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A New Analytical Method for Solving Hamilton-Jacobi-Bellman Equation

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Abstract

In this paper, we apply a modification of variational iteration method using He's polynomials for a class of nonlinear optimal control problems which are converted to the Hamilton-Jacobi-Bellman equations (HJB) and present a convergence theorem of the method. The proposed modification is made by introducing He's polynomials in the correction functional. The suggested algorithm is quite efficient and is practically well suited for using in these problems. Some examples are given to demonstrate the simplicity and efficiency of the proposed method.

AMS Subject Classification: **34B99**
Keywords: Optimal control problem; Homotopy perturbation method; Variational iteration method; Numerical solution.

1 . Introduction

Theory and application of optimal control have been widely used in different fields such as biomedicine [1], aircraft systems [2], robotic [3], etc. However, optimal control of nonlinear systems is a challenging task which has been studied extensively for decades. Methods of solving nonlinear optimal control problems (OCP's) can be divided into two categories. The first category, which contains direct methods, converts the problem into a nonlinear programming by using the discretization or parameterization techniques [4, 5]. The second category contains indirect methods and leads to the Hamilton-Jacobi-Bellman (HJB) equation, based on dynamic programming [6], or nonlinear two-point boundary value problem (TPBVP), based on the Pontryagin's maximum principle [7]. In general, the HJB equation is a nonlinear partial differential equation that is hard to solve in most cases. An excellent literature review on the methods for approximating the solution of HJB equation is provided in [8]. Besides, nonlinear TPBVP has no analytical solution except for a few simple cases. Thus, many researches have been devoted to find an approximate solution for the nonlinear TPBVP's.

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Recently, successive approximation approach (SAA) and sensitivity approach have been introduced in [9] and [10], respectively. In those, a sequence of nonhomogeneous linear time-varying TPBVP's is solved instead of directly solving the nonlinear TPBVP derived from the maximum principle. However, solving time-varying equations is much more difficult than solving time-invariant ones.
He[15, 16] developed the variational iteration and homotopy perturbation methods for solving linear, nonlinear, initial and boundary value problems. The homotopy perturbation method was developed by merging two techniques: the standard homotopy and the perturbation. The basic motivation of this paper is to dicsuss convergence the variational iteration method coupled with He's polynomials (MVIM) [13, 14, 18] and applying it for finding the solution of a class of nonlinear optimal control problems. In this algorithm, the correct functional is developed [11, 16, 19, 20, 21] and the Lagrange multipliers are calculated optimally via variational theory. The use of Lagrange multipliers reduces the successive application of the integral operator and the cumbersome of huge computational work while still maintaining a very high level of accuracy. Finally, the He's polynomials are introduced in the correct functional and the comparison of like powers of p gives solutions of various orders. This paper is organized as follows. In Section 2, we introduce the nonlinear time-variant HJB equation. In Section 3, we explain the modified variational method (MVIM) and convergence analysis of this method is discussed. In Section 4, numerical examples are simulated to show the reasonableness of our theory and demonstrate the high performance of proposed method. Finally, Some conclusions are summarized in the last section.

2 . Nonlinear time-variant HJB equation

In this section, we Consider a nonlinear control system described by

$$\dot{x}(t) = a(x(t),u(t),t), \tag{1}$$

where $x(t)$ is a state vector; $u(t)$ is a control signal. The objective is to find the optimal control law $u^*(t)$, which minimizes the following cost function:

$$J = h(x(t_f),t_f) + \int_0^{t_f} g(x(\tau),u(\tau),\tau)d\tau. \tag{2}$$

In this cost function, h and g are arbitrary convex functions and t_f is final time of system operation. Using dynamic programming approach, we introduce a new variable as

$$j(x(t),t,u(\tau)) = h(x(t_f),t_f) + \int_0^{t_f} g(x(\tau),u(\tau),\tau)d\tau,$$
$$t \le \tau \le t_f, \quad t_0 \le t \le t_f. \tag{3}$$

