using the fully implicit finite difference scheme are precise and reliable.

1. Introduction

The non-linear generalized regularized long wave (GRLW) equation has the form,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \varepsilon u^m \frac{\partial u}{\partial x} - \mu \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{1}$$

where u is the wave amplitude, μ and ε are positive parameter and m is a positive integer. Physical boundary conditions require $u \to 0$ as $x \to \pm\infty$. This equation is very important in physics since it describes phenomena with weak nonlinearity and dispersion waves including nonlinear transverse waves in shallow water, ion acoustic and magnetohydrodynamic waves in plasma, and phonon packets in nonlinear cyrstals. Their solutions are kinds of solitary waves called solitons whose shapes are not affected by collision. A special case of Eq. (1) for $m = 1$ is the regularized long wave (RLW) equation derived for long waves propagating with dispersion processes, has the following form,

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \varepsilon u \frac{\partial u}{\partial x} - \mu \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) = 0. \tag{2}$$

The equation was first obtained by Peregrine [1]. In this paper, we consider another special case of the generalized regularized long wave (GRLW) equation for $m = 2$, called the modified regularized long wave (MRLW) equation given by

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \varepsilon u^2 \frac{\partial u}{\partial x} - \mu \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) = 0. \tag{3}$$

In literature, many numerical methods have been proposed and implemented for approximating solution of the MRLW equation. Khalifa et al. [2] solved the MRLW equation by collocation method using cubic B-splines finite element. Adomian decomposition method is applied for the modified regularized long wave equation by Khalifa et al. [3]. The homotopy perturbation method is used to implement the equation by Achouri et al. [4]. Raslan developed a new algorithm based on the collocation method to solve the MRLW equation [5]. Collocation methods using sextic B-splines have been developed for solving numerically the modified regularized long wave equation by Hassan et al. [6]. Raslan et al. [7] used B-spline finite element method to solve the equation. Dereli [8] solved the equation by using the meshless method based on collocation with well-known radial basis functions. Numerical scheme based on quartic B-spline collocation method is designed for the solution of MRLW equation by Haq et al. [9]. Dereli [10] used meshless kernel based method of lines by using radial basis functions for the modified regularized long wave equation. A homotopy analysis method employed to obtain approximate numerical solution of the equation by Khan et al. [11]. Karakoc et al. [12] solved the MRLW equation by a Petrov-Galerkin finite element method. Dag et al. [13] used a collocation method based on an extended cubic B-spline function for the numerical solution of the equation.

On the other hand, the fully implicit finite difference schemes are high-accuracy schemes for the numerical solution of the nonlinear problems. Bahadır employed the fully implicit finite difference method to compute an approximation to the solution of 1D [14] and 2D [15] Burgers' equations. İnan and Bahadır used the method for solving numerically the equal width wave equation [17].

In this paper, we develop a fully implicit finite difference scheme for solving the MRLW equation. To show efficiency and validity of the algorithm we apply this technique to several test examples and comparisons with the solutions obtained by other methods.

This paper is organized in four sections. In Section 2, we introduce the fully implicit finite difference scheme for the MRLW equation. Numerical examples are given in Section 3. Section 4 contains some conclusions.

2. Fully implicit finite difference method for solving the MRLW equation

The discretization is done by the finite differences with the implicit approach of solutions. Solution domain is discretized into cells described by the nodes set (x_i, t_n) in which $x_i = ih$, $(i = 0, 1, 2, ..., N)$ and $t_n = nk$, $(n = 0, 1, 2, ...)$, h is the spatial mesh size and k is the time step.

