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F_m-contractive and F_m-expanding mappings in M-metric spaces

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Abstract

Inspired by the work of Górnicki in his recent article [J. Górnicki, Fixed Point Theory Appl., **2017** (2017), 10 pages], where he introduced a new class of self mappings called F-expanding mappings, in this paper we introduce the concept of F_m-contractive and F_m-expanding mappings in M-metric spaces. Also, we prove the existence and uniqueness of fixed point for such mappings.

Keywords: M-metric spaces, F_m-contractive, F_m-expanding mappings.

2010 MSC: 54H25, 47H10.

1. Introduction

In [2], M-metric space was introduced, which is an extension of partial metric spaces, and it has many applications. In this paper, we introduce the notion of F_m-contractive and F_m-expanding mappings in M-metric space, where we prove that self mappings on a complete M-metric spaces that are F_m-contractive have a unique fixed point. Also, we show that surjective self mappings on a complete M-metric spaces that are F_m-expanding mappings in M-metric spaces have a unique fixed point.

This article is organized as follows. In this section we recall the concept of M-metric spaces. In Section 2, we present the concept of F_m-contraction along with a fixed point theorem which we are going to support it by an example. In the Section 3, we introduce the concept of F_m-expanding mappings. In Section 4 we show that the results of [7] and [3], are direct consequences of our results. In the last section, we present some open questions.

Notation 1.1 ([2]).

1. $m_{x,y} := \min\{m(x,x), m(y,y)\}$;
2. $M_{x,y} := \max\{m(x,x), m(y,y)\}$.

Definition 1.2 ([2]). Let X be a nonempty set, if the function $m : X^2 \to R^+$, for all $x, y, z \in X$, satisfies the following conditions:

(1) $m(x,x) = m(y,y) = m(x,y)$ if and only if $x = y$;

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(2) $m_{x,y} \leqslant m(x,y)$;
(3) $m(x,y) = m(y,x)$;
(4) $(m(x,y) - m_{x,y}) \leqslant (m(x,z) - m_{x,z}) + (m(z,y) - m_{z,y})$,

then the pair (X, m) is called an M-metric space.

Examle 1.3. Let $X := [0, \infty)$. Then

$$m(x,y) = \frac{x+y}{2} \quad \text{on} \quad X$$

is an M-metric space.

Examle 1.4. Let $X = \{1,2,3\}$ and define

$$m(1,1) = 1, \; m(2,2) = 9, \; m(3,3) = 5, \qquad m(1,2) = m(2,1) = 10,$$
$$m(1,3) = m(3,1) = 7, \qquad m(3,2) = m(2,3) = 7.$$

Note that (X, m) is an M-metric space that is not a partial metric space.

Notice that, we can construct a metric space from M-metric space.

Examle 1.5 ([2])**.** If m be an M-metric space, then the following functions

1. $m^w(x,y) = m(x,y) - 2m_{x,y} + M_{x,y}$,
2. $m^s(x,y) = m(x,y) - m_{x,y}$ when $x \neq y$ and $m^s(x,y) = 0$ if $x = y$

are ordinary metrics.

As mentioned in [2], each M-metric on set X generates a T_0 topology τ_m on X. The set

$$\{B_m(x,\epsilon) : x \in X, \epsilon > 0\} \text{ where } B_m(x,\epsilon) = \{y \in X \mid m(x,y) < m_{x,y} + \epsilon\} \text{ for all } x \in X \text{ and } \epsilon > 0,$$

forms a base of τ_m.