Suppose that we have

$$V(x(t),t) = J^*(x(t),t) = \min_{\substack{u(\tau)\\ t\le\tau\le t_f}} \{h(x(t_f),t_f) + \int_t^{t_f} g(x(\tau),u(\tau),\tau)d\tau\}. \tag{4}$$

Therefore, we have

$$V(x(t),t) = \min_{\substack{u(\tau)\\ t\le\tau\le t_f}} \{h(x(t_f),t_f) + \int_t^{t+\Delta t} g(x(\tau),u(\tau),\tau)d\tau$$
$$+ \int_{t+\Delta t}^{t_f} g(x(\tau),u(\tau),\tau)d\tau\}. \tag{5}$$

According to the principle of optimality, we have

$$V(x(t),t) = \min_{\substack{u(\tau)\\ t\le\tau\le t_f}} \{\int_t^{t+\Delta t} g(x(\tau),u(\tau),\tau)d\tau + V(x(t+\Delta t),t+\Delta t)\}. \tag{6}$$

Therefore, using Taylor series we have

$$V(x(t),t) = \min_{\substack{u(\tau)\\ t\le\tau\le t_f}} \{\int_t^{t+\Delta t} g(x(\tau),u(\tau),\tau)d\tau + V(x(t),t),$$

$$+\frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial x}[x(t+\Delta t)-x(t)] + H.O.T\}. \tag{7}$$

If we suppose that Δt is small enough then $\tau \to t$ and we have

$$V(x(t),t) = \min_{u(\tau)} \{g\Delta t + V(x,t) + \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial x}a(x,u,t)\Delta t + O(\Delta t)\}. \tag{8}$$

By dividing both sides of Equation (8) by Δt, we have

$$-\frac{\partial V}{\partial t} = \min_{u(\tau)} \{g(x(t),u(t),t) + \frac{\partial V}{\partial x}a(x,u,t)\}. \tag{9}$$

This nonlinear time-variant differential equation called HJB equation. We have the following boundary condition

$$J^*(x(t_f),t_f) = V(x(t_f),t_f) = h(x(t_f),t_f), \tag{10}$$

by introducing the Hamiltonian function

$$H(x,u,V_x,t) = g(x(t),u(t),t) + \frac{\partial V}{\partial x}a(x,u,t), \tag{11}$$

we have

$$H(x,u^*,V_x,t) = \min_{u(\tau)} \quad H(x,u,V_x,t). \tag{12}$$

Therefore, by substituting of Hamiltonian function (12) in Equation (9), we have

$$-\frac{\partial V}{\partial t} = H(x,u^*(x,V_x,t),V_x,t). \tag{13}$$

3 . New analytic method

3.1 Description of method

The variational iteration method, which provides an analytical approximate solution, is applied to various nonlinear problems [20, 21]. In this section, we present an alternative approach of VIM. This approach can be implemented, in a reliable and efficient way, to handle the nonlinear differential equation,

$$L[u(t)] + N[u(t)] = g(t), \quad t > 0, \tag{14}$$

where $L = \frac{d^m}{dt^m}, m \in \mathrm{N}$, is a linear operator as, N a nonlinear operator and $g(t)$ is the source inhomogeneous term, subject to the initial conditions,

$$u^{(k)}(0) = c_k,\ k = 0,1,2,\cdots,m-1. \tag{15}$$

Where c_k is a real number. According to the He's variational iteration method, we can construct a correction functional for (14) as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\{Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\}d\tau, \quad n \geq 0,$$

where λ is a general Lagrangian multiplier [17], which can be identified optimally via variational theory. Here, we apply restricted variations to nonlinear term Nu, in this case we can easily determine the Lagrange multiplier. Making the above functional stationary, noticing that $\delta\tilde{u}_n = 0$,