Eq. (3) can be written in the following form:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\varepsilon}{3}\frac{\partial u^3}{\partial x} - \mu \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{4}$$

The derivatives of Eq. (4) can be approximated as

$$\frac{\partial u}{\partial t} \cong \frac{U_i^{n+1} - U_i^n}{k}$$

$$\frac{\partial u}{\partial x} \cong \frac{1}{2}\left[\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2h} + \frac{U_{i+1}^n - U_{i-1}^n}{2h}\right]$$

$$\frac{\partial u^3}{\partial x} \cong \frac{1}{2}\left[\frac{\left(U_{i+1}^{n+1}\right)^3 - \left(U_{i-1}^{n+1}\right)^3}{2h} + \frac{\left(U_{i+1}^n\right)^3 - \left(U_{i-1}^n\right)^3}{2h}\right]$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) \cong \frac{1}{k}\left[\frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{h^2} - \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2}\right].$$

Putting these approximations in (4) we get

$$\frac{U_i^{n+1} - U_i^n}{k} + \frac{1}{4h}\left[U_{i+1}^{n+1} - U_{i-1}^{n+1} + U_{i+1}^n - U_{i-1}^n\right]$$

$$+ \frac{\varepsilon}{12h}\left[\left(U_{i+1}^{n+1}\right)^3 - \left(U_{i-1}^{n+1}\right)^3 + \left(U_{i+1}^n\right)^3 - \left(U_{i-1}^n\right)^3\right]$$

$$- \frac{\mu}{kh^2}\left[U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} - U_{i+1}^n + 2U_i^n - U_{i-1}^n\right] = 0 \tag{5}$$

which is valid for values of i lying in the interval $1 \le i \le N-1$. Where U_i^n denotes the finite difference approximation at the grid point (x_i, t_n) to the exact solution $u(x,t)$. Eq. (5) is a system of nonlinear difference equations. Let us consider this nonlinear system of equations in the form

$$\mathbf{F}(\mathbf{V}) = \mathbf{0} \tag{6}$$

where $\mathbf{F} = [f_1, f_2, \ldots, f_{N-1}]^T$ and $\mathbf{V} = [U_1^{n+1}, U_2^{n+1}, \ldots, U_{N-1}^{n+1}]^T$. Newton's method applied to Eq. (6) results in the following iteration:

1. Set $\mathbf{V}^{(0)}$, an initial guess.
2. For $m = 0, 1, 2, \ldots$ until convergence do:

   Solve $J(\mathbf{V}^{(m)})\delta^{(m)} = -F(\mathbf{V}^{(m)})$;

Set $\mathbf{V}^{(m+1)} = \mathbf{V}^{(m)} + \delta^{(m)}$ where $J(\mathbf{V}^{(m)})$ is the Jacobian matrix which is evaluated analytically. The solution at the previous time-step is taken as the initial estimate. The Newton's iteration at each time-step is stopped when $\left\|\mathbf{F}(\mathbf{V}^{(m)})\right\|_\infty \le 10^{-5}$. The convergence is generally obtained in two or three iterations.

3. Test Problems and Discussion

In this section, some test problems have been considered to illustrate the performance of the method described in previous section.

The accuracy of the method is measured by using the L_2 and L_∞ norms defined by

$$L_2 = \|u - U\|_2 = \left(h \sum_{i=0}^{N} |u_i - U_i|^2 \right)^{\frac{1}{2}},$$

(7)

$$L_\infty = \|u - U\|_\infty = \max_{0 \le i \le N} |u_i - U_i|.$$

Where u and U represent the exact and approximate solutions, respectively. We also examined our results by calculating the following three conserved quantities corresponding to mass, momentum and energy [16], respectively [10].

$$I_1 = \int_{-\infty}^{+\infty} u\,dx \cong h \sum_{i=1}^{N} U_i^n,$$

$$I_2 = \int_{-\infty}^{+\infty} \left(u^2 + \mu(u_x)^2 \right) dx \cong h \sum_{i=1}^{N} \left[(U_i^n)^2 + \mu((U_x)_i^n)^2 \right]$$

(8)

$$I_3 = \int_{-\infty}^{+\infty} \left[u^4 - \mu(u_x)^2 \right] dx \cong h \sum_{i=1}^{N} \left[(U_i^n)^4 - \mu((U_x)_i^n)^2 \right]$$

To give a clear overview of the methodology the following examples will be discussed.

