Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci., 18 (2018), 282–293
Research Article

Online: ISSN 2008-949x

Journal of Mathematics and Computer Science

Journal Homepage: www.tjmcs.com - www.isr-publications.com/jmcs

On upper and lower (τ_1, τ_2)-precontinuous multifunctions

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Abstract

This paper deals with the concepts of upper and lower (τ_1, τ_2)-precontinuous multifunctions. Some characterizations of upper and lower (τ_1, τ_2)-precontinuous multifunctions are investigated. The relationships between upper and lower (τ_1, τ_2)-precontinuous multifunctions and the other types of continuity are discussed.

Keywords: $\tau_1\tau_2$-preopen, upper (τ_1, τ_2)-precontinuous multifunction, lower (τ_1, τ_2)-precontinuous multifunction.

2010 MSC: 54C08, 54C60, 54E55.

1. Introduction

Continuity and multifunctions are basic topics in the theory of classical point set topology and several branches of mathematics. It is well-known that multifunctions play a very important role not in functional analysis but also in mathematical economics, control theory, and fuzzy topology. Semi-open sets, preopen sets, α-open sets, β-open sets, and δ-open sets play an important role in the researches of generalizations of continuity on topological spaces. By using these sets several authors introduced and studied various types of work forms of continuity for functions and multifunctions. Levine [10] introduced the notion of semi-open sets and semi-continuity in topological spaces. Maheshwari and Prasad [11] extended the notions of semi-open sets and semi-continuity to the bitopological setting. Bose [3] further investigated several properties of semi-open sets and semi-continuity in bitopological spaces. In 1982, Mashhour et al. [12] introduced the notions of preopen sets and percontinuity in topological spaces. Jelić [7] generalized the notions of preopen sets and precontinuity to the setting of bitopological spaces. Khedr et al. [8] generalized the notion of semi-preopen sets to bitopological spaces and defined semi-precontinuity in bitopological spaces. In 2008, Ekici et al. [5] introduced the notion of contra-continuous multifunctions. Recently, Ekici et al. [6] introduced and studied two new concepts namely, contra-precontinuous and almost contra-precontinuous multifunctions which are containing the class of contra-continuous multifunctions and contained in the class of weakly precontinuous multifunctions. Noiri and Popa [13] introduced

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doi: 10.22436/jmcs.018.03.04

Received: 2017-08-07 *Revised:* 2018-01-26 *Accepted:* 2018-02-23

the notion of weakly precontinuous functions in bitopological spaces and obtained several characterizations and some properties of weakly precontinuous functions. The purpose of the present paper is to introduce the notions of upper and lower (τ_1, τ_2)-precontinuous multifunctions and investigate some characterizations of upper and lower (τ_1, τ_2)-precontinuous multifunctions. Furthermore, the relationships between upper and lower (τ_1, τ_2)-precontinuous multifunctions and the other types of continuity are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-semi-open (Resp. $\tau_1\tau_2$-regular open [1], $\tau_1\tau_2$-regular closed [4], $\tau_1\tau_2$-preopen [7]) if $A \subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(A))$ (Resp. $A = \tau_1\text{-Int}(\tau_2\text{-Cl}(A))$, $A = \tau_1\text{-Cl}(\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A))$). The complement of $\tau_1\tau_2$-semi-open (Resp. $\tau_1\tau_2$-preopen) set is said to be $\tau_1\tau_2$-semi-closed (Resp. $\tau_1\tau_2$-preclosed). The $\tau_1\tau_2$-semi-closure (Resp. $\tau_1\tau_2$-preclosure [8]) of A is defined by the intersection of $\tau_1\tau_2$-semi-closed (Resp. $\tau_1\tau_2$-preclosed) sets containing A and is denoted by $\tau_1\tau_2$-sCl(A) (Resp. $\tau_1\tau_2$-pCl(A)). The $\tau_1\tau_2$-semi-interior (Resp. $\tau_1\tau_2$-preinterior [13]) of A is defined by the union of $\tau_1\tau_2$-semi-open (Resp. $\tau_1\tau_2$-preopen) sets contained in A and is denoted by $\tau_1\tau_2$-sInt(A) (Resp. $\tau_1\tau_2$-pInt(A)). By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [2], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Then F is said to be surjection if $F(X) = Y$, or equivalent, if for each $y \in Y$ there exists $x \in X$ such that $y \in F(x)$ and F is called injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

Lemma 2.1 ([13])**.** *Let (X, τ_1, τ_2) be a bitopological space and $\{A_\alpha \mid \alpha \in \nabla\}$ a family of subsets of X. The following properties are hold.*

(1) *If A_α is $\tau_1\tau_2$-preopen for each $\alpha \in \nabla$, then $\bigcup_{\alpha \in \nabla} A_\alpha$ is $\tau_1\tau_2$-preopen.*
(2) *If A_α is $\tau_1\tau_2$-preclosed for each $\alpha \in \nabla$, then $\bigcap_{\alpha \in \nabla} A_\alpha$ is $\tau_1\tau_2$-preclosed.*

Lemma 2.2 ([13])**.** *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties are hold.*

(1) $\tau_1\tau_2$-pInt(A) *is $\tau_1\tau_2$-preopen.*
(2) $\tau_1\tau_2$-pCl(A) *is $\tau_1\tau_2$-preclosed.*

Lemma 2.3 ([13])**.** *For a subset A of a bitopological space (X, τ_1, τ_2), $x \in \tau_1\tau_2$-pCl(A) if and only if $U \cap A \neq \emptyset$ for every $\tau_1\tau_2$-preopen set U containing x.*

Lemma 2.4 ([13])**.** *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties are hold.*

(1) $X - \tau_1\tau_2\text{-pInt}(A) = \tau_1\tau_2\text{-pCl}(X - A)$.
(2) $X - \tau_1\tau_2\text{-pCl}(A) = \tau_1\tau_2\text{-pInt}(X - A)$.

A subset A of a bitopolgical space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*closed* if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is said to be $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets containing A is called $\tau_1\tau_2$-*closure* of A and denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets contained in A is called $\tau_1\tau_2$-*interior* of A and denoted by $\tau_1\tau_2$-Int(A). A subset N of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*neighborhood* (Resp. $\tau_1\tau_2$-*preneighborhood*) of $x \in X$ if there exists a $\tau_1\tau_2$-open (Resp. $\tau_1\tau_2$-preopen) set V of (X, τ_1, τ_2) such that $x \in V \subseteq N$.