Definition 1.6. Let (X, m) be an M-metric space. Then

1) a sequence $\{x_n\}$ in X converges to a point x if and only if

$$\lim_{n\to\infty} (m(x_n, x) - m_{x_n,x}) = 0;$$

2) a sequence $\{x_n\}$ in X is said to be m-Cauchy sequence if and only if

$$\lim_{n,m\to\infty} (m(x_n, x_m) - m_{x_n,x_m}) \text{ and } \lim_{n\to\infty} (M_{x_n,x_m} - m_{x_n,x_m})$$

exist and finite;

3) an M-metric space is said to be complete if every m-Cauchy sequence $\{x_n\}$ converges to a point x such that

$$\lim_{n\to\infty} (m(x_n, x) - m_{x_n,x}) = 0 \text{ and } \lim_{n\to\infty} (M_{x_n,x} - m_{x_n,x}) = 0.$$

Next, we state the following lemmas.

Lemma 1.7 ([2])**.** *Assume that* $x_n \to x$ *and* $y_n \to y$ *as* $n \to \infty$ *in an M-metric space* (X, m). *Then*

$$\lim_{n\to\infty} (m(x_n, y_n) - m_{x_n,y_n}) = m(x,y) - m_{x,y}.$$

Lemma 1.8 ([2])**.** *Assume that* $x_n \to x$ *in an M-metric space* (X, m). *Then*

$$\lim_{n\to\infty} (m(x_n, y) - m_{x_n,y}) = m(x,y) - m_{x,y}.$$

2. F_m-contraction in M-metric spaces

First, we give the definition of the following family of functions.

Definition 2.1. Let $\mathbb{F}$ be the family of all functions $F; (0,\infty) \to \mathbb{R}$ such that:

(F_1) F is strictly increasing;
(F_2) for each sequence $\{\alpha_n\}$ in $(0,\infty)$ the following holds,

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F_3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

The following is an example of some functions that satisfy the conditions (F_1), (F_2), and (F_3) of Definition 2.1.

Examle 2.2.

1. $F : (0,\infty) \to \mathbb{R}$ defined by $F(x) = \ln(x)$;
2. $F : (0,\infty) \to \mathbb{R}$ defined by $F(x) = \ln(x) + x$;
3. $F : (0,\infty) \to \mathbb{R}$ defined by $F(x) = -\dfrac{1}{\sqrt{x}}$;
4. $F : (0,\infty) \to \mathbb{R}$ defined by $F(x) = \ln(x^2 + x)$.

It is not difficult to see that these three functions satisfy the conditions (F_1), (F_2), and (F_3) of Definition 2.1.

Now, we give the definition of an F_m-contraction.

Definition 2.3. Let (X, m) be a complete M-metric space. A self mapping T on X is said to be an F_m-contraction on X if there exist $F \in \mathbb{F}$ and $t > 0$ such that for all $x, y \in X$ the following holds:

$$m(Tx, Ty) > 0 \Rightarrow t + F(m(Tx, Ty)) \leqslant F(m(x, y)).$$

We start by proving the following lemma about F_m-contractive self mapping on M-metric spaces.

Lemma 2.4. *Let (X, m) be an M-metric space, and T be an F_m-contractive self mapping on X. Consider the sequence $\{x_n\}_{n\geqslant 0}$ defined by $x_{n+1} = Tx_n$. If $x_n \to u$ as $n \to \infty$, then $Tx_n \to Tu$ as $n \to \infty$.*

Proof. First, note that if $m(Tx_n, Tu) = 0$, then $m_{Tx_n,Tu} = 0$ and that is due to the fact that $m_{Tx_n,Tu} \leqslant m(Tx_n, Tu)$, which implies that

$$m(Tx_n, Tu) - m_{Tx_n,Tu} \to 0 \text{ as } n \to \infty \text{ and hence } Tx_n \to Tu \text{ as } n \to \infty.$$

So we may assume that $m(Tx_n, Tu) > 0$, by the F_m-contractive property of T we deduce that $m(Tx_n, Tu) < m(x_n, u)$. Now, if $m(u, u) \leqslant m(x_n, x_n)$ and by the F_m-contractive property it is easy see that $m(x_n, x_n) \to 0$, which implies that $m(u, u) = 0$ and since $m(Tu, Tu) < m(u, u) = 0$ we deduce that $m(Tu, Tu) = m(u, u) = 0$, and $m(x_n, u) \to 0$, on the other we have $m(Tx_n, Tu) \leqslant m(x_n, u) \to 0$. Hence, $m(Tx_n, Tu) - m_{Tu,Tx_n} \to 0$ and thus $Tx_n \to Tu$.