3.1. Motion of single solitary wave

We first model the motion of a single solitary wave of the MRLW equation. The solitary wave analytical solution of the MRLW equation (3) is

$$u(x,t) = A\,\mathrm{sech}(p(x - (1 + c)t - x^*))$$

(9)

where $p = \sqrt{c/\mu(1+c)}$. This solution corresponds to a solitary wave of amplitude $A = \sqrt{6c/\varepsilon}$ and initially centered on the peak position x^*. The initial and boundary conditions are set to:
$u(x,0) = A\,\mathrm{sech}\,(p(x - x^*))$ and $u \to 0$ as $x \to \pm\infty$, respectively.

The analytical values of conservation quantities can be found as [2]

$$I_1 = \frac{\pi A}{p}, \; I_2 = \frac{2A^2}{p} + \frac{2\mu p A^2}{3}, \; I_3 = \frac{4A^2}{3p\varepsilon}\left(A^2\varepsilon - 3\mu p^2\right)$$

(10)

To allow comparison with the previous method parameters are taken as $\mu = 1$ and $\varepsilon = 6$. In the first case, we choose the parameters $c = 1$, $h = 0.2$, $k = 0.025$ and $x^* = 40$ over the solution domain $0 \le x \le 100$. The analytical values for the invariants are $I_1 = 4.442883$, $I_2 = 3.299832$ and $I_3 = 1.414214$. The numerical solutions and L_2 and L_∞ error norms are found for these parameters are presented in Table 1. The growths of the invariants I_1, I_2 and I_3 from their initial values to $t = 10$ are less than 5×10^{-13},

1×10^{-6} and 1×10^{-5}, respectively. The comparisons of numerical invariants with some previous results and analytical values of the invariants at $t = 10$ for these parameters are given in Table 2. Figure 1 illustrates the motion of a single solitary wave for this case at different times. It is observed from Figure 1 that the single solitary wave moved to the right with constant amplitude and shape as time increases.

Table 1. Invariants and error norms for single solitary wave with $c = 1$.

t	I_1	I_2	I_3	L_2	L_∞
0	4.442883	3.299777	1.414268		
1	4.442883	3.299795	1.414236	0.0064435	0.0048730
2	4.442883	3.299811	1.414206	0.0116747	0.0076591
3	4.442883	3.299816	1.414188	0.0158906	0.0158906
4	4.442883	3.299817	1.414176	0.0195888	0.0114470
5	4.442883	3.299815	1.414167	0.0230392	0.0131584
6	4.442883	3.299812	1.414160	0.0263702	0.0148681
7	4.442883	3.299809	1.414153	0.0296452	0.0165691
8	4.442883	3.299805	1.414147	0.0328959	0.0182659
9	4.442883	3.299800	1.414142	0.0361388	0.0199612
10	4.442883	3.299796	1.414136	0.0393822	0.0216564

Table 2. The comparisons of results for single solitary wave at $t = 10$.

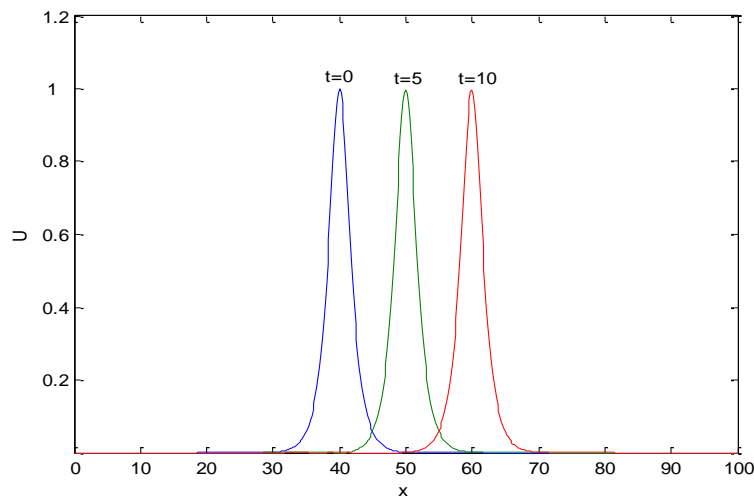| Method	I_1	I_2	I_3
Analytical	4.442883	3.299832	1.414214
Present	4.442883	3.299796	1.414136
Khalifa et al. [2]	4.44288	3.29983	1.41420
Raslan [5] ($h = 0.1$, $k = 0.025$)	4.445176	3.302476	1.417411
Dereli [8] G	4.44280	3.29957	1.41395
Karakoc et al. [12]	4.4431758	3.3003023	1.4146927

Figure 1: Single solitary wave of the equation with $c = 1$.