Lemma 2.5. *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold.*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ *and* $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.
(2) *If* $A \subseteq B$, *then* $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.
(3) $\tau_1\tau_2\text{-}Cl(A)$ *is* $\tau_1\tau_2$*-closed.*
(4) A *is* $\tau_1\tau_2$*-closed if and only if* $A = \tau_1\tau_2\text{-}Cl(A)$.
(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

Lemma 2.6. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold.*

(1) $\tau_1\tau_2\text{-}sCl(A) = \tau_1\text{-}Int(\tau_2\text{-}Cl(A)) \cup A$.
(2) *If A is $\tau_1\tau_2$-open, then* $\tau_1\tau_2\text{-}sCl(A) = \tau_1\text{-}Int(\tau_2\text{-}Cl(A))$.

Proof.
(1) Since $\tau_1\tau_2\text{-sCl}(A)$ is $\tau_1\tau_2$-semi-closed, we have $\tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\tau_2\text{-sCl}(A))) \subseteq \tau_1\tau_2\text{-sCl}(A)$. Therefore, $\tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \subseteq \tau_1\tau_2\text{-sCl}(A)$ and hence, $\tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \cup A \subseteq \tau_1\tau_2\text{-sCl}(A)$. To establish the opposite inclusion we observe that

$$\begin{aligned}\tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \cup A)) &= \tau_1\text{-Int}(\tau_2\text{-Cl}(A) \cup \tau_2\text{-Cl}(\tau_1\text{-Int}(\tau_2\text{-Cl}(A))))\\ &\subseteq \tau_2\text{-Cl}(A) \cup \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(\tau_2\text{-Cl}(A))))\\ &= \tau_2\text{-Cl}(A) \cup \tau_1\text{-Int}(\tau_2\text{-Cl}(A)) = \tau_2\text{-Cl}(A).\end{aligned}$$

Thus,

$$\tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \cup A)) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \cup A.$$

Hence, $\tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \cup A$ is $\tau_1\tau_2$-semi-closed and so $\tau_1\tau_2\text{-sCl}(A) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A)) \cup A$.

(2) Let A be a $\tau_1\tau_2$-open set. Then $A = \tau_1\text{-Int}(\tau_2\text{-Int}(A)) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A))$ and by (1), we have $\tau_1\tau_2\text{-sCl}(A) = \tau_1\text{-Int}(\tau_2\text{-Cl}(A))$. □

The following example shows that the converse of (2) in the above lemma is not true in general.

Example 2.7. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{c\}, X\}$. Then $\tau_1\tau_2\text{-sCl}(\{a, b\}) = \tau_1\text{-Int}(\tau_2\text{-Cl}(\{a, b\})$ but $\{a, b\}$ is not $\tau_1\tau_2$-open.

3. Some characterizations

We begin this section by introducing the notions of upper and lower (τ_1, τ_2)-precontinuous multifunctions.

Definition 3.1. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:

(1) *upper* (τ_1, τ_2)*-precontinuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau_1\tau_2$-preopen set U containing x such that $F(U) \subseteq V$;
(2) *lower* (τ_1, τ_2)*-precontinuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-preopen set U containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$;
(3) *upper* (Resp. *lower*) (τ_1, τ_2)*-precontinuous* if F has this property at each point of X.

The following theorems give some characterizations of upper and lower (τ_1, τ_2)-precontinuous multifunctions.

Theorem 3.2. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F *is upper* (τ_1, τ_2)*-precontinuous;*
(2) $F^+(V)$ *is* $\tau_1\tau_2$*-preopen in* X *for any* $\sigma_1\sigma_2$*-open set* V *of* Y*;*
(3) $F^-(H)$ *is* $\tau_1\tau_2$*-preclosed in* X *for any* $\sigma_1\sigma_2$*-closed set* H *of* Y*;*
(4) $\tau_1\tau_2$*-pCl*$(F^-(B)) \subseteq F^-(\sigma_1\sigma_2$*-Cl*$(B))$ *for any subset* B *of* Y*;*
(5) *for each* $x \in X$ *and each* $\sigma_1\sigma_2$*-neighborhood* V *of* $F(x)$*,* $F^+(V)$ *is a* $\tau_1\tau_2$*-preneighborhood of* x*;*
(6) *for each* $x \in X$ *and each* $\sigma_1\sigma_2$*-neighbourhood* V *of* $F(x)$*, there exists a* $\tau_1\tau_2$*-preneighborhood* U *of* x *such that* $F(U) \subseteq V$*;*
(7) $F^+(\sigma_1\sigma_2$*-Int*$(B)) \subseteq \tau_1\tau_2$*-pInt*$(F^+(B))$ *for any subset* B *of* Y*.*

Proof.

(1)⇒(2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. By (1), there exists a $\tau_1\tau_2$-preopen set U_x containing x, such that $U_x \subseteq F^+(V)$. It follows that $F^+(V) = \cup_{x \in F^+(V)} U_x$. By Lemma 2.1, we have $F^+(V)$ is $\tau_1\tau_2$-preopen in X.

(2)⇔(3): It follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

(3)⇒(4): For any subset B of Y, $\sigma_1\sigma_2$-Cl$(B))$ is $\sigma_1\sigma_2$-closed in Y. By (3), $F^-(\sigma_1\sigma_2\text{-Cl}(B)))$ is $\tau_1\tau_2$-preclosed in X. Therefore, we obtain $\tau_1\tau_2\text{-pCl}(F^-(B)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(B))$.

(4)⇒(3): Let H be any $\sigma_1\sigma_2$-closed set of Y. By (4), $\tau_1\tau_2\text{-pCl}(F^-(H)) \subseteq F^-(\sigma_1\sigma_2\text{-Cl}(H)) = F^-(H)$ and hence, $F^-(H)$ is $\tau_1\tau_2$-preclosed in X.

