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AN ITERATIVE METHOD FOR SEMIGROUPS OF NONEXPANSIVE MAPPINGS

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Abstract

We introduce an iterative method for finding a common fixed point of a semigroup of infinite family of nonexpansive mappings in Hilbert space, with respect to a sequence of left regular means defined on an appropriate space of bounded real valued functions of the semigroup. we prove the strong convergence of the proposed iterative algorithm to the unique solution of a variational inequality, which is the optimality condition for a minimization problem.

Keywords: Hilbert space, Amenable semigroups, Common fixed point, Nonexpansive mappings.

1. Introduction

Let H be a real Hilbert space. Assume A is strongly positive, that is there is a constant $\bar{\gamma}$ with the property:

$$\langle Ax, x\rangle \geq \bar{\gamma}\| x\|^2.$$

Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings of H into itself, we shall assume that $F := \bigcap_{i=1}^{\infty} FixT_i \neq \emptyset$, and let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of nonnegative real numbers in $[0,1]$. for $n \geq 1$, define a mapping W_n of H into itself as follows:

$$
\begin{aligned}
&U_{n,n+1} = I \\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I \\
&\quad . \\
&\quad . \\
&\quad . \\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I \\
&W_n := U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I \qquad (1.1)
\end{aligned}
$$

Y.Yao in [12] introduced an iterative algorithm to appropriate the common fixed points of an infinite family of nonexpansive self mapping in a real Hilbert space as follows:
Let $x_0 \in H$ is arbitrarily chosen and

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n x_n$$

Where W_n is a sequence defined by (1.1), f be a contraction on H with coeffecient $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$

Under the assumption that sequences $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

$(C_1): \lim_{n\to\infty} \alpha_n = 0,$

$(C_2): \sum_{n=1}^{\infty} \alpha_n = \infty,$

$(C_3): 0 < \liminf \beta_n < \limsup \beta_n < 1,$

Yao proved that the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \qquad x \in F. \qquad (1.2)$$

In this paper motivated and inspired by Yao, we introduce a composite iteration schem as follows:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)T_{\mu_n} W_n x_n \qquad (1.3)$$

Let S be a semigroup, $\varphi = \{T_t : t \in S\}$ is a nonexpansive semigroup on H such that $Fix\,\varphi = \bigcap_{t \in S} Fix T_t \neq \emptyset$, X is a left invariant subspace of the space of all bounded real valued functions defined on S such that $1 \in X$, the mapping $t \to \langle T_t(x), y\rangle$ is an element of X for each $x, y \in H$ and $\{\mu_n\}$ is a sequence of means on X. Our purpose here is to introduce the general iterative algorithm for approximating the common fixed points of left amenable semigroup of nonexpansive mapping and infinite family of nonexpansive mappings which also solve some variational inequalities.

2. PRELIMINARIES

Let S be a semigroup and let B(S) be the space of all bounded real valued functions defined on S with the supremum norm. For $s \in S$ and $f \in B(S)$ we write l_s and r_s on B(S) by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for each $t \in S$ and $f \in B(S)$. Let X be a subspace of B(S) containing 1 and X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), i.e $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (res $\mu(r_s f) = \mu(f)$ for each $s \in S$ and $f \in X$. X is said to be left (resp. right) amenable if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable.
As is well known, B(S) is amenable when S is a commutative semigroup or a solvable group.
A net $\{\mu_\alpha\}$ of means on X is said to be strongly left regular if $\lim_\alpha \| l_s^* \mu_\alpha - \mu_\alpha \| = 0$ for each $s \in S$ where l_s^* is the adjoint operator of l_s.
Let C be a nonempty closed and convex subset of a reflexive Banach space E. A family $\varphi = \{T_t : t \in S\}$ is called a nonexpansive semigroup on C if for each $t \in S$ the mapping $T_t : C \to C$ is nonexpansive and $T_{st} = T_s o T_t$ for each $s, t \in S$. We denote by $Fix(\varphi)$ the set of common fixed points of φ.