If $m(u, u) \geqslant m(x_n, x_n)$ and once again by the F_m-contractive property it is easy to see that $m(x_n, x_n) \to 0$, which implies that $m_{x_n,u} \to 0$. Hence, $m(x_n, u) \to 0$ and since $m(Tx_n, Tu) < m(x_n, u) \to 0$ we deduce that $m(Tx_n, Tu) - m_{Tu,Tx_n} \to 0$ and thus $Tx_n \to Tu$ as desired. □

Theorem 2.5. *Let (X, m) be a complete M-metric space and let $T : X \to X$ be an F_m-contraction. Then T has a unique fixed point u in X, and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to u.*

Proof. First of all, we claim that if T has a fixed point then it is unique. To see this, assume that there exist $u, v \in X$ such that $Tu = u \neq v = Tv$. If $m(Tu, Tv) = 0$, and without loss of generality suppose that

$m_{u,v} = m(u,u)$, then
$$m(Tu,Tv) = 0 = m(u,u).$$
Now, if $m(v,v) = 0$, then $u = v$. So, assume that $m(v,v) > 0$, this implies that
$$F(m(v,v)) = F(m(Tv,Tv)) \leqslant F(m(v,v)) - t < F(m(v,v)),$$
which leads to a contradiction. Therefore, $m(v,v) = 0$ and thus $u = v$. So, now we may assume that $m(u,v) > 0$. Hence, by using the fact that T is an F_m-contraction, we deduce that
$$0 < t \leqslant F(m(u,v)) - F(m(Tu,Tv)) = 0,$$
which leads to a contradiction. Therefore, if T has a fixed point then it is unique.

Next, we show that T has a fixed point. So, let $x_0 \in X$ and define a sequence $\{x_n\}$ as follows
$$x_1 = Tx_0, \quad x_2 = Tx_1 = T^2x_0, \ldots, x_{n+1} = Tx_n, \ldots \,.$$
If there exists a natural number i such that $x_{i+1} = x_i$, then we are done and x_i is the fixed point of T in X.

Secondly, assume that $m(x_n,x_n) = 0$ for some n, we want to show that in this case
$$m(x_m,x_m) = 0 \text{ for all } m > n.$$
So, assume that $m(x_n,x_n) = 0$ and $m(x_{n+1},x_{n+1}) \neq 0$ by the F_m-contractive property of T we obtain
$$F(m(x_{n+1},x_{n+1})) = F(m(Tx_n,Tx_n)) \leqslant F(m(x_n,x_n)) - t \leqslant F(m(x_n,x_n)),$$
but F is an increasing function. Therefore,
$$0 = m(x_n,x_n) \geqslant m(x_{n+1},x_{n+1}).$$
Hence, by induction on n, we get
$$\text{if } m(x_n,x_n) = 0 \text{ then } m(x_m,x_m) = 0 \text{ for all } m > n.$$
Also, note that it is not difficult to see that if $m > n$, then we have $m_{x_n,x_m} = m(x_m,x_m)$, to see this, assume that $m_{x_n,x_m} = m(x_n,x_n)$. Hence, if $m(x_n,x_n) = 0$, then by the above claim we obtain $m(x_m,x_m) = 0$, and if $m(x_n,x_n) > 0$, then $m(x_m,x_m) > 0$, thus
$$\begin{aligned} F(m(x_m,x_m)) &= F(m(Tx_{m-1},Tx_{m-1})) \\ &\leqslant F(m(x_{m-1},x_{m-1})) - t \\ &\vdots \\ &\leqslant F(m(x_n,x_n)) - (m-n)t \\ &< F(m(x_n,x_n)) \end{aligned}$$
but F is an increasing function. Therefore, if $m > n$, we have $m_{x_n,x_m} = m(x_m,x_m)$.