(2)⇒(5): Let $x \in X$ and V be a $\sigma_1\sigma_2$-neighborhood of $F(x)$. There exists a $\sigma_1\sigma_2$-open set G of Y such that $F(x) \subseteq G \subseteq V$. Then we have $x \in F^+(G) \subseteq F^+(V)$. By (2), $F^+(G)$ is $\tau_1\tau_2$-preopen in X and hence, $F^+(V)$ is a $\tau_1\tau_2$-preneighborhood of x.

(5)⇒(6): Let $x \in X$ and V be any $\sigma_1\sigma_2$-neighborhood of $F(x)$. By (5), $F^+(V)$ is a $\tau_1\tau_2$-preneighborhood of x. Put $U = F^+(V)$, then U is a $\tau_1\tau_2$-preneighborhood of x and $F(U) \subseteq V$.

(6)⇒(1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. Then V is a $\sigma_1\sigma_2$-neighborhood of $F(x)$ and by (6), there exists a $\tau_1\tau_2$-preneighborhood U of x such that $F(U) \subseteq V$. Therefore, there exists a $\tau_1\tau_2$-preopen set W such that $x \in W \subseteq U$ and so $F(W) \subseteq V$.

(2)⇒(7): For any subset B of Y, $\sigma_1\sigma_2$-Int(B) is $\sigma_1\sigma_2$-open in Y. By (2), $F^+(\sigma_1\sigma_2\text{-Int}(B))$ is $\tau_1\tau_2$-preopen in X. Therefore, $F^+(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-pInt}(F^+(B))$.

(7)⇒(2): Let V be any $\sigma_1\sigma_2$-open of Y. Then $F^+(V) = F^+(\sigma_1\sigma_2\text{-Int}(V)) \subseteq \tau_1\tau_2\text{-pInt}(F^+(V))$ and hence, $F^+(V)$ is $\tau_1\tau_2$-preopen in X. □

Theorem 3.3. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) F *is lower* (τ_1, τ_2)*-precontinuous;*
(2) $F^-(V)$ *is* $\tau_1\tau_2$*-preopen in* X *for any* $\sigma_1\sigma_2$*-open set* V *of* Y*;*
(3) $F^+(H)$ *is* $\tau_1\tau_2$*-preclosed in* X *for any* $\sigma_1\sigma_2$*-closed set* H *of* Y*;*
(4) *for each* $x \in X$ *and each* $\sigma_1\sigma_2$*-neighborhood* V *which intersects* $F(x)$*,* $F^-(V)$ *is a* $\tau_1\tau_2$*-preneighborhood of* x*;*
(5) *for each* $x \in X$ *and each* $\sigma_1\sigma_2$*-neighborhood* V *which intersects* $F(x)$*, there exists a* $\tau_1\tau_2$*-preneighborhood* U *of* x *such that* $F(z) \cap V \neq \emptyset$ *for any* $z \in U$*;*
(6) $\tau_1\tau_2$*-pCl*$(F^+(B)) \subseteq F^+(\tau_1\tau_2$*-Cl*$(B))$ *for any subset* B *of* Y*;*
(7) $F^-(\sigma_1\sigma_2$*-Int*$(B)) \subseteq \tau_1\tau_2$*-pInt*$(F^-(B))$ *for any subset* B *of* Y*.*

Proof. The proof is similar to that of Theorem 3.2. □

Definition 3.4. The $\tau_1\tau_2$*-prefrontier* of a subset A of a bitopological space (X,τ_1,τ_2), denoted by $\tau_1\tau_2$-pfr(A), is defined by $\tau_1\tau_2\text{-pfr}(A) = \tau_1\tau_2\text{-pCl}(A) \cap \tau_1\tau_2\text{-pCl}(X - A) = \tau_1\tau_2\text{-pCl}(A) - \tau_1\tau_2\text{-pInt}(A)$.

Theorem 3.5. *The set of all points* x *of* X *at which a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is not upper* (τ_1, τ_2)*-precontinuous is identical with the union of the* $\tau_1\tau_2$*-prefrontier of the upper inverse images of* $\sigma_1\sigma_2$*-open sets containing* $F(x)$*.*

Proof. Let $x \in X$ at which F is not upper (τ_1, τ_2)-precontinuous. There exists a $\sigma_1\sigma_2$-open set V of Y containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $\tau_1\tau_2$-preopen set U containing x. Then we have $x \in \tau_1\tau_2\text{-pCl}(X - F^+(V)) = X - \tau_1\tau_2\text{-pInt}(F^+(V))$ and $x \in F^+(V)$. Hence, we obtain $x \in \tau_1\tau_2\text{-pfr}(F^+(V))$.

Conversely, suppose that V is $\sigma_1\sigma_2$-open set of Y containing $F(x)$ such that $x \in \tau_1\tau_2\text{-pfr}(F^+(V))$. If F is (τ_1, τ_2)-upper precontinuous at x, there exists a $\tau_1\tau_2$-preopen set U containing x such that $U \subseteq F^+(V)$. This implies that $x \in \tau_1\tau_2\text{-pInt}(F^+(V))$. This is a contradiction and hence, F is not upper (τ_1, τ_2)-precontinuous. □

Theorem 3.6. *The set of all points x of X at which a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not lower (τ_1, τ_2)-precontinuous is identical with the union of the $\tau_1\tau_2$-prefrontier of the lower inverse images of $\sigma_1\sigma_2$-open sets meeting $F(x)$.*

Proof. The proof is similar to that of Theorem 3.5. □

Definition 3.7. Let A be a subset of a bitopological space (X, τ_1, τ_2). The set

$$\cap\{G \mid A \subseteq G \text{ and } G \text{ is } \tau_1\tau_2\text{-open}\}$$

is called the $\tau_1\tau_2$-*kernel* of A and is denoted by $\tau_1\tau_2\text{-ker}(A)$.

