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Some Refinements of Jensen's Inequality on Product Spaces

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Abstract

In this paper, we give some refinements of the classical Jensen's inequality which generalizes some results already obtained in literatures.

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1. Introduction

In [3] J. Rooin refined the classical Jensen's inequality as

$$\varphi\left(\int_X f d\mu\right) \leq \int_Y \varphi\left(\int_X f(x)\omega(x,y)d\mu(x)\right) d\lambda(y) \leq \int_X (\varphi \circ f) d\mu, \tag{1.1}$$

where (X, A, μ) and (Y, B, λ) are two probability measure spaces, $\omega: X \times Y \to [0, \infty)$ is a weight function on $X \times Y$, I is an interval of the real line, $f \in L^1(\mu)$, $f(x) \in I$ for all $x \in X$ and φ is a real-valued convex function on I.

Also, in [2], the authors proved a generalization of the classical Jensen's inequality by Riemann-Stieltjes integration for two convex functions defined on an interval of $\mathbb{R}$.

In this paper, we generalize the above papers to a very general case by considering a more general abstract space i.e the L^p spaces and two fuctions in this space.

2. Main Results

We refine the classical Jensen's inequality on the L^p spaces and show how our result generalizes those in literature.

Theorem 2.1. Let X be a measure space, with measure μ. Let $f \in L^p(\mu)$ and $g \in L^q(\mu)$. Suppose φ is any convex function and $\frac{1}{p}+\frac{1}{q}=1$, where $1<p<\infty$ and $1<q<\infty$ then the following inequality holds

$$\varphi\left(\int_X fg dx\right) \le \left(\int_X \varphi \circ f^p dx\right)^{\frac{1}{p}}\left(\int_X \varphi \circ g^q dx\right)^{\frac{1}{q}}. \tag{2.1}$$

Proof:

Let

$$A=\left(\int_X \varphi \circ f^p dx\right)^{\frac{1}{p}}, \quad B=\left(\int_X \varphi \circ g^q dx\right)^{\frac{1}{q}}$$

The case when $A=0$ is trivial. Also $A>0$ and B=∞ is trivial. So we consider the case $0<A<\infty$, $0<B<\infty$. We set

$$F=\frac{\varphi \circ f}{A}, \quad G=\frac{\varphi \circ g}{B}$$

Now,

$$\int_X F^p dx = \int_X \frac{\varphi \circ f^p dx}{\int_X \varphi \circ f^p dx} = \int_X \frac{\varphi \circ g^q dx}{\int_X \varphi \circ g^q dx} = \int_X G^q dx = 1.$$

Let $x \in X \ni 0<F(x)<\infty$ and $0<G(x)<\infty$ implies that $\exists \quad s,t \in R \ni F(x)=e^{\frac{s}{p}}, \quad G(x)=e^{\frac{t}{q}}$. This implies

$$e^{\frac{s}{p}+\frac{t}{q}} \le p^{-1}e^s + q^{-1}e^t$$

$$F(x)G(x) \le p^{-1}F^p(x) + q^{-1}G^q(x) \quad \forall x \in X \tag{2.2}$$

Integrating both sides of (2.2), to obtain

$$\int_X \frac{\varphi \circ f}{\left(\int_X \varphi \circ f^p dx\right)^{\frac{1}{p}}} \frac{\varphi \circ g}{\left(\int_X \varphi \circ g^q dx\right)^{\frac{1}{q}}} dx \le 1.$$

This implies

$$\int_X \varphi\circ f \quad \varphi\circ g dx \le \left(\int_X \varphi\circ f^p dx\right)^{\frac{1}{p}}\left(\int_X \varphi\circ g^q dx\right)^{\frac{1}{q}}.$$

That is

$$\int_X \varphi(fg)dx \le \int_X \varphi\circ f \quad \varphi\circ g dx \le \left(\int_X \varphi\circ f^p dx\right)^{\frac{1}{p}}\left(\int_X \varphi\circ g^q dx\right)^{\frac{1}{q}}.$$

$$\varphi\left(\int_X (fg)dx\right) \le \int_X \varphi(fg)dx \le \int_X \varphi\circ f\ \varphi\circ g dx \le \left(\int_X \varphi\circ f^p dx\right)^{\frac{1}{p}}\left(\int_X \varphi\circ g^q dx\right)^{\frac{1}{q}}.$$

Remark 2.2. If φ is an identity function then Theorem 2.1 gives Theorem 3.5 in [4]. For simplicity, we state it as Corollary 2.3.

Corollary 2.3. Let X be a measurable space, with measure μ. Let f and g be measurable functions on X with range [0, ∞]. Suppose φ is any identity function and $\frac{1}{p}+\frac{1}{q}=1$, where $1<p,q<\infty$. Then the following inequality holds

$$\left(\int_X fg dx\right) \le \left(\int_X f^p dx\right)^{\frac{1}{p}}\left(\int_X g^q dx\right)^{\frac{1}{q}}.$$