Now, suppose that $m(x_{n+1},x_n) = 0$ for some n, this implies that $m_{x_n,x_{n+1}} = 0$. We know that $m_{x_n,x_{n+1}} = m(x_{n+1},x_{n+1}) = 0$. Thus, by the above argument we have $m(x_{n+2},x_{n+2}) = 0$. Thus, now we have two cases, either $m(x_{n+1},x_{n+2}) = 0$ and in this case it is easy to see that $x_{n+1} = x_{n+2}$ and that is x_{n+1} is the fixed point, or $m(x_{n+1},x_{n+2}) > 0$, again by the F_m-contractive property of T we have
$$F(m(x_{n+1},x_{n+2})) = F(m(Tx_n,Tx_{n+1})) \leqslant F(m(x_n,x_{n+1})) - t < F(m(x_n,x_{n+1})) = F(0),$$
but the fact that F is an increasing function leads us to a contradiction.

Hence, now we can assume that $m(x_n, x_{n+1}) > 0$ for all n. Let $B_n = m(x_n, x_{n+1})$, hence

$$F(B_n) \leqslant F(B_{n-1}) - t \leqslant F(B_{n-2}) - 2t \leqslant \cdots \leqslant F(B_0) - nt.$$

Thus, $F(B_n) \to -\infty$ as $n \to \infty$. Hence, by (F_2) we get

$$\lim_{n\to\infty} B_n = 0$$

and by (F_3) there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} B_n^k F(B_n) = 0.$$

Thereby,

$$B_n^k F(B_n) - B_n^k F(B_0) \leqslant B_n^k [F(B_0) - nt] - B_n^k F(B_0) = -B_n^k nt \leqslant 0.$$

Hence,

$$\lim_{n\to\infty} nB_n^k = 0.$$

Therefore, there exists a natural number n_0 such that $nB_n^k \leqslant 1$ for all $n > n_0$. Thus, we deduce that

$$B_n \leqslant \frac{1}{n^{\frac{1}{k}}} \text{ for all } n > n_0.$$

Now, let n, m be integers such that $m > n > n_0$. First, notice the following fact about the triangle inequality of the M-metric spaces,

$$(m(x,y) - m_{x,y}) \leqslant (m(x,z) - m_{x,z}) + (m(z,y) - m_{z,y}) \leqslant m(x,z) + m(z,y).$$

Thus, it is not difficult to see that

$$m(x_n, x_m) - m_{x_n,x_m} \leqslant B_n + B_{n+1} + B_{n+2} + \cdots + B_m < \sum_{i=n}^{\infty} B_i \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is a convergent series, we deduce that $m(x_n, x_m) - m_{x_n,x_m}$ converges as $n, m \to \infty$. Now, if $M_{x_n,x_m} = 0$, then $m_{x_n,x_m} = 0$, which implies that $M_{x_n,x_m} - m_{x_n,x_m} = 0$. So, we may assume that $M_{x_n,x_m} > 0$, this implies that $m(x_n, x_n) > 0$.

Now, let $\eta_n = m(x_n, x_n)$, hence

$$F(\eta_n) \leqslant F(\eta_{n-1}) - t \leqslant F(\eta_{n-2}) - 2t \leqslant \cdots \leqslant F(\eta_0) - nt.$$

Thus, $F(\eta_n) \to -\infty$ as $n \to \infty$. Hence, by (F_2) we get

$$\lim_{n\to\infty} \eta_n = 0$$

and by (F_3) there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \eta_n^k F(\eta_n) = 0.$$

Thereby,

$$\eta_n^k F(\eta_n) - \eta_n^k F(\eta_0) \leqslant \eta_n^k [F(\eta_0) - nt] - \eta_n^k F(\eta_0) = -\eta_n^k nt \leqslant 0.$$