Lemma 3.8. *For subsets A, B of a bitopological space (X, τ_1, τ_2), the following properties hold.*

(1) $A \subseteq \tau_1\tau_2\text{-}ker(A)$.
(2) *If $A \subseteq B$, then $\tau_1\tau_2\text{-}ker(A) \subseteq \tau_1\tau_2\text{-}ker(B)$.*
(3) *If A is $\tau_1\tau_2$-open, then $\tau_1\tau_2\text{-}ker(A) = A$.*
(4) *$x \in \tau_1\tau_2\text{-}ker(A)$ if and only if $A \cap H \neq \emptyset$ for any $\tau_1\tau_2$-closed set H containing x.*

Theorem 3.9. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction. If $F^+(\sigma_1\sigma_2\text{-}ker(A)) \subseteq \tau_1\tau_2\text{-}pInt(F^+(A))$ for every subset A of Y, then F is upper (τ_1, τ_2)-precontinuous.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y. By Lemma 3.8, $F^+(V) = F^+(\sigma_1\sigma_2\text{-ker}(V)) \subseteq \tau_1\tau_2\text{-pInt}(F^+(V))$ and hence, $\tau_1\tau_2\text{-pInt}(F^+(V)) = F^+(V)$. This shows that $F^+(V)$ is $\tau_1\tau_2$-preopen. By Theorem 3.2, F is upper (τ_1, τ_2)-precontinuous. □

The converse of above theorem is not true in general, which follows from the following example.

Example 3.10. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{b\}, X\}$ and $\tau_2 = \{\emptyset, \{b, c\}, X\}$. Let $Y = \{-2, -1, 0, 1, 2\}$ with topologies $\sigma_1 = \{\emptyset, \{-1\}, \{1\}, \{-1, 1\}, Y\}$ and $\sigma_2 = \{\emptyset, \{-1, 1\}, Y\}$. Define $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ as follows: $F(a) = \{0\}$, $F(b) = \{-1, 1\}$ and $F(c) = \{-2, 2\}$. Then F is upper (τ_1, τ_2)-precontinuous but

$$F^+(\sigma_1\sigma_2\text{-ker}(\{-1, 0, 1\})) \nsubseteq \tau_1\tau_2\text{-pInt}(F^+(\{-1, 0, 1\})).$$

Theorem 3.11. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction. If $F^-(\sigma_1\sigma_2\text{-}ker(A)) \subseteq \tau_1\tau_2\text{-}pInt(F^-(A))$ for every subset A of Y, then F is lower (τ_1, τ_2)-precontinuous.*

Proof. The proof is similar to that of Theorem 3.9. □

Definition 3.12. A collection $\mathfrak{U}$ of subsets of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*locally finite* if every $x \in X$ has a $\tau_1\tau_2$-neighborhood which intersects only finitely many elements of $\mathfrak{U}$.

Example 3.13. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then $\mathfrak{U} = \{\{a\}, \{b\}, \{a, b\}, X\}$ is $\tau_1\tau_2$-locally finite.

Definition 3.14. A subset A of a bitopological space (X, τ_1, τ_2) is said to be:

(1) $\tau_1\tau_2$-*paracompact* if every cover of A by $\tau_1\tau_2$-open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$-open sets of X and is $\tau_1\tau_2$-locally finite in X;

(2) $\tau_1\tau_2$-*regular* if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2\text{-Cl}(V) \subseteq U$.

Example 3.15. In Example 3.13, $\{a, b\}$ is $\tau_1\tau_2$-paracompact.

Example 3.16. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\emptyset, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{b, c\}, X\}$. Then $\{a, c\}$ is $\tau_1\tau_2$-regular.

Lemma 3.17. *If A is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $\tau_1\tau_2$-open neighborhood of A, then there exists a $\tau_1\tau_2$-open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$.*

Proof. The proof is similar to that [9, Theorem 2.5]. □

For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, by $\text{ClF}_{\circledast} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ we denote a multifunction defined as follows: $\text{ClF}_{\circledast}(x) = \sigma_1\sigma_2\text{-Cl}(F(x))$ for each $x \in X$. Similarly, we can define $\text{pClF}_{\circledast}$.

Lemma 3.18. *If $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a multifunction such that $F(x)$ is $\tau_1\tau_2$-regular and $\tau_1\tau_2$-paracompact for each $x \in X$, then $G^+(V) = F^+(V)$ for each $\sigma_1\sigma_2$-open set V of Y, where G denotes $\text{pClF}_{\circledast}$ or $\text{ClF}_{\circledast}$.*

Proof. Let V be any $\sigma_1\sigma_2$-open set V of Y and $x \in G^+(V)$. Then $G(x) \subseteq V$ and $F(x) \subseteq G(x) \subseteq V$. Therefore, we have $x \in F^+(V)$ and hence, $G^+(V) \subseteq F^+(V)$. On the other hand, let $x \in F^+(V)$. Then $F(x) \subseteq V$ and by Lemma 3.17 there exists a $\sigma_1\sigma_2$-open set U of Y such that $F(x) \subseteq \sigma_1\sigma_2\text{-Cl}(U) \subseteq U \subseteq V$; hence $G(x) \subseteq \sigma_1\sigma_2\text{-Cl}(F(x)) \subseteq V$. Therefore, we have $x \in G^+(V)$ and so $F^+(V) \subseteq G^+(V)$. Consequently, we obtain $G^+(V) = F^+(V)$. □

Theorem 3.19. *Let $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $x \in X$. Then the following properties are equivalent:*

(1) *F is upper (τ_1, τ_2)-percontinuous;*
(2) *$\text{pClF}_{\circledast}$ is upper (τ_1, τ_2)-percontinuous;*
(3) *$\text{ClF}_{\circledast}$ is upper (τ_1, τ_2)-precontinuous.*

Proof. We put $G = \text{ClF}_{\circledast}$ or $\text{pClF}_{\circledast}$ in the sequel. Suppose that F is upper (τ_1, τ_2)-precontinuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $G(x)$. By Lemma 3.18, we have $x \in G^+(V) = F^+(V)$ and hence, there exists a $\tau_1\tau_2$-preopen set U containing x such that $F(U) \subseteq V$. Since $F(z)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $z \in U$, by Lemma 3.17, there exists a $\tau_1\tau_2$-open set W such that $F(z) \subseteq W \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$; hence $G(z) \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$ for each $z \in U$. Therefore, we obtain $G(U) \subseteq V$. This shows that G is upper (τ_1, τ_2)-precontinuous.