Lemma 2.1. [1, 4] Let f be a function of semigroup S into a reflexive Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact and let X be a subspace of B(S) containing all functions $t \to \langle f(t), x^* \rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element f_μ in E such that $\langle f_\mu, x^* \rangle = \mu_t \langle f(t), x^* \rangle$, for all $x^* \in E^*$. Moreover, if μ is a mean on X, then

$$\int f(t)\,d\mu(t) \in \overline{co}\{f(t) : t \in S\}.$$

We can write f_μ by $\int f(t)\,d\mu(t)$.

Lemma 2.2. [1, 4] Let C be a nonempty closed convex subset of a Hilbert space

H, $\varphi = \{T_t : t \in S\}$ be semigroup from C into C such that $F(\varphi) \neq \emptyset$ and the mapping $t \to \langle T_t(x), y \rangle$ be an element of X for each $x \in C$ and $y \in H$, and μ be a mean on X. If we write $T_\mu(x)$, instead of $\int T_t(x) d\mu(t)$, then the followings hold:

(i) T_μ is nonexpansive mapping from C into C.

(ii) $T_\mu(x) = x$, for each $x \in Fix(\varphi)$.

(iii) $T_\mu(x) \in \overline{co}\{T_t(x) : t \in S\}$, for each $x \in C$.

(iv) If μ is left invariant, then T_μ is a nonexpansive retraction from C onto $Fix(\varphi)$.

Lemma 2.3. [3] Let C be a nonempty closed convex subset of H and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.

Recall the metric (nearest point) projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: give $x \in H$, $P_C(x)$ is the only point in C with the property:

$$\| x - P_C(x) \| = \inf\{ \| x - y \| \quad : y \in C\}$$

It is well-known that P_C is a nonexpansive mapping of H onto C.

Lemma 2.4. [3] Let C be a nonempty convex subset of a Hilbert space H and P_C be the metric projection mapping from H onto C. Let $x \in H$ and $y \in C$ then, the followings are equivalent:

i) $y = P_C(x)$

ii) $\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C$.

Lemma 2.5. [9] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in [0, 1] with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Suppose $x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n$, for all integers $n \geq 0$, and $\limsup_{n\to\infty}(\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|) \leq 0$.

Then, $\lim_{n\to\infty} \| y_n - x_n \| = 0$.

The following lemma is an immediate consequence of the inner product on H.

Lemma 2.6. For all $x, y \in H$, there holds the inequality

$$\| x + y \|^2 \leq \| x \|^2 + 2\langle y, x + y \rangle.$$

Lemma 2.7. [5] Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be an infinite family of nonexpansive mappings with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$, $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_i \leq b < 1, \forall i \geq 1$. Considering W_n which is defined by (1.1) we have:

(1) W_n is nonexpansive and $Fix(W_n) = \bigcap_{i=1}^{n} Fix(T_i)$ for each $n \geq 1$

(2) for each $x \in C$ and for each positive integer j, $\lim_{n\to\infty} U_{n,j}x$ exists.
(3) The mapping $W : C \to C$, defined by

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \forall x \in C.$$

is a nonexpansive mapping satisfying $Fix(W) = \bigcap_{i=1}^{\infty} Fix(T_i)$ and is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$.

Lemma 2.8. [8] Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be a countable family of nonexpansive mappings with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$, $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_i \le b < 1, \forall i \ge 1$. If D is any bounded subset of C, then
$\lim_{n\to\infty} \sup_{x\in D} \| Wx - W_n x\| = 0.$

Lemma 2.9. [3] Let $\{a_n\}$ be a sequence of nonnegative real numbers such that

$$a_{n+1} \le (1-b_n)a_n + b_n c_n, \quad \forall n \ge 0$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying the following conditions

(i) $\{b_n\} \subset [0,1], \sum_{n=0}^{\infty} b_n = \infty$

(ii) either $\limsup_{n\to\infty} c_n \le 0$ or $\sum_{n=0}^{\infty} |b_n c_n| < \infty$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. [2] Assume that A is a strongly positive bounded operator on a Hilbert space H with the coefficient $\bar{\gamma} > 0$ and $0 < \rho < \| A\|^{-1}$.
Then, $\| I - \rho A\| \le 1 - \rho\bar{\gamma} > 0$.