Hence,

$$\lim_{n\to\infty} n\eta_n^k = 0.$$

Therefore, there exists a natural number n_0 such that $n\eta_n^k \leqslant 1$ for all $n > n_0$. Thus, we deduce that

$$\eta_n \leqslant \frac{1}{n^{\frac{1}{k}}} \text{ for all } n > n_0.$$

Therefore, we obtain

$$m(x_n, x_n) - m(x_m, x_m) \leqslant \eta_n + \eta_{n+1} + \eta_{n+2} + \cdots + \eta_m < \sum_{i=n}^{\infty} \eta_i \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is a convergent series, we deduce that $m(x_n, x_n) - m(x_m, x_m)$ converges as $n, m \to \infty$ and that is

$$M_{x_n, x_m} - m_{x_n, x_m} \text{ converges as desired.}$$

Therefore, $\{x_n\}$ is an m-Cauchy sequence, and using the fact that (X, m) is an m-complete M-metric space, we deduce that $\{x_n\}$ converges to some $u \in X$.

Since $m(x_n, x_{n+1}) > 0$ and by F_m-contractive property of T, one can easily deduce that $m(x_n, Tx_n) \to 0$ and $m(Tu, Tu) < m(u, u)$. Now, using the fact that $m_{x_n, Tx_n} \to 0$ and by Lemmas 1.7 and 1.8, we deduce that $m(u, Tu) = m_{u,Tu} = m(Tu, Tu)$. Now, by Lemmas 1.7, 1.8, 2.4, and $x_n = Tx_{n-1} \to u$, we deduce that

$$0 = \lim_{n\to\infty} (m(x_n, Tx_n) - m_{x_n, Tx_n}) = \lim_{n\to\infty} (m(x_n, x_{n-1}) - m_{x_n, Tx_n}) = m(u, u) - m_{u,Tu}.$$

Therefore, $m(u, u) = m_{u,Tu}$. Hence, $m(u, u) = m_{u,Tu} = m(Tu, Tu)$ and that is $Tu = u$ as required. □

Next, we present the following example.

Examle 2.6. Let $X := [1, \infty)$ and

$$m(x, y) = \frac{x + y}{2} \text{ for all } X.$$

First, note that (X, m) is a complete M-metric space. Now, consider the function

$$F : (0, \infty) \to \mathbb{R} \text{ defined by } F(x) = \ln(x).$$

Notice that $F \in \mathbb{F}$.

Next, let $T : X \to X$ such that $Tx = \frac{x+1}{2}$. Since $x \in [1, \infty)$, which implies that $x + y > 2$ for all $x, y \in X$. Hence,

$$m(x, y) - m(Tx, Ty) = \frac{x + y}{2} - \frac{x + y + 2}{4} = \frac{x + y - 2}{4} > 0.$$

Also, we have $m(x, y) > 0$ for all $x, y \in X$ and given the fact that F is an increasing function, we deduce that T is an F_m-contraction. Therefore, by Theorem 2.5, T has a unique fixed point in X, in this case the fixed point is 1.

3. F_m-expanding in M-metric spaces

First, we give the definition of F_m-expanding self mapping on M-metric spaces.

Definition 3.1. Let (X, m) be an M-metric spaces. We say that a self mapping T on X is F_m-expanding if there exists $F \in \mathbb{F}$ and $t > 0$ such that for all $x, y \in X$ the following holds:

$$m(x, y) > 0 \Rightarrow F(m(Tx, Ty) \geqslant F(m(x, y)) + t.$$

Next, we present the following useful lemma.

Lemma 3.2 ([3]). *If a self mapping* T *on* X *is surjective, then there exists a self mapping* $T^* : X \to X$, *such that the map* $(T \text{ o } T^*)$ *is the identity map on* X.

Theorem 3.3. *Let* (X, m) *be a complete* M*-metric space and let* $T : X \to X$ *be a surjective* F_m*-expanding map. Then* T *has a unique fixed point in* X.