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta\int_0^t \lambda(\tau)\{Lu_n(\tau) - g(\tau)\}d\tau,$$

yields the following Lagrange multipliers,

$$\lambda = -1 \quad for \quad m = 1,$$
$$\lambda = \tau - t, \quad for \quad m = 2,$$

and in general,

$$\lambda = \frac{(-1)^m}{(m-1)!}(\tau - t)^{(m-1)}, \quad for \;\; m \geq 1.$$

Now we assume that the solution of (14) can be expressed as a series of the power of p

$$v = v_0 + p\, v_1 + p^2 v_2 + \cdots. \tag{16}$$

Substituting $p = 1$ in (16), yields the approximate solution of (14) as follows

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{17}$$

The method considers the nonlinear term $N[v]$ as

$$N(v) = \sum_{i=0}^{+\infty} p^i H_i = H_0 + pH_1 + p^2 H_2 + \cdots,$$

where H_n's are the so-called He's polynomials [13], which can be calculated by using the formula

$$H_n(v_0, v_1, \cdots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left(N\left(\sum_{i=0}^{n} p^i v_i\right)\right)_{p=0}, \;\; n = 0,1,2,\cdots.$$

Now, we apply a series of the power of p and then using He's polynomials we have

$$\begin{aligned}\sum_{n=0}^{\infty} p^n v_n &= u_0 + p\int_0^t \lambda(\tau)\left[N(\sum_{n=0}^{\infty} p^n v_n(\tau)) - g(\tau)\right]d\tau \\ &= u_0 + p\int_0^t \lambda(\tau)\left[\sum_{n=0}^{\infty} p^n H_n - g(\tau)\right]d\tau,\end{aligned} \tag{18}$$

which is the modified variational iteration method using He's polynomials. Now, equating coefficients of like powers of p, we have

$$p^0 : v_0 = u_0,$$

$$p^1 : v_1 = \int_0^t \frac{(-1)^m}{(m-1)!}(\tau - t)^{(m-1)}\left[H_0(v_0) - g(\tau)\right]d\tau,$$

$$p^2 : v_2 = \int_0^t \frac{(-1)^m}{(m-1)!}(\tau - t)^{(m-1)}\left[H_1(v_0, v_1)\right]d\tau,$$

$$\vdots$$
$$p^j : v_j = \int_0^t \frac{(-1)^m}{(m-1)!}(\tau - t)^{(m-1)}\left[H_{j-1}(v_0, v_1, \cdots, v_{j-1})\right]d\tau,$$
$$\vdots \tag{19}$$

The zeroth (initial) approximation $v_0 = u_0$ can be freely chosen if it satisfies the initial and boundary conditions of the problem. The success of the method depends on the proper selection of the initial approximation v_0. However, using the initial values $u^{(k)}(0) = c_k, k = 0,1,2,\cdots, m-1$ are preferably used for the selective zeroth approximation v_0 as will be seen later. In our alternative approach we select the initial approximation v_0 as

$$v_0(t) = \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k. \tag{20}$$

3.2 Convergence of method

Theorem 3.1 *If the series solution* $u(t) = \sum_{n=0}^{+\infty} v_n(t)$*, defined in(17),converges, then it is an exact solution of the nonlinear problem (14).*

Proof. Substituting $p = 1$ and $u_0(t) = \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k$ in (18), suppose that the series solution (17) converges, say $\phi(t) = \sum_{n=0}^{+\infty} v_n(t)$, then we have

$$\phi(t) = \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k + \int_0^t \frac{(-1)^m}{(m-1)!}(\tau - t)^{(m-1)}\left[N(\phi(\tau)) - g(\tau)\right]d\tau, \tag{21}$$

Applying the operator $L = \frac{d^m}{dt^m}, m \in \mathrm{N}$, to both sides of Eq.(21) and then using Leibnitz rule we obtain

$$L[\phi(t)] = -\left[N(\phi(t)) - g(t)\right].$$

Consequently, we have

$$L[\phi(t)] + \left[N(\phi(t)) - g(t)\right] = 0.$$

Therefore, we can observe that $\phi(t) = \sum_{n=0}^{+\infty} v_n(t)$ is an exact solution of problem (14). This completes the proof of Theorem (3.2).