In the second case, the parameters are chosen $c = 0.05$, $x^* = 40$, with space step $h = 0.1$, time step $k = 0.1$ and with range $[0, 80]$. The analytical values of the invariants are $I_1 = 3.219174$, $I_2 = 0.465532$ and $I_3 = 0.008001$. Invariants and error norms for a single solitary wave are presented in Table 3. It is observed that from Table 3, error norms are satisfactorily small. The growths in I_1, I_2 and I_3 from their initial values are 4×10^{-5}, 6×10^{-9} and 1×10^{-8}, respectively. These values indicate that our scheme provides high accuracy.

Table 3. Invariants and error norms for single solitary wave with $c = 0.05$.

t	I_1	I_2	I_3	L_2	L_∞
0	3.218504	0.465532	0.008001		
0.2	3.218503	0.465532	0.008001	0.000005	0.000002
0.4	3.218501	0.465532	0.008001	0.000009	0.000004
0.6	3.218499	0.465532	0.008001	0.000015	0.000007
0.8	3.218494	0.465532	0.008001	0.000019	0.000010
1.0	3.218489	0.465532	0.008001	0.000024	0.000013
1.2	3.218483	0.465532	0.008001	0.000029	0.000016
1.4	3.218475	0.465532	0.008001	0.000034	0.000019
1.6	3.218466	0.465532	0.008001	0.000039	0.000022
1.8	3.218456	0.465532	0.008001	0.000047	0.000026
2.0	3.218444	0.465532	0.008001	0.000049	0.000029

3.2. Interaction of two solitary waves

Secondly, the interaction process of two solitary waves traveling in the same direction is studied by using the initial condition

$$u(x,0) = A_1 \text{sech}(p_1(x - x_1^*)) + A_2 \text{sech}(p_2(x - x_2^*)) \tag{11}$$

where $A_j = \sqrt{6c_j/\varepsilon}$, $p_j = \sqrt{c_j/\mu(1 + c_j)}$, $j = 1,2$ and the boundary conditions are $u \to 0$ as $x \to \pm\infty$. To allow comparison with the previous method parameters are taken as $\mu = 1$ and $\varepsilon = 6$.

The analytical values of the invariant quantities are [2]

$$I_1 = \sum_{i=1}^{2} \frac{\pi A_i}{p_i},$$

$$I_2 = \sum_{i=1}^{2} \left(\frac{2c_i}{p_i} + \frac{2\mu p_i c_i}{3} \right), \tag{12}$$

$$I_3 = \sum_{i=1}^{2} \left(\frac{4c_i^2}{3p_i} - \frac{2\mu p_i c_i}{3} \right).$$

Firstly, the parameters are taken as $c_1 = 4$, $c_2 = 1$, $x_1^* = 25$, $x_2^* = 55$, $h = 0.1$ and $k = 0.025$ over the solution domain $0 \leq x \leq 250$, then the amplitudes are in ratio 2:1, where $A_1 = 2A_2$. The results are tabulated in Table 4. The analytical values of the invariants are $I_1 = 11.467698$, $I_2 = 14.629243$ and $I_3 = 22.880466$ and the growth of invariants I_1, I_2 and I_3 are 1×10^{-7}, 0.025137 and 0.113372, respectively. Comparisons of the invariants for this case are documented in Table 5. Figure 2 shows the interaction of two solitary waves for these parameters at different time levels. It can be seen from Figure 2 that the larger solitary wave has passed the smaller solitary wave as time increases. Then, two solitary waves regain their original shape after the interaction.