Conversely, suppose that G is upper (τ_1, τ_2)-precontinuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $G(x)$. By Lemma 3.18, we have $x \in F^+(V) = G^+(V)$ and hence $G(x) \subseteq V$. There exists a $\tau_1\tau_2$-preopen set U containing x such that $F(U) \subseteq V$. Therefore, we obtain $U \subseteq G^+(V) = F^+(V)$ and so $F(U) \subseteq V$. This shows that F is upper (τ_1, τ_2)-precontinuous. □

Lemma 3.20. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, $G^-(V) = F^-(V)$ for each $\sigma_1\sigma_2$-open set V of Y, where G denotes $\text{pClF}_{\circledast}$ or $\text{ClF}_{\circledast}$.*

Proof. Let V be any $\sigma_1\sigma_2$-open set V of Y and $x \in G^-(V)$. Then $G(x) \cap V \neq \emptyset$ and hence, $F(x) \cap V \neq \emptyset$ since V is $\sigma_1\sigma_2$-open. Thus, we have $x \in F^-(V)$ and so $G^-(V) \subseteq F^-(V)$. On the other hand, let $x \in F^-(V)$. Then, we have $\emptyset \neq F(x) \cap V \subseteq G(x) \cap V$ and hence, $x \in G^-(V)$. Therefore, $F^-(V) \subseteq G^-(V)$. Consequently, we obtain $G^-(V) = F^-(V)$. □

Theorem 3.21. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is lower (τ_1, τ_2)-percontinuous;*
(2) *$\text{pClF}_{\circledast}$ is lower (τ_1, τ_2)-percontinuous;*

(3) *Cl*$F_{\circledast}$ *is lower* (τ_1, τ_2)*-precontinuous.*

Proof. The proof is similar to that of Theorem 3.19. □

Definition 3.22. A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-compact* (Resp. $\tau_1\tau_2$*-precompact*) if every cover of X by $\tau_1\tau_2$-open (Resp. $\tau_1\tau_2$-preopen) sets of X has a finite subcover.

Remark 3.23. Every $\tau_1\tau_2$-precompact is $\tau_1\tau_2$-compact, but the converse need not be true, as this may be seen from the following example.

Example 3.24. Let $X = \mathbb{Z}$ with topologies $\tau_1 = \{\emptyset, \{1\}, X\}$ and $\tau_2 = \{\emptyset, X\}$. Then (X, τ_1, τ_2) is $\tau_1\tau_2$-compact but it is not $\tau_1\tau_2$-precompact since $\{\{n\} \mid n \in \mathbb{Z}\}$ is $\tau_1\tau_2$-preopen cover of X which has no finite subcover.

Theorem 3.25. *Let* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be an upper* (τ_1, τ_2)*-precontinuous surjective multifunction such that* $F(x)$ *is* $\sigma_1\sigma_2$*-compact for each* $x \in X$*. If* X *is* $\tau_1\tau_2$*-precompact, then* Y *is* $\sigma_1\sigma_2$*-compact.*

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a $\sigma_1\sigma_2$-open cover of Y. For each $x \in X$, $F(x)$ is $\sigma_1\sigma_2$-compact and there exists a finite subset $\nabla(x)$ of ∇ such that $F(x) \subseteq \cup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Set $V(x) = \cup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Since F is upper (τ_1, τ_2)-precontinuous, there exists a $\tau_1\tau_2$-preopen set $U(x)$ containing x such that $F(U(x)) \subseteq V(x)$. The family $\{U(x) \mid x \in X\}$ is a $\tau_1\tau_2$-preopen cover of X and there exists a finite number of points, say, $x_1, x_2, \ldots, x_n$ in X such that $X = \cup\{U(x_i) \mid 1 \leqslant i \leqslant n\}$. Therefore, we have

$$Y = F(X) = F(\bigcup_{i=1}^{n} U(x_i)) = \bigcup_{i=1}^{n} F(U(x_i)) \subseteq \bigcup_{i=1}^{n} V(x_i) = \bigcup_{i=1}^{n} \cup_{\alpha\in\nabla(x_i)} V_\alpha.$$

This shows that Y is $\sigma_1\sigma_2$-compact. □

Let $\{(X_\gamma, \tau_1(\gamma), \tau_2(\gamma)) \mid \gamma \in \Gamma\}$ be a family of bitopological spaces. Let $(X^*, \tau_1^*, \tau_2^*)$ be the product space, where $X^* = \prod_{\gamma\in\Gamma} X_\gamma$ and τ_i^* denotes the product topology of $\{\tau_i(\gamma) \mid \gamma \in \Gamma\}$ for $i = 1, 2$.

Lemma 3.26 ([8])**.** *Let* A_γ *be a non-empty subset of* X_γ *for* $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_n$*. Then* $A = \prod_{k=1}^{n} A_{\gamma_k} \times \prod_{\gamma\neq\gamma_k} X_\gamma$ *is* $\tau_1^*\tau_2^*$*-preopen in* X^* *if and only if* A_{γ_k} *is* $\tau_1(\gamma_k)\tau_2(\gamma_k)$*-preopen in* X_{γ_k} *for each* $k = 1, 2, \ldots, n$*.*

Let $\{(X_\gamma, \tau_1(\gamma), \tau_2(\gamma)) \mid \gamma \in \Gamma\}$ and $\{(Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma)) \mid \gamma \in \Gamma\}$ be two arbitrary families of bitopological spaces with the same set of indices. Let $F_\gamma : (X_\gamma, \tau_1(\gamma), \tau_2(\gamma)) \to (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$ be a multifunction for each $\gamma \in \Gamma$. Let $F^* : (X^*, \tau_1^*, \tau_2^*) \to (Y^*, \sigma_1^*, \sigma_2^*)$ be the product multifunction defined by $F^*(\{x_\gamma\}) = \prod_{\gamma\in\Gamma} F_\gamma(x_\gamma)$ for each $\{x_\gamma\}$ in $X^* = \prod_{\gamma\in\Gamma} X_\gamma$, where τ_i^* and σ_i^* denote the product topologies for $i = 1, 2$.