4 . Application

In order to assess the advantages and the accuracy of MVIM for solving nonlinear optimal control problems, we will consider the following examples.

Example 4.1 *Consider a single-input scalar system as follows [22]:*

$$\dot{x} = -x(t) + u(t), \tag{22}$$

$$J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt, \tag{23}$$

The corresponding Hamiltonian function will be

$$H(x,u,V_x,t)=\frac{1}{2}x^2(t)+\frac{1}{2}u^2(t)+\frac{\partial V(x,t)}{\partial x}[-x(t)+u(t)]. \tag{24}$$

For finding u^*, we have

$$\frac{\partial H}{\partial u}=u(t)+\frac{\partial V}{\partial x}=0. \tag{25}$$

Therefore, we obtain

$$u^*(t)=-\frac{\partial V}{\partial x}. \tag{26}$$

Because $\frac{\partial^2 H}{\partial u^2}=1>0$, u^* is a minimum and acceptable. Now, by substituting u^* in HJB equation, we have the following equation:

$$-\frac{\partial V}{\partial t}=\frac{1}{2}x^2-\frac{1}{2}\left(\frac{\partial V(x,t)}{\partial x}\right)^2-x\frac{\partial V(x,t)}{\partial x}, \tag{27}$$

$$V(x(1),1)=0. \tag{28}$$

or

$$\frac{\partial V}{\partial t}=-\frac{1}{2}x^2(t)+\frac{1}{2}\left(\frac{\partial V(x,t)}{\partial x}\right)^2+x\frac{\partial V(x,t)}{\partial x}, \tag{29}$$

$$V(x(1),1)=0. \tag{30}$$

For this problem, we have the exact solution of state $x(t)$ and the control $u(t)$ as follows

$$x(t)=cosh(\sqrt{2}t)+\beta sinh(\sqrt{2}t),$$
$$u(t)=(1+\sqrt{2}\beta)cosh(\sqrt{2}t)+(\sqrt{2}+\beta)sinh(\sqrt{2}t),$$

where

$$\beta=-\frac{cosh(\sqrt{2})+\sqrt{2}sinh(\sqrt{2})}{\sqrt{2}cosh(\sqrt{2})+sinh(\sqrt{2})}\approx -0.98,$$

for the sake of simplicity, we assume that $u^*(x,t)=-k(t)x(t)$, where

$$k(t)=-\frac{(1+\sqrt{2}\beta)cosh(\sqrt{2}t)+(\sqrt{2}+\beta)sinh(\sqrt{2}t)}{cosh(\sqrt{2}t)+\beta sinh(\sqrt{2}t)}.$$

To solve Eq. (29) by means of MVIM, as a mention in section (3.1) we construct a correction functional

$$V_{n+1}(x,\tau)=V_n(x,\tau)-\int_1^t\left(\frac{\partial V_n(x,\tau)}{\partial t}-\frac{1}{2}(\frac{\partial V_n(x,\tau)}{\partial x})^2-x\frac{\partial V_n(x,\tau)}{\partial x}+\frac{1}{2}x^2(\tau)\right)d\tau,$$

By applying the MVIM we have

$$\sum_{i=0}^{\infty}p^i w_i(x,t)=V_0(x,t)-p\int_1^t\left[-\frac{1}{2}\sum_{i=0}^{\infty}p^i H_i-x\frac{\partial(\sum_{i=0}^{\infty}p^i w_i(x,\tau))}{\partial x}+\frac{1}{2}x^2(\tau)\right]d\tau,$$

where

$$H_i = \frac{1}{i!}\frac{\partial^i}{\partial p^i}\left(\frac{\partial(\sum_{k=0}^{i} p^k w_k(x,t))}{\partial x}\right)^2_{p=0},$$