Table 4. Invariants for interaction of two solitary waves.

t	I_1	I_2	I_3
0	11.467698	14.629194	22.880514
1	11.467698	14.626621	22.870153
2	11.467698	14.624007	22.859699
3	11.467698	14.621398	22.849280
4	11.467698	14.618792	22.838876
5	11.467698	14.616187	22.828464
6	11.467698	14.613584	22.818009
7	11.467698	14.610848	22.806149
8	11.467698	14.607107	22.784129
9	11.467698	14.598231	22.715753
10	11.467698	14.604057	22.767142

Table 5. The comparisons of results for interaction of two solitary waves at $t = 10$.

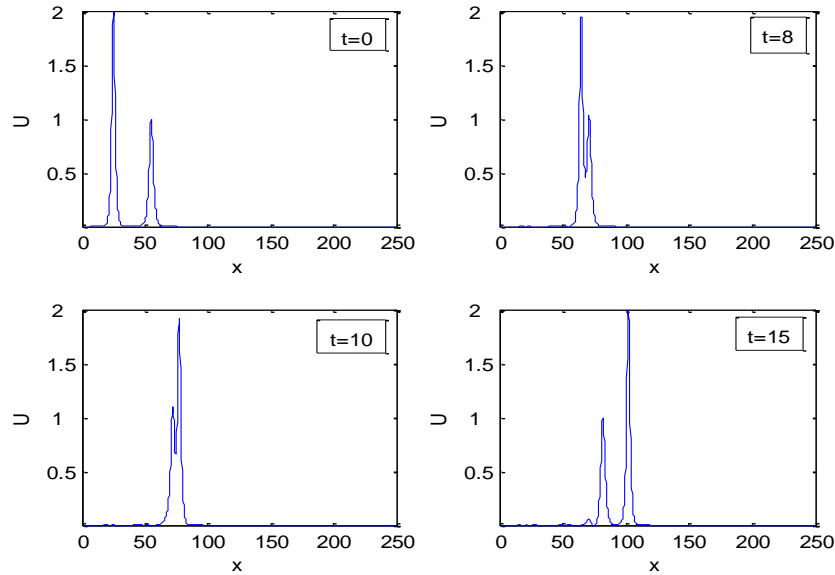| Method	I_1	I_2	I_3
Analytical	11.467698	14.629243	22.880466
Present	11.467698	14.604057	22.767142
Raslan et al. [7] ($h = 0.2$, $k = 0.025$)	11.460050	14.608160	22.842790
Haq et al. [9] ($h = 0.2$, $k = 0.025$)	11.467700	14.603687	22.771773

Figure 2: Interaction of two solitary waves at selected times.

Secondly, we chose $c_1 = 1$, $c_2 = 0.5$, $x_1^* = 10$, $x_2^* = 40$, $h = 0.2$ and $k = 0.05$ over the solution domain $0 \leq x \leq 100$. Invariants for interaction of two solitary waves are given in Table 6. The analytical values of the invariants are $I_1 = 8.29053243$, $I_2 = 5.22433254$ and $I_3 = 1.79911374$. The growths of invariants I_1, I_2 and I_3 are 1×10^{-3}, 6×10^{-5} and 8×10^{-5}, respectively.

Table 6. Invariants for interaction of two solitary waves.

t	I_1	I_2	I_3
0	8.287956	5.224284	1.799162
2	8.292722	5.224238	1.799022
4	8.294685	5.224171	1.798906
6	8.296623	5.224095	1.798812
8	8.298449	5.224015	1.798727
10	8.299991	5.223934	1.798644
12	8.301186	5.223852	1.798562
14	8.302087	5.223768	1.798483
16	8.302801	5.223699	1.798376
18	8.303417	5.223601	1.798331
20	8.303961	5.223505	1.798281

3.3. Interaction of three solitary waves

Thirdly, the interaction process of three solitary waves traveling in the same direction is studied by using the initial condition

$$u(x,0) = A_1 \text{sech}(p_1(x - x_1^*)) + A_2 \text{sech}(p_2(x - x_2^*))$$
$$+ A_3 \text{sech}(p_3(x - x_3^*))$$