3. Main results

Theorem 3.1. Let f be a contraction on H with coefficient $0 < \alpha < 1$. Let A be a strongly positive operator on H with coefficient $\bar{\gamma} > 0$. Let $\{\mu_n\}$ be a left regular sequence of means on X such that $\lim_{n\to\infty} \| \mu_{n+1} - \mu_n\| = 0$. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive selfmapping of H such that $F := \bigcap_{i=1}^{\infty} FixT_i \bigcap Fix(\varphi) \neq \emptyset$ and $T_i(Fix\,\varphi) \subset Fix(\varphi)$, for all $i \in N$. Let $x_0 \in H$, $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and let $\{x_n\}$ be generated by the iterative algorithm (1.3), where W_n is a sequence defined by (1.1), and $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0, 1] satisfying conditions (C1), (C2), (C3).Then $\{x_n\}$ converges strongly $x^* \in F$, which also uniquely solves the variational inequality (1.2).

Proof. We shall divide the proof into several steps.

Step1. The sequence $\{x_n\}$ is bounded.

Since A is strongly positive operator on H, then

$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}$.

$\|(1-\beta_n)I - \alpha_n A\| = \sup\{1-\beta_n - \alpha_n \langle Ax, x\rangle : x \in H, \|x\| = 1\} \le 1-\beta_n - \alpha_n \bar{\gamma}$.

Let $p \in F$, by lemma 2.2 in [1], [10], we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(T_{\mu_n}W_n x_n - p)\| \\ &\le \alpha_n\|\gamma f(x_n) - \gamma f(p)\| + \alpha_n\|\gamma f(p) - Ap\| + \beta_n\|x_n - p\| \\ &+ (1-\beta_n - \alpha_n\bar{\gamma})\|T_{\mu_n}W_n x_n - p\| \le (1-(\bar{\gamma} - \gamma\alpha)\alpha_n)\|x_n - p\| + \alpha_n|\gamma f(p) - Ap\|\end{aligned}$$

It follows from induction that

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}\} = M$$

Step 2. $\lim_{n\to\infty}\|x_n - T_{\mu_n}w_n x_n\| = 0$.

$\|x_n - T_{\mu_n}w_n x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - T_{\mu_n}w_n x_n\|$ we have:

$\|x_{n+1} - T_{\mu_n}w_n x_n\| \le \alpha_n\|\gamma f(x_n) - AT_{\mu_n}w_n x_n\| + \beta_n\|x_n - T_{\mu_n}w_n x_n\|$ so we have

$$\|x_n - T_{\mu_n}w_n x_n\| \le \frac{1}{1-\beta_n}[\alpha_n\|\gamma f(x_n) - AT_{\mu_n}w_n x_n\| + \|x_n - x_{n+1}\|]$$

Since W_n, T_{μ_n} are nonexpansive, we have

$\alpha_n\|\gamma f(x_n) - AT_{\mu_n}w_n x_n\| \le \alpha_n((\gamma\alpha + \|A\|)M_0 + \|\gamma f(p) - Ap\|)$ take

$L_0 = (\gamma\alpha + \|A\|)M_0 + \|\gamma f(p) - Ap\|)$ since $\lim_{n\to\infty}\alpha_n = 0$, there exists $K_0 \in N$ such that $\alpha_n < \frac{\delta}{L_0}$,

for all $n > K_0$ therefore we have

$$\|x_n - T_{\mu_n}w_n x_n\| \le \frac{1}{1-\beta_n}[\delta + \|x_n - x_{n+1}\|] \quad \forall n > K_0 \qquad (3.1)$$

It sufficient to show that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$

Define

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n, \qquad n \ge 0$$

Observe that from the definition of y_n, we obtain

$$\begin{aligned}y_{n+1} - y_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}, \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}[\gamma f(x_{n+1}) - AT_{\mu_{n+1}}W_{n+1}x_{n+1}] - \frac{\alpha_n}{1-\beta_n}[\gamma f(x_n) - AT_{\mu_n}W_n x_n] \\ &+ T_{\mu_{n+1}}W_{n+1}x_{n+1} - T_{\mu_n}W_n x_n\end{aligned}$$