Theorem 3.27. *If* $F^* : (X^*, \tau_1^*, \tau_2^*) \to (Y^*, \sigma_1^*, \sigma_2^*)$ *is upper* (τ_1^*, τ_2^*)*-precontinuous, then*

$$F_\gamma : (X_\gamma, \tau_1(\gamma), \tau_2(\gamma)) \to (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$$

is upper $(\tau_1(\gamma), \tau_2(\gamma))$*-precontinuous for each* $\gamma \in \Gamma$*.*

Proof. Let V_γ be a $\sigma_1(\gamma)\sigma_2(\gamma)$-open set in Y_γ. Since F^* is upper (τ_1^*, τ_2^*)-precontinuous and Lemma 3.26, we have $F^{*+}(V_\gamma \times \prod_{\gamma\neq\alpha} Y_\alpha) = F_\gamma^+(V_\gamma) \times \prod_{\gamma\neq\alpha} X_\alpha$ is a $\tau_1^*\tau_2^*$-preopen set in X^* and hence, $F_\gamma^+(V_\gamma)$ is a $\tau_1(\gamma)\tau_2(\gamma)$-preopen set in X_γ. This shows that F_γ is upper $(\tau_1(\gamma), \tau_2(\gamma))$-precontinuous. □

Theorem 3.28. *If* $F^* : (X^*, \tau_1^*, \tau_2^*) \to (Y^*, \sigma_1^*, \sigma_2^*)$ *is lower* (τ_1^*, τ_2^*)*-precontinuous, then*

$$F_\gamma : (X_\gamma, \tau_1(\gamma), \tau_2(\gamma)) \to (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$$

is lower $(\tau_1(\gamma), \tau_2(\gamma))$*-precontinuous for each* $\gamma \in \Gamma$*.*

Proof. The proof is similar to that of Theorem 3.27. □

Theorem 3.29. *Let* (X, τ_1, τ_2) *and* $(Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$ *be bitopological spaces for each* $\gamma \in \Gamma$. *Let*

$$F_\gamma : (X, \tau_1, \tau_2) \to (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$$

be a multifunction for each $\gamma \in \Gamma$ *and* $F : (X, \tau_1, \tau_2) \to (Y^*, \sigma_1^*, \sigma_2^*)$ *a multifunction defined by* $F(x) = \prod_{\gamma\in\Gamma} F_\gamma(x)$ *for each* $x \in X$. *If* F *is upper* (τ_1, τ_2)*-precontinuous, then* F_γ *is upper* (τ_1, τ_2)*-precontinuous for each* $\gamma \in \Gamma$.

Proof. Let $x \in X$, $\gamma \in \Gamma$ and V_γ be any $\sigma_1(\gamma)\sigma_2(\gamma)$-open set in Y_γ containing $F_\gamma(x)$. Then, we have $\pi_\gamma^{-1}(V_\gamma) = V_\gamma \times \prod_{\gamma\neq\alpha} Y_\alpha$ is a $\sigma_1^*\sigma_2^*$-open set of Y^* containing $F(x)$, where

$$\pi_\gamma : (Y^*, \sigma_1^*, \sigma_2^*) \to (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$$

is the projection for each $\gamma \in \Gamma$. Since F is upper (τ_1, τ_2)-precontinuous, there exists a $\tau_1\tau_2$-preopen set U of X containing x such that $F(U) \subseteq \pi_\gamma^{-1}(V_\gamma)$. Therefore, we obtain $F_\gamma(U) \subseteq \pi_\gamma(F(U)) \subseteq \pi_\gamma(\pi_\gamma^{-1}(V_\gamma)) = V_\gamma$. This shows that F_γ is upper (τ_1, τ_2)-precontinuous for each $\gamma \in \Gamma$. □

Theorem 3.30. *Let* (X, τ_1, τ_2) *and* $(Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$ *be bitopological spaces for each* $\gamma \in \Gamma$. *Let*

$$F_\gamma : (X, \tau_1, \tau_2) \to (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$$

be a multifunction for each $\gamma \in \Gamma$ *and* $F : (X, \tau_1, \tau_2) \to (Y^*, \sigma_1^*, \sigma_2^*)$ *a multifunction defined by* $F(x) = \prod_{\gamma\in\Gamma} F_\gamma(x)$ *for each* $x \in X$. *If* F *is lower* (τ_1, τ_2)*-precontinuous, then* F_γ *is lower* (τ_1, τ_2)*-precontinuous for each* $\gamma \in \Gamma$.

Proof. The proof is similar to that of Theorem 3.29. □

Definition 3.31. A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-connected* (Resp. $\tau_1\tau_2$*-preconnected*) if X cannot be written as the union of two non-empty disjoint $\tau_1\tau_2$-open (Resp. $\tau_1\tau_2$-preopen) sets.

Remark 3.32. Every $\tau_1\tau_2$-preconnected is $\tau_1\tau_2$-connected, but the converse need not be true, as this may be seen from the following example.

Example 3.33. Let $X = \{a, b\}$ with topologies $\tau_1 = \{\emptyset, \{b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, X\}$. Then (X, τ_1, τ_2) is $\tau_1\tau_2$-connected but it is not $\tau_1\tau_2$-preconnected.

Definition 3.34. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$*-clopen* if A is both $\tau_1\tau_2$-open and $\tau_1\tau_2$-closed.

Theorem 3.35. *If* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is an upper or lower* (τ_1, τ_2)*-precontinuous surjective multifunction such that* $F(x)$ *is* $\sigma_1\sigma_2$*-connected for each* $x \in X$ *and* (X, τ_1, τ_2) *is* $\tau_1\tau_2$*-preconnected, then* (Y, σ_1, σ_2) *is* $\sigma_1\sigma_2$*-connected.*

Proof. Suppose that (Y, σ_1, σ_2) is not $\sigma_1\sigma_2$-connected. There exist non-empty $\sigma_1\sigma_2$-open sets U and V of Y such that $U \cap V = \emptyset$ and $U \cup V = Y$. Since $F(x)$ is $\sigma_1\sigma_2$-connected for each $x \in X$, either $F(x) \subseteq U$ or $F(x) \subseteq V$. If $x \in F^+(U \cup V)$, then $F(x) \subseteq U \cup V$ and so $x \in F^+(U) \cup F^+(V)$. Moreover, since F is surjective, there exist x and y in X such that $F(x) \subseteq U$ and $F(y) \subseteq V$; hence $x \in F^+(U)$ and $y \in F^+(V)$. Therefore, we obtain the following:

(1) $F^+(U) \cup F^+(V) = F^+(U \cup V) = X$;
(2) $F^+(U) \cap F^+(V) = F^+(U \cap V) = \emptyset$;
(3) $F^+(U) \neq \emptyset$ and $F^+(V) \neq \emptyset$.