(13)

where $A_j = \sqrt{6c_j/\varepsilon}$, $p_j = \sqrt{c_j/\mu(1+c_j)}$, $j = 1,2,3$ and the boundary conditions are $u \to 0$ as $x \to \pm\infty$. To allow comparison with the previous method parameters are taken as $\mu = 1$ and $\varepsilon = 6$. The analytical values of the invariant quantities are [2]

$$I_1 = \sum_{i=1}^{3} \frac{\pi A_i}{p_i},$$

$$I_2 = \sum_{i=1}^{3} \left(\frac{2c_i}{p_i} + \frac{2\mu p_i c_i}{3} \right), \tag{14}$$

$$I_3 = \sum_{i=1}^{3} \left(\frac{4c_i^2}{3p_i} - \frac{2\mu p_i c_i}{3} \right).$$

In this case, following parameters are used: $c_1 = 4$, $c_2 = 1$, $c_3 = 0.25$, $x_1^* = 15$, $x_2^* = 45$, $x_3^* = 60$, $h = 0.2$ and $k = 0.025$ over the solution domain $0 \le x \le 250$. Thus the amplitudes are in the ratio 4:2:1 where $A_1 = 2A_2 = 4A_3$. The analytical values of invariants for the interaction of three waves are $I_1 = 14.980105$, $I_2 = 15.821812$ and $I_3 = 22.992270$. Numerical values of the invariants are obtained by fully implicit finite difference method are recorded in Table 7. The growths in I_1, I_2 and I_3 are 4×10^{-5}, 0.014989 and 0.146276, respectively. The comparison of invariants for these parameters are given in Table 8. Figure 3 presents interaction of these solitary waves to $t = 25$. Figure 3 shows that the largest solitary wave has passed the smallest solitary wave as time increases.

Table 7. Invariants for interaction of three solitary waves.

t	I_1	I_2	I_3
0	14.980098	15.836732	23.008932
1	14.980112	15.834559	22.997699
2	14.980115	15.832109	22.987443
3	14.980119	15.829650	22.977288
4	14.980122	15.827206	22.967134
5	14.980126	15.824793	22.956927
6	14.980129	15.822343	22.946991
7	14.980133	15.820419	22.934121
8	14.980136	15.819951	22.908805
9	14.980139	15.820619	22.832402
10	14.980141	15.821743	22.862656

Table 8. The comparisons of results for interaction of three solitary waves $t = 10$.

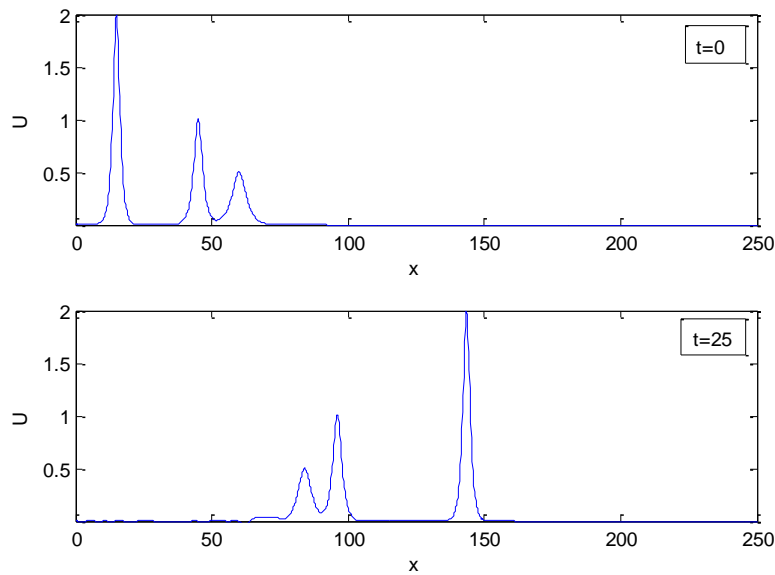| Method	I_1	I_2	I_3
Analytical	14.980105	15.821812	22.992269
Present	14.980141	15.821743	22.862656
Khalifa et al. [2]	13.6891	15.6514	22.8388
Raslan et al. [7]	14.930390	15.822500	22.964190
Dereli [8] G ($h = 0.5$, $k = 0.01$)	14.9794	15.8049	22.8703

Figure 3: Interaction of three solitary waves at selected times.