This implies that

$\| y_{n+1} - y_n \|$

$\leq [\frac{\alpha_{n+1}}{1-\beta_{n+1}}[\| \gamma f(x_{n+1})\| + \| AT_{\mu_{n+1}}W_{n+1}x_{n+1}\|] - \frac{\alpha_n}{1-\beta_n}[\| \gamma f(x_n)\| + \| AT_{\mu_n}W_n x_n\|] +$

$\| T_{\mu_{n+1}}W_{n+1}x_{n+1} - T_{\mu_n}W_n x_n\|$ (3.2)

We have

$\| T_{\mu_{n+1}}W_{n+1}x_{n+1} - T_{\mu_n}W_n x_n\|$

$\leq \| T_{\mu_{n+1}}W_{n+1}x_{n+1} - T_{\mu_{n+1}}W_n x_n\| + \| T_{\mu_{n+1}}W_n x_n - T_{\mu_n}W_n x_n\|$

$\leq \| W_{n+1}x_{n+1} - W_n x_n\| + \| T_{\mu_{n+1}}W_n x_n - T_{\mu_n}W_n x_n\|$

$\leq \| W_{n+1}x_{n+1} - W_{n+1}x_n\| + \| W_{n+1}x_n - W_n x_n\| + \| T_{\mu_{n+1}}W_n x_n - T_{\mu_n}W_n x_n\|$

$\leq \| x_{n+1} - x_n\| + \| W_{n+1}x_n - W_n x_n\| + \| T_{\mu_{n+1}}W_n x_n - T_{\mu_n}W_n x_n\|$ (3.3)

Since T_i and $U_{n,i}$ are nonexpansive, from (1.1), we have

$\| W_{n+1}x_n - W_n x_n\| = \| \lambda_1 T_1 U_{n+1,2}x_n - \lambda_1 T_1 U_{n,2}x_n\| \leq \lambda_1 \| U_{n+1,2}x_n - U_{n,2}x_n\| \ldots\ldots \leq M_1 \prod_{i=1}^{n} \lambda_i$

(3.4)

Where $M_1 \geq 0$ is an appropriate constant such that $\| U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \leq M_1$ using (2.3) and (2.4) we have $\| T_{\mu_{n+1}}W_{n+1}x_{n+1} - T_{\mu_n}W_n x_n\|$

$$\leq \| x_{n+1} - x_n\| + M_1 \prod_{i=1}^{n} \lambda_i + \| T_{\mu_{n+1}}W_n x_n - T_{\mu_n}W_n x_n\| \quad (3.5)$$

Substituting (2.5) in (2.2), we have

$\| y_{n+1} - y_n\| - \| x_{n+1} - x_n\| \leq$

$\frac{\alpha_{n+1}}{1-\beta_{n+1}}[\| \gamma f(x_{n+1})\| + \| AT_{\mu_{n+1}}W_{n+1}x_{n+1}\|] - \frac{\alpha_n}{1-\beta_n}[\| \gamma f(x_n)\| + \| AT_{\mu_n}W_n x_n\|] +$

$M_1 \prod_{i=1}^{n} \lambda_i + \| T_{\mu_{n+1}}W_n x_n - T_{\mu_n}W_n x_n\|$

Which implies that (noting that (C1) and $0 < \lambda_i \leq b < 1,\ for\ i \geq 1$)

$$\limsup_{n\to\infty} \| y_{n+1} - y_n\| - \| x_{n+1} - x_n\| \leq 0$$

Hence by Lemma 2.5, we have $\lim_{n\to\infty} \| y_n - x_n\| = 0$. Consequently

$\lim_{n\to\infty} \| x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\| y_n - x_n\| = 0$. This together with (2.1) implies that

$\lim_{n\to\infty} \| x_n - T_{\mu_n} w_n x_n\| = 0$.