Next, we show that $F^+(U)$ and $F^+(V)$ are $\tau_1\tau_2$-preopen in X. (i) Let F be upper (τ_1, τ_2)-precontinuous. By Theorem 3.2, we obtain $F^+(U)$ and $F^+(V)$ are $\tau_1\tau_2$-preopen in X. (ii) Let F be lower (τ_1, τ_2)-precontinuous. By Theorem 3.3, we have $F^+(U)$ is $\tau_1\tau_2$-preclosed in X since U is $\sigma_1\sigma_2$-clopen in Y. Therefore, $F^+(V)$ is $\tau_1\tau_2$-preopen in X. Similarly, we obtain $F^+(U)$ is $\tau_1\tau_2$-preopen in X. Consequently, (X, τ_1, τ_2) is not $\tau_1\tau_2$-preconnected. This completes the proof. □

Definition 3.36. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:

(1) *upper almost* (τ_1, τ_2)*-precontinuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \subseteq V$, there exists a $\tau_1\tau_2$-preopen set U containing x such that $F(U) \subseteq \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V))$;

(2) *lower almost* (τ_1, τ_2)*-precontinuous* at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-preopen set U containing x such that $F(z) \cap \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V)) \neq \emptyset$ for each $z \in U$;

(3) *upper almost* (Resp. *lower almost*) (τ_1, τ_2)*-precontinuous* if F has this property at each point of X.

Remark 3.37. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:

$$\text{upper } (\tau_1, \tau_2)\text{-precontinuity} \Rightarrow \text{upper almost } (\tau_1, \tau_2)\text{-precontinuity.}$$

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 3.38. Let $X = \{1, 2, 3\}$ with topologies $\tau_1 = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, X\}$ and $\tau_2 = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, X\}$. Let $Y = \{a, b, c, d, e\}$ with topologies $\sigma_1 = \{\emptyset, \{a, b, c, d\}, Y\}$ and $\sigma_2 = \{\emptyset, \{a, b, c, d\}, Y\}$. A multifunction

$$F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is defined as follows: $F(1) = \{c\}$, $F(2) = \{b, d\}$, and $F(3) = \{a, e\}$. Then F is upper almost (τ_1, τ_2)-precontinuous but F is not upper (τ_1, τ_2)-precontinuous.

Remark 3.39. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following implication holds:

$$\text{lower } (\tau_1, \tau_2)\text{-precontinuity} \Rightarrow \text{lower almost } (\tau_1, \tau_2)\text{-precontinuity.}$$

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 3.40. Let $X = \{1, 2\}$ with topologies $\tau_1 = \{\emptyset, X\}$ and $\tau_2 = \{\emptyset, \{2\}, X\}$. Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\emptyset, \{a\}, \{b, c\}, Y\}$ and $\sigma_2 = \{\emptyset, \{a\}, Y\}$. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is defined as follows: $F(1) = \{a\}$ and $F(2) = \{b, c\}$. Then F is lower almost (τ_1, τ_2)-precontinuous but F is not lower (τ_1, τ_2)-precontinuous.

Theorem 3.41. *For a multifunction* $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) *F is upper almost* (τ_1, τ_2)*-precontinuous at* $x \in X$*;*

(2) $x \in \tau_1\tau_2\text{-}pInt(F^+(\sigma_1\text{-}Int(\sigma_2\text{-}Cl(V))))$ *for every* $\sigma_1\sigma_2$*-open set V of Y containing* $F(x)$*;*

(3) $x \in \tau_1\tau_2\text{-}pInt(F^+(\sigma_1\sigma_1\text{-}sCl(V)))$ *for every* $\sigma_1\sigma_2$*-open set V of Y containing* $F(x)$*;*

(4) $x \in \tau_1\tau_2\text{-}pInt(F^+(V))$ *for every* $\sigma_1\sigma_2$*-regular open set V of Y containing* $F(x)$*;*

(5) *for each* $\sigma_1\sigma_2$*-regular open set V of Y containing* $F(x)$*, there exists a* $\tau_1\tau_2$*-preopen set U containing x such that* $F(U) \subseteq V$*.*

Proof.

(1)⇒(2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. There exists a $\tau_1\tau_2$-preopen set U containing x such that $F(U) \subseteq \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V))$. Thus, we have $x \in U \subseteq F^+(\sigma_1\text{-Int}(\sigma_2\text{-Cl}(V)))$ and hence, $x \in \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\text{-Int}(\sigma_2\text{-Cl}(V))))$.

(2)⇒(3): This follows from Lemma 2.6.

(3)⇒(4): Let V be a $\sigma_1\sigma_2$-regular open set of Y containing $F(x)$. Then it follows from Lemma 2.6 that $V = \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V)) = \sigma_1\sigma_2\text{-sCl}(V)$.

(4)⇒(5): Let V be a $\sigma_1\sigma_2$-regular open set of Y containing $F(x)$. By (4), $x \in \tau_1\tau_2\text{-pInt}(F^+(V))$, and so there exists a $\tau_1\tau_2$-preopen set U containing x such that $x \in U \subseteq F^+(V)$; hence $F(U) \subseteq V$.

(5)⇒(1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $F(x)$. Since $\sigma_1\text{-Int}(\sigma_2\text{-Cl}(V))$ is $\sigma_1\sigma_2$-regular open, there exists a $\tau_1\tau_2$-preopen set U containing x such that $F(U) \subseteq \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V))$. This shows that F is upper almost (τ_1, τ_2)-precontinuous at $x \in X$. □

Theorem 3.42. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is lower almost (τ_1, τ_2)-precontinuous at $x \in X$;*
(2) *$x \in \tau_1\tau_2\text{-pInt}(F^-(\sigma_1\text{- Int}(\sigma_2\text{-Cl}(V))))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$;*
(3) *$x \in \tau_1\tau_2\text{-pInt}(F^-(\sigma_1\sigma_1\text{-sCl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \emptyset$;*
(4) *$x \in \tau_1\tau_2\text{-pInt}(F^-(V))$ for every $\sigma_1\sigma_2$-regular open set V of Y such that $F(x) \cap V \neq \emptyset$;*
(5) *for each $\sigma_1\sigma_2$-regular open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\tau_1\tau_2$-preopen set U containing x such that $U \subseteq F^-(V)$.*

Proof. The proof is similar to that of Theorem 3.41. □

The following theorems give some characterizations of upper and lower almost (τ_1, τ_2)-precontinuous multifunctions.