Proof. Since T is surjective, by Lemma 3.2, we know that there exists a self mapping $T^* : X \to X$, such that the map $(T \text{ o } T^*)$ is the identity map on X. Now, consider $x, y \in X$ such that $m(T^*x, T^*y) > 0$ and let $z = T^*x$ and $w = T^*y$. Hence,

$$m(z, w) > 0.$$

First, notice the following fact,

$$Tz = T(T^*x) = x \text{ and } Tw = T(T^*y) = y.$$

Now, by applying the F_m-expanding property of T we get

$$F(m(Tz, Tw) \geqslant F(m(z, w)) + t.$$

Therefore,

$$F(m(x, y) \geqslant F(m(T^*x, T^*y)) + t.$$

Hence, T^* is a an F_m-contraction self mapping on X. Thus, by Theorem 2.5, T^* has a unique fixed point say $u \in X$. Using the fact that $Tu = T(T^*u) = u$ we deduce that $Tu = u$, that is u is a fixed point of T. Now, assume that there exist $u \neq v \in X$ such that $Tu = u$ and $Tv = v$, where u is also the unique fixed point of T^*. Let $x \in X$ such that $v = T^*x$. Thus,

$$x = T(T^*x) = Tv = v, \text{ but } v = T^*x \text{ which implies that } v = T^*v.$$

Hence, v is a fixed point of T^*, and since T^* has a unique fixed point we deduce that $u = v$ as desired. □

Remark 3.4. We want to bring to reader's attention that if T is not surjective, the result in Theorem 3.3 is false. For example, Let $X = (0, \infty)$ and $m : X^2 \to \mathbb{R}^+$ defined by $m(x, y) = \frac{x+y}{2}$, note that (X, m) is an M-metric space. Now, consider the map $T : X \to X$ defined by $Tx = 5x + 4$. Note that T satisfies the condition

$$m(Tx, Ty) \geqslant 2m(x, y) \text{ for all } x, y \in X.$$

Therefore, it satisfies all the hypothesis of Theorem 3.3, except that T is not surjective in X, and T does not have a fixed point in X.

We finish this section by an example of an F_m-expanding mapping in a complete M-metric space.

Examle 3.5. Let $X := [1, \infty)$ and

$$m(x, y) = \frac{x + y}{2} \text{ for all } X.$$

First, note that (X, m) is a complete M-metric space. Now, consider the function

$$F : (0, \infty) \to \mathbb{R} \text{ defined by } F(x) = \ln(x).$$

Notice that $F \in \mathbb{F}$. Next, let $T : X \to X$ such that $Tx = x^3 + x - 1$. Since $x \in [1, \infty)$, which implies that $x^2 + y^3 > 2$ for all $x, y \in X$. Hence,

$$m(Tx, Ty) - m(x, y) = \frac{x^3 + x - 1 + y^3 + y - 1}{2} - \frac{x + y}{2} = \frac{x^3 + y^3 - 2}{2} > 0.$$

Since we have $m(x, y) > 0$ for all $x, y \in X$ and F is an increasing function, we deduce that T is an F_m-expanding self mapping on X. It is not difficult to see that T is also a surjective map. Therefore, by Theorem 3.3, T has a unique fixed point in X, in this case the fixed point is 1.

4. Consequences

First, we remind the definition of partial metric space which was introduced by Matthews in [5], and it is a very useful extension of metric spaces. However, Shahzad in [4], cleared some issues about partial metric spaces, which was a big misunderstanding for many authors.

Definition 4.1. Let X be a nonempty set and $p : X \times X \longrightarrow [0, +\infty)$. We say that (X, p) is a partial metric spaces if the following conditions are satisfied for all $x, y, z \in X$,

1. $x = y$ if and only if $p(x, y) = p(x, x) = p(y, y)$;
2. $p(x, x) \leqslant p(x, y)$;
3. $p(x, y) = p(y, x)$;
4. $p(x, z) \leqslant p(x, y) + p(y, z) - p(y, y)$.