Theorem 2.4. Let (X,A,μ) and (Y,B,λ) be two measure spaces and $\omega\colon X\times Y\to[0,\infty)$ be a weight function on $X\times Y$ such that

$$\int_X \omega(x,y)d\mu(x) = 1 \quad \forall \quad y\in Y,$$

$$\int_Y \omega(x,y)d\lambda(y) = 1 \quad \forall \quad x\in X.$$

If I is a measurable space, $f,g\in L^p(\mu), f(x)\in I \quad \forall \quad x\in X$ and φ is a convex function in I, then

$$\varphi\left(\int_X fg d\mu\right) \le \left[\int_Y \varphi\left(\int_X f^p(x)\omega(x,y)d\mu(x)\right)d\lambda(y)\right]^{\frac{1}{p}}$$

$$\times\left[\int_Y \varphi\left(\int_X g^q(x)\omega(x,y)d\mu(x)\right)d\lambda(y)\right]^{\frac{1}{q}}$$

$$\le \left[\int_X \varphi\circ f^p d\mu\right]^{\frac{1}{p}}\left[\int_X \varphi\circ g^q d\mu\right]^{\frac{1}{q}}.$$

Proof:

The functions ω and $(x,y) \to f(x)$ and so $(x,y) \to f^p(x)\omega(x,y)$ is product-measurable on $X \times Y$. The same thing goes for $g(x)$. We prove the first inequality. Clearly,

$$\left(\int_X \int_Y |f(x)|^p \omega(x,y) d\lambda(y) d\mu(x)\right)^{\frac{1}{p}} = \left(\int_X |f(x)|^p \left(\int_Y \omega(x,y) d\lambda(y)\right) d\mu(x)\right)^{\frac{1}{p}}$$

$$= \left(\int_X |f(x)|^p d\mu(x)\right)^{\frac{1}{p}}$$

$$= ||f||_{L^p(\mu)} \leq \infty.$$

Similarly for$g(x)$, we have

$$\left(\int_X \int_Y |g(x)|^q \omega(x,y) d\lambda(y) d\mu(x)\right)^{\frac{1}{q}} = \left(\int_X |g(x)|^q \left(\int_Y \omega(x,y) d\lambda(y)\right) d\mu(x)\right)^{\frac{1}{q}}$$

$$= \left(\int_X |g(x)|^q d\mu(x)\right)^{\frac{1}{q}}$$

$$= ||g||_{L^p(\mu)} \leq \infty.$$

By Fubini's theorem we know that $(x,y) \to f^p(x)\omega(x,y)$ on $X \times Y$ belongs to $L^p(\mu \times \lambda)$. By the same argument, $(x,y) \to g^q(x)\omega(x,y)$ belongs to $L^p(\mu \times \lambda)$. Next, we define $F: Y \to R$ and $G: Y \to R$ by

$$F(y) = \left(\int_X f^p(x)\omega(x,y) d\mu(x)\right)^{\frac{1}{p}},$$

$$G(y) = \left(\int_X g^q(x)\omega(x,y) d\mu(x)\right)^{\frac{1}{q}}.$$

Now,

$$\left[\int_Y \varphi\left(\int_X f^p(x)\omega(x,y) d\mu(x)\right) d\lambda(y)\right]^{\frac{1}{p}} \left[\int_Y \varphi\left(\int_X g^q(x)\omega(x,y) d\mu(x)\right) d\lambda(y)\right]^{\frac{1}{q}}$$

$$= \left[\int_Y (\varphi \circ F^p)(y)d\lambda(y)\right]^{\frac{1}{p}} \left[\int_Y (\varphi \circ G^q)(y)d\lambda(y)\right]^{\frac{1}{q}}.$$

Using Theorem 2.1 we obtain

$$\left[\int_Y (\varphi \circ F^p)(y)d\lambda(y)\right]^{\frac{1}{p}} \left[\int_Y (\varphi \circ G^q)(y)d\lambda(y)\right]^{\frac{1}{q}} \geq \varphi\left(\int_Y F(y)d\lambda(y) \int_y G(y)d\lambda(y)\right)$$

$$= \varphi\left(\int_Y \left(\int_X f^p(x)\omega(x,y)d\mu(x)\right)^{\frac{1}{p}} \left(\int_X g^q(x)\omega(x,y)d\mu(x)\right)^{\frac{1}{q}} d\lambda(y)\right)$$

$$\geq \left[\varphi\left(\int_Y F^p(y)d\lambda(y)\right)\right]^{\frac{1}{p}} \left[\varphi\left(\int_Y G^q(y)d\lambda(y)\right)\right]^{\frac{1}{q}}$$

$$= \left[\varphi\left(\int_Y \left(\int_X f^p(x)\omega(x,y)d\mu(x)d\lambda(y)\right)\right)\right]^{\frac{1}{p}} \left[\varphi\left(\int_Y \left(\int_X g^q(x)\omega(x,y)d\mu(x)d\lambda(y)\right)\right)\right]^{\frac{1}{q}}$$

$$= \left[\varphi\left(\int_X f^p(x)\left(\int_Y \omega(x,y)d\lambda(y)\right)d\mu(x)\right)\right]^{\frac{1}{p}} \left[\varphi\left(\int_X g^q(x)\left(\int_Y \omega(x,y)d\lambda(y)\right)d\mu(x)\right)\right]^{\frac{1}{q}}$$

$$= \left[\varphi\left(\int_X f^p(x)d\mu(x)\right)\right]^{\frac{1}{p}} \left[\varphi\left(\int_X g^q(x)d\mu(x)\right)\right]^{\frac{1}{q}}$$

$$\geq \varphi\left(\int_X fg\,d\mu(x)\right).$$

Remark 2.5. Theorem 2.4 refines the result obtained by Hewitt and Stromberg on page 202 of [1] and also generalizes [3].

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