Theorem 3.43. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is upper almost (τ_1, τ_2)- precontinuous;*
(2) *$F^+(V) \subseteq \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\text{-Int}(\sigma_2\text{-Cl}(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;*
(3) *$\tau_1\tau_2\text{-pCl}(F^-(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(K)))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;*
(4) *$F^+(V)$ is $\tau_1\tau_2$-preopen in X for every $\sigma_1\sigma_2$-regular open set V of Y;*
(5) *$F^-(K)$ is $\tau_1\tau_2$-preclosed in X for every $\sigma_1\sigma_2$-regular closed set V of Y.*

Proof.

(1)⇒(2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then $F(x) \subseteq V$. By Theorem 3.41, we have $x \in \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\text{-Int}(\sigma_2\text{-Cl}(V))))$. This shows that $F^+(V) \subseteq \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\text{-Int}(\sigma_2\text{-Cl}(V))))$.

(2)⇒(3): Let K be any $\sigma_1\sigma_2$-closed set of Y. Then $Y - K$ is $\sigma_1\sigma_2$-open in Y and by (2) we have

$$\begin{aligned}X - F^-(K) = F^+(Y-K) &\subseteq \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\text{-Int}(\sigma_2\text{-Cl}(Y-K))))\\ &= \tau_1\tau_2\text{-pInt}(X - F^-(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(K))))\\ &= X - \tau_1\tau_1\text{-pCl}(F^-(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(K)))).\end{aligned}$$

Hence, we obtain $\tau_1\tau_2\text{-pCl}(F^-(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(K)))) \subseteq F^-(K)$.

(3)⇒(4): Let V be any $\sigma_1\sigma_2$-regular open set of Y. Then we have

$$\begin{aligned}F^+(V) = X - F^-(Y-V) &\subseteq X - \tau_1\tau_2\text{-pCl}(F^-(\sigma_1\text{-Cl}(\sigma_2\text{-Int}(Y-V))))\\ &= X - \tau_1\tau_2\text{-pCl}(F^-(Y - \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V))))\\ &= \tau_1\tau_2\text{-pInt}(F^+(\sigma_1\text{-Int}(\sigma_2\text{-Cl}(V)))).\end{aligned}$$

Therefore, we obtain $F^+(V) \subseteq \tau_1\tau_2\text{-pInt}(F^+(V))$ and hence $F^+(V)$ is $\tau_1\tau_2$-preopen in X.

(4)⇒(5): It follows from the fact that $F^+(Y-K) = X - F^-(K)$ for any subset K of Y.

(5)⇒(1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-regular open set of Y containing $F(x)$. Since $Y - V$ is $\sigma_1\sigma_2$-regular closed, by (5) we have $F^-(Y - V) = X - F^+(V)$ is $\tau_1\tau_2$-preclosed in X and hence $F^+(V)$ is $\tau_1\tau_2$-preopen. Put $U = F^+(V)$. Then U is a $\tau_1\tau_1$-preopen set of X containing x such that $F(U) \subseteq V$. It follows from Theorem 3.41 that F is upper almost (τ_1, τ_2)-precontinuous. □

Theorem 3.44. *For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *F is lower almost (τ_1, τ_2)- precontinuous;*
(2) *$F^-(V) \subseteq \tau_1\tau_2$-pInt$(F^-(\sigma_1$-Int$(\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;*
(3) *$\tau_1\tau_2$-pCl$(F^+(\sigma_1$-Cl$(\sigma_2$-Int$(K)))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;*
(4) *$F^-(V)$ is $\tau_1\tau_2$-preopen in X for every $\sigma_1\sigma_2$-regular open set V of Y;*
(5) *$F^+(K)$ is $\tau_1\tau_2$-preclosed in X for every $\sigma_1\sigma_2$-regular closed set V of Y.*

Proof. The proof is similar to that of Theorem 3.43. □

Theorem 3.45. *If $F^* : (X^*, \tau_1^*, \tau_2^*) \to (Y^*, \sigma_1^*, \sigma_2^*)$ is upper almost (τ_1^*, τ_2^*)-precontinuous, then*

$$F_\gamma : (X_\gamma, \tau_1(\gamma), \tau_2(\gamma)) \to (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$$

is upper almost $(\tau_1(\gamma), \tau_2(\gamma))$-precontinuous for each $\gamma \in \Gamma$.

Proof. Let V_γ be a $\sigma_1(\gamma)\sigma_2(\gamma)$-regular open set in Y_γ. Since F^* is upper almost (τ_1^*, τ_2^*)-precontinuous and Lemma 3.26, we have $F^{*+}(V_\gamma \times \prod_{\gamma \neq \alpha} Y_\alpha) = F_\gamma^+(V_\gamma) \times \prod_{\gamma \neq \alpha} X_\alpha$ is a $\tau_1^*\tau_2^*$-preopen set in X^* and hence, $F_\gamma^+(V_\gamma)$ is a $\tau_1(\gamma)\tau_2(\gamma)$-preopen set in X_γ. This shows that F_γ is upper almost $(\tau_1(\gamma), \tau_2(\gamma))$-precontinuous. □

Theorem 3.46. *If $F^* : (X^*, \tau_1^*, \tau_2^*) \to (Y^*, \sigma_1^*, \sigma_2^*)$ is lower almost (τ_1^*, τ_2^*)-precontinuous, then*

$$F_\gamma : (X_\gamma, \tau_1(\gamma), \tau_2(\gamma)) \to (Y_\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$$

is lower almost $(\tau_1(\gamma), \tau_2(\gamma))$-precontinuous for each $\gamma \in \Gamma$.

Proof. The proof is similar to that of Theorem 3.45. □

Acknowledgment

This research was financially supported by Mahasarakham University.

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