Next, we give a brief description of the topology of partial metric spaces.

1. A sequence $\{x_n\}_{n=0}^{\infty}$ of elements in X is called p-Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and finite.
2. A partial metric space (X, p) is called complete if for each p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ there exists $z \in X$ such that
$$p(z, z) = \lim_{n\to\infty} p(z, x_n) = \lim_{n,m\to\infty} p(x_n, x_m).$$
3. A sequence x_n in a partial metric space (X, p) is called 0-Cauchy if
$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$
4. We say that (X, p) is 0-complete if every 0-Cauchy in X converges to a point $x \in X$ such that $p(x, x) = 0$.

Since M-metric spaces is a generalization of partial metric spaces, and that is every M-metric is a partial metric but the converse not always true, for instance the M-metric presented in Example 3.5 is not a partial metric space. More examples can be found in [1].

Definition 4.2. Let (X, p) be a complete partial metric space. A self mapping T on X is said to be an F_p-contraction on X if there exist $F \in \mathbb{F}$ and $t > 0$ such that for all $x, y \in X$ the following holds:

$$p(Tx, Ty) > 0 \Rightarrow t + F(p(Tx, Ty)) \leqslant F(p(x, y)).$$

Definition 4.3. Let (X, p) be an partial metric space. We say that a self mapping T on X is F_p-expanding if there exists $F \in \mathbb{F}$ and $t > 0$ such that for all $x, y \in X$ the following holds:

$$p(x, y) > 0 \Rightarrow F(p(Tx, Ty) \geqslant F(p(x, y)) + t.$$

Remark 4.4. Notice that,

$$\text{if a map } T \text{ is } F_p\text{-contractive on } X, \text{ then } T \text{ is } F_m\text{-contractive on } X.$$

Also,

$$\text{if a map } T \text{ is } F_p\text{-expanding on } X, \text{ then } T \text{ is } F_m\text{-expanding on } X.$$

Therefore, the following are consequences of our results in the previous two sections.

Corollary 4.5. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be an F_p-contraction. Then T has a unique fixed point u in X, and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to u.*

Corollary 4.6. *Let (X, m) be a complete partial metric space and let $T : X \to X$ be a surjective F_p-expanding map. Then T has a unique fixed point in X.*

Similarly, it is not difficult to see most the results of [7] and [3] are direct consequences of our results.

5. Conclusion

In closing, we want to present some open questions.

Question 5.1. *Let* (X, m) *be an* M*-metric space,* $F \in \mathbb{F}$, $t > 0$, *and* T *be a self mapping on* X, *such that for every* $x, y \in X$ *we have*

$$m(Tx, Ty) > 0 \Rightarrow t + F(m(Tx, Ty) \leqslant F(\max\{m(x, y), m(x, Tx), m(y, Ty), \frac{m(x, Ty) + m(y, Tx)}{2}\}).$$

Does T *have a unique fixed point on* X*?*

In [6], M_s-metric spaces were introduced.

Notation 5.2.

1. $m_{s_{x,y,z}} := \min\{m_s(x, x, x), m_s(y, y, y), m_s(z, z, z)\}$;
2. $M_{s_{x,y,z}} := \max\{m_s(x, x, x), m_s(y, y, y), m_s(z, z, z)\}$.

Definition 5.3 ([6])**.** An M_s-metric space on a nonempty set X is a function $m_s : X^3 \to R^+$ if for all $x, y, z, t \in X$ we have

1. $m_s(x, x, x) = m_s(y, y, y) = m_s(z, z, z) = m_s(x, y, z)$ if and only if $x = y = z$;
2. $m_{s_{x,y,z}} \leqslant m_s(x, y, z)$;
3. $m_s(x, x, y) = m_s(y, y, x)$;
4. $(m_s(x, y, z) - m_{s_{x,y,z}}) \leqslant (m_s(x, x, t) - m_{s_{x,x,t}}) + (m_s(y, y, t) - m_{s_{y,y,t}}) + (m_s(z, z, t) - m_{s_{z,z,t}})$,

then the pair (X, m_s) is called an M_s-metric space.