Now by comparison of like powers and equating the coefficients of the terms with the identical powers of p we have

$$p^0 : w_0(x,t) = V_0(x,t),$$
$$p^1 : w_1(x,t) = -\int_1^t\left(-\frac{1}{2}(\frac{\partial w_0(x,\tau)}{\partial x})^2 - x(\frac{\partial w_0(x,\tau)}{\partial x}) + \frac{1}{2}x^2(\tau)\right)d\tau,$$
$$p^2 : w_2(x,t) = \int_1^t -\frac{1}{2}(\frac{\partial w_0(x,\tau)}{\partial x})(\frac{\partial w_1(x,\tau)}{\partial x}) + x(\frac{\partial w_1(x,\tau)}{\partial x})d\tau,$$
$$\vdots$$
$$p^j : w_j(x,t) = -\int_1^t\left(-\frac{1}{2}\sum_{k=0}^{j-1}(\frac{\partial w_k(x,\tau)}{\partial x})(\frac{\partial w_{j-k-1}(x,\tau)}{\partial x}) + x(\frac{\partial w_{j-1}(x,\tau)}{\partial x})\right)d\tau.$$

If we take $w_0(x,t) = 0$, we have

$$w_1(x,t) = -\frac{1}{2}x^2(t-1),$$
$$w_2(x,t) = -\frac{1}{2}x^2(t^2-1) + x^2(t-1),$$
$$w_3(x,t) = -\frac{1}{6}x^2(t^3-1) + \frac{1}{6}x^2(t^2-1) - \frac{1}{2}x^2(t-1),$$
$$w_4(x,t) = \frac{1}{6}x^2(t^4-1) - \frac{2}{3}x^2(t^3-1) + x^2(t^2-1) - \frac{2}{3}x^2(t-1),$$
$$\vdots$$

In figure (1) and table (1) we compare the results of MVIM with the exact solution of $k(t)$. This confirm that the proposed method yields excellent results.

Table 1: Comparison between the exact and MVIM solution of $k(t)$, for n =15.

t	MVIM Solution	Analytic Solution	Absolute Error
0.0	3.8010e-001	3.8582e-001	5.7169e-003
0.2	3.6438e-001	3.6460e-001	2.1416e-004
0.4	3.2801e-001	3.2801e-001	2.8398e-006
0.6	2.6588e-001	2.6588e-001	5.6802e-009
0.8	1.6306e-001	1.6306e-001	1.1410e-013
1.0	-1.7396e-015	1.3634e-015	3.1030e-015

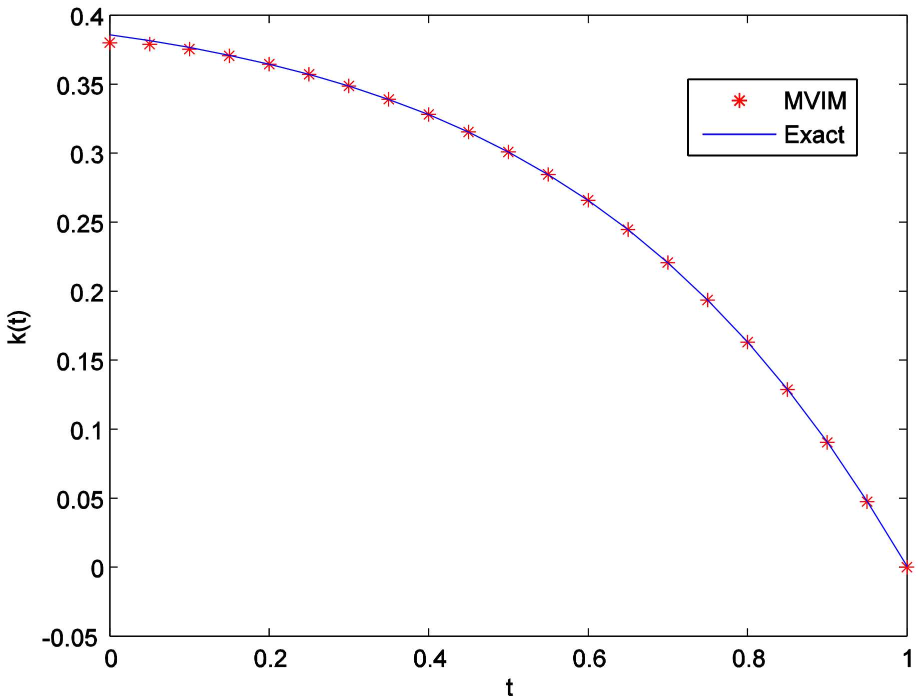

Fig. 1. The exact and MVIM approximate solution of $k(t)$.