3.4. The Maxwellian initial condition

Finally, we use the initial condition

$$u(x,0) = \exp(-(x-7)^2) \tag{15}$$

and the boundary conditions are $u \to 0$ as $x \to \pm\infty$ to analyse the evolution of an Maxwellian pulse into solitary waves.

In this test, we study for values of $\mu = 0.1$, $\mu = 0.04$, $\mu = 0.015$, $\mu = 0.01$ and $\varepsilon = 6$ through the interval $0 \leq x \leq 100$.

Table 9 gives the invariants obtained from the present method for $h = 0.1$, $k = 0.001$. The growth of I_1, I_2, I_3 are given in Table 10. Figure 4 shows wave profiles for the Maxwellian initial condition at $t = 15$ for $0 \leq x \leq 100$. This figure contains figures for $\mu = 0.1$, $\mu = 0.04$, $\mu = 0.015$ and $\mu = 0.01$, respectively. As can be seen from Figure 4, μ is reduced more and more solitary waves are formed, as expected.

Table 9. Invariants for the Maxwellian initial condition.

t	μ	I_1	I_2	I_3	μ	I_1	I_2	I_3
0.01	0.1	1.772454	1.378632	0.760909	0.015	1.772454	1.272108	0.867433
0.03		1.772454	1.378623	0.760918		1.772454	1.272078	0.867463
0.05		1.772454	1.378606	0.760936		1.772454	1.272015	0.867526
0.01	0.04	1.772454	1.303440	0.836102	0.01	1.772454	1.265842	0.873699
0.03		1.772454	1.303420	0.836121		1.772454	1.265808	0.873734
0.05		1.772454	1.303382	0.836159		1.772454	1.265737	0.873805

Table 10. The growth of invariants for the Maxwellian initial condition.

μ	I_1	I_2	I_3
0.1	1×10^{-16}	2×10^{-5}	2×10^{-5}
0.04	2×10^{-15}	5×10^{-5}	5×10^{-5}
0.015	1×10^{-15}	1×10^{-5}	1×10^{-5}
0.01	1×10^{-15}	1×10^{-4}	1×10^{-4}
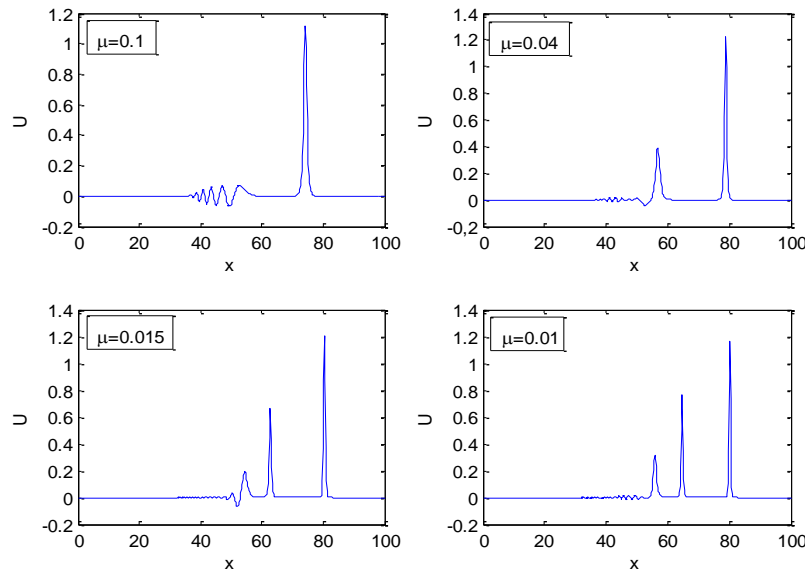
Figure 4: Maxwellian initial condition for $k = 0.025$, $h = 0.2$ at $t = 15$.

4. Conclusion

In this paper, a fully implicit finite difference method has been successfully applied to finding the solution of the MRLW equation. Numerical tests for single solitary wave, interaction of two solitary waves, interaction of three solitary waves and Maxwellian initial condition are considered by the fully implicit finite difference method. The numerical results demonstrate that the present method is quite accurate and readily implemented in the solution of the MRLW equation.

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