Examle 5.4. Let $X = \{1, 2, 3\}$ and define the M_s-metric space m_s on X by $m_s(1, 2, 3) = 6$, $m_s(1, 1, 2) = m_s(2, 2, 1) = m_s(1, 1, 1) = 8$, $m_s(1, 1, 3) = m_s(3, 3, 1) = m_s(3, 3, 2) = m_s(2, 2, 3) = 7$, $m_s(2, 2, 2) = 9$, and $m_s(3, 3, 3) = 5$. It is not difficult to see that (X, m_s) is an M_s-metric space.

Definition 5.5 ([6])**.** Let (X, m_s) be a M_s-metric space. Then

1) a sequence $\{x_n\}$ in X converges to a point x if and only if

$$\lim_{n\to\infty}(m_s(x_n, x_n, x) - m_{s x_n, x_n, x}) = 0;$$

2) a sequence $\{x_n\}$ in X is said to be m_s-Cauchy sequence if and only if

$$\lim_{n,m\to\infty}(m_s(x_n, x_n, x_m) - m_{s x_n, x_n, x_m}) \text{ and } \lim_{n\to\infty}(M_{s x_n, x_n, x_m} - m_{s x_n, x_n, x_m})$$

exist and finite;

3) an M_s-metric space is said to be complete if every m_s-Cauchy sequence $\{x_n\}$ converges to a point x such that

$$\lim_{n\to\infty}(m_s(x_n, x_n, x) - m_{s x_n, x_n, x}) = 0 \text{ and } \lim_{n\to\infty}(M_{s x_n, x_n, x} - m_{s x_n, x_n, x}) = 0.$$

Question 5.6. *Let* (X, m) *be an* M_s*-metric space,* $k > 1$, *and* T *be a surjective self mapping on* X, *such that for every* $x, y, z \in X$ *we have*

$$m_s(Tx, Ty, Tz) \geqslant k m_s(x, y, z).$$

Does T *have a unique fixed point on* X*?*

Question 5.7. *Let* (X, m) *be an* M_s*-metric space,* $F \in \mathbb{F}$, $t > 0$, *and* T *be a self mapping on* X, *such that for every* $x, y \in X$ *we have*

$$m_s(x, Tx, y) > 0 \Rightarrow F(m_s(Tx, T^2x, Ty)) \geqslant F(m_s(x, Tx, y)) + t.$$

Does T *have a unique fixed point on* X*?*

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References

[1] K. Abodayeh, N. Mlaiki, T. Abdeljawad, W. Shatanawi, *Relations between partial metric spaces and M-metric spaces, Caristi Kirk's Theorem in M-metric type spaces*, J. Math. Anal., **7** (2016), 1–12. 4

[2] M. Asadi, E. Karapınar, P. Salimi, *New extension of* p*-metric spaces with some fixed-point results on* M*-metric spaces*, J. Inequal. Appl., **2014** (2014), 9 pages. 1, 1.1, 1.2, 1.5, 1, 1.7, 1.8

[3] J. Górnicki, *Fixed points theorems for F-expanding mappings*, Fixed Point Theory Appl., **2017** (2017), 10 pages. 1, 3.2, 4

[4] R. H. Haghi, S. Rezapour, N. Shahzad, *Be careful on partial metric fixed point results*, Topology Appl., **160** (2013), 450–454. 4

[5] S. G. Matthews, *Partial metric topology*, Ann. New York Acad. Sci., **728** (1994), 183–197. 4

[6] N. Mlaiki, N. Souayah, K. Abodayeh, T. Abdeljawad, *Contraction principles in* M_s*-metric spaces*, J. Nonlinear Sci. Appl., **10** (2017), 575–582. 5, 5.3, 5.5

[7] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012** (2012), 6 pages. 1, 4