Example 4.2 *Consider the following purely mathematical optimal control problem:*

$$\dot{x} = x(t) + u(t), \tag{31}$$

$$J = x^2(t_f) + \int_0^{t_f} u^2(t)dt. \tag{32}$$

The corresponding Hamiltonian function will be

$$H(x,u,V_x,t) = u^2(t) + V_x(x,t)[x+u]. \tag{33}$$

For finding u^*, we have

$$\frac{\partial H}{\partial u} = 2u(t) + V_x(x,t) = 0. \tag{34}$$

Therefore, we obtain

$$u^*(t) = -\frac{1}{2}V_x(x,t). \tag{35}$$

Because $\dfrac{\partial^2 H}{\partial u^2} = 2 > 0$, u^* is a minimum and acceptable. Now, by substituting u^* in HJB equation, we have the following equation:

$$-V_t = -\frac{1}{4}V_x^2 + xV_x, \tag{36}$$

$$V(x(t_f),t_f)=x^2(t_f).$$

In Chapter 19 of [12], authors obtained the solution of the above HJB equation in form:

$$V(x,t,t_f)=\frac{2x^2}{1+e^{2(t-t_f)}}. \tag{37}$$

for sake of simplicity, let $u^*(x,t)=k(t)x(t),$ where

$$k(t)=\frac{-2}{1+e^{2(t-t_f)}}.$$

To solve Eq. (36) by means of MVIM when $t_f=1$, as a mention in section (3.1) we construct a correction functional

$$V_{n+1}(x,\tau)=V_n(x,\tau)-\int_1^t\left(\frac{\partial V_n(x,\tau)}{\partial t}-\frac{1}{4}(\frac{\partial V_n(x,\tau)}{\partial x})^2+x\frac{\partial V_n(x,\tau)}{\partial x}\right)d\tau,$$

By applying the MVIM we have

$$\sum_{i=0}^{\infty}p^iw_i(x,t)=V_0(x,t)-p\int_1^t\left[-\frac{1}{4}\sum_{i=0}^{\infty}p^iH_i+x\frac{\partial(\sum_{i=0}^{\infty}p^iw_i(x,\tau))}{\partial x}\right]d\tau,$$

where

$$H_i=\frac{1}{i!}\frac{\partial^i}{\partial p^i}\left(\frac{\partial(\sum_{k=0}^{i}p^kw_k(x,t))}{\partial x}\right)^2_{p=0},$$

Now by comparison of like powers and equating the coefficients of the terms with the identical powers of p we have

$$p^0: w_0(x,t)=V_0(x,t),$$

$$p^1: w_1(x,t)=-\int_1^t\left(-\frac{1}{4}(\frac{\partial w_0(x,\tau)}{\partial x})^2+x(\frac{\partial w_0(x,\tau)}{\partial x})\right)d\tau,$$

$$p^2: w_2(x,t)=\int_1^t-\frac{1}{4}(\frac{\partial w_0(x,\tau)}{\partial x})(\frac{\partial w_1(x,\tau)}{\partial x})+x(\frac{\partial w_1(x,\tau)}{\partial x})d\tau,$$

$$\vdots$$

$$p^j: w_j(x,t)=-\int_1^t\left(-\frac{1}{4}\sum_{k=0}^{j-1}(\frac{\partial w_k(x,\tau)}{\partial x})(\frac{\partial w_{j-k-1}(x,\tau)}{\partial x})+x(\frac{\partial w_{j-1}(x,\tau)}{\partial x})\right)d\tau.$$

If we take $w_0(x,t)=x^2$, we have

$$w_1(x,t)=-x^2(t-1),$$

$$w_2(x,t)=0,$$

$$w_3(x,t)=-\frac{1}{3}x^2(t^3-1)-x^2(t^2-1)+x^2(t-1),$$

$$w_4(x,t)=0,$$

$$w_5(x,t)=0,$$

$$w_6(x,t) = -\frac{2}{15}x^2(t^5-1)+\frac{2}{3}x^2(t^4-1)-\frac{4}{3}x^2(t^3-1)+\frac{4}{3}x^2(t^2-1)-\frac{2}{3}x^2(t^{-}1),$$

$$\vdots$$

In figure (2) and table (2) we compare the results of MVIM with the exact solution of $k(t)$. This confirm that the proposed method yields excellent results.

Table 2: Comparison between the exact and MVIM solution of $k(t)$, for n =20.

t	MVIM Solution	Analytic Solution	Absolute Error
0.0	-1.7615	-1.7616	6.8965e-005
0.2	-1.6640	-1.6640	7.0978e-007
0.4	-1.5370	-1.5370	1.8553e-009
0.6	-1.3799	-1.3799	4.0035e-013
0.8	-1.1974	-1.1974	0
1.0	-1.0000	-1.0000	3.3307e-016

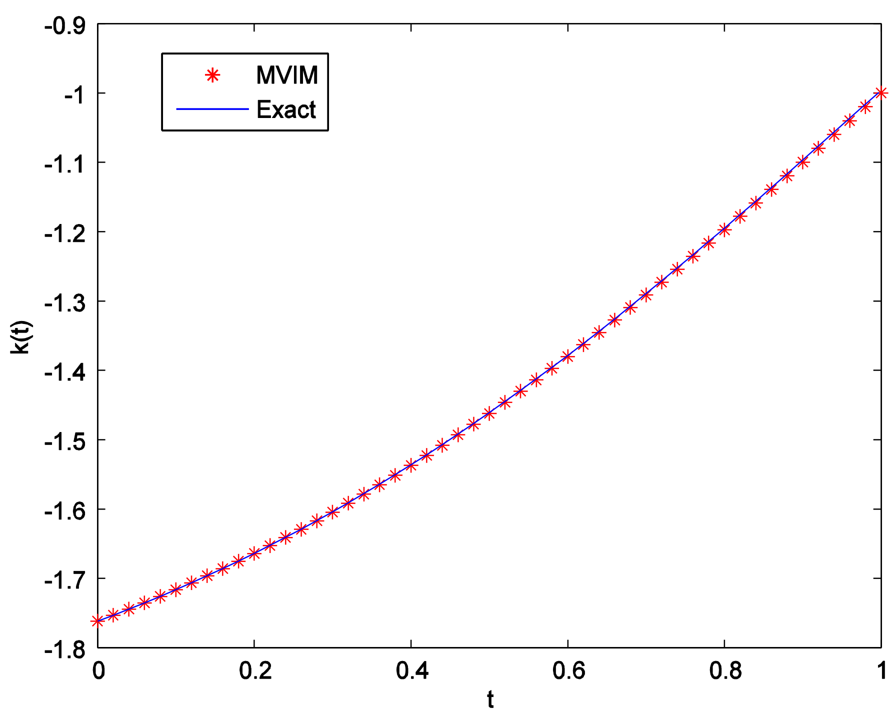

Fig. 1. The exact and MVIM approximate solution of $k(t)$.

5 . Conclusion

In this paper, we present a reliable algorithm based on the MVIM to solve nonlinear optimal control problem. The obtained solution is compared with the exact solution. The examples show that the MVIM is clearly very efficient and powerful technique in finding the solutions of the nonlinear optimal control problems.The computations associated with the example in this paper were performed using Matlab 7.

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