Step 3. $\limsup_{n\to\infty} \sup_{y\in D} \| T_{\mu_n} y - T_t T_{\mu_n} y\| = 0, \quad \forall t \in S$

Set $D = \{y \in H : \| y - p\| \leq M_0\}$. We point out that D is a bounded closed convex set, $\{x_n\} \subset D$ being invariant under φ and W_n, for all $n \in N$

Let $\varepsilon > 0$. By Theorem 1.2 in [6], there exists $\delta > 0$ such that

$$\overline{co}F_\delta(T_t;D)+B_\delta\subset F_\varepsilon(T_t;D),\quad \forall t\in S. \tag{3.6}$$

Also by Corollary 1.1 in [6], there exists a natural number N such that

$$\left\|\frac{1}{N+1}\sum_{i=0}^{N}T_{t^i s}y-T_t\left(\frac{1}{N+1}\sum_{i=0}^{N}T_{t^i s}y\right)\right\|\le\delta,\forall t,s\in S,\ y\in D \tag{3.7}$$

Let $t\in S$. Since $\{\mu_n\}$ is strongly left regular, there exists $N_0\in \mathrm{N}$ such that

$$\|\mu_n-l_{t^i}^*\mu_n\|\le\frac{\delta}{(M_0+\|x^*\|)},\qquad \forall n\ge N_0, i=1,2,\ldots,N.$$

Then we have

$$\sup_{y\in D}\left\|T_{\mu_n}y-\int\frac{1}{N+1}\sum_{i=0}^{N}T_{t^i s}y\,d\mu_n(s)\right\|=\sup_{y\in D}\sup_{\|z\|=1}\left|\langle T_{\mu_n}y,z\rangle-\left\langle\int\frac{1}{N+1}\sum_{i=0}^{N}T_{t^i s}y\,d\mu_n(s),z\right\rangle\right|$$

$$=\sup_{y\in D}\sup_{\|z\|=1}\left|\frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s-\frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s\langle T_{t^i s}y,z\rangle\right|$$

$$\le\frac{1}{N+1}\sum_{i=0}^{N}\sup_{y\in D}\sup_{\|z\|=1}(\mu_n)_s\langle T_{t^i s}y,z\rangle-(l_{t^i}^*\mu_n)_s\langle T_s y,z\rangle|$$

$$\le\max_{i=1,2,\ldots,N}\|\mu_n-l_{t^i}^*\mu_n(M_0+\|p\|)\le\delta,\qquad\forall n\ge N_0. \tag{3.8}$$

By Lemma 2.2, we have

$$\int\frac{1}{N+1}\sum_{i=0}^{N}T_{t^i s}y\,d\mu_n(s)\in\overline{co}\left\{\int\frac{1}{N+1}\sum_{i=0}^{N}T_{t^i}(T_s y):s\in S\right\}. \tag{3.9}$$

It follows from 2.6, 2.7, 2.8 and 2.9 that

$$T_{\mu_n}y\in\overline{co}\left\{\frac{1}{N+1}\sum_{i=0}^{N}T_{t^i s}y:s\in S\right\}+B_\delta\subset\overline{co}F_\delta(T_t;D)+B_\delta\subset F_\epsilon(T_t;D)\qquad\forall y\in D, n\ge N_0.$$

Therefore, $\limsup_{n\to\infty}\sup_{y\in D}\|T_{\mu_n}y-T_tT_{\mu_n}y\|\le\epsilon$. Since $\epsilon>0$ is arbitrary, we can proof step 3.

Step 4. $\lim_{n\to\infty}\|x_n-T_t x_n\|=0,\forall t\in S$.

We have shown in Step 2 that

$$\alpha_n\|\gamma f(x_n)-AT_{\mu_n}w_n x_n\|\le\alpha_n((\gamma\alpha+\|A\|)M_0+\|\gamma f(p)-Ap\|)\le\frac{\delta}{L_0}L_0=\delta \text{ for all } n>K_0$$

Using Step 2 we can assume, there exists $K_1\in N$ such that for all $n\ge K_1$,

$$\beta_n(x_n-T_{\mu_n}w_n x_n)\in B_{\frac{\delta}{2}}$$

From Lemma 2.7 we have $\|W_n x_n-p\|=\|W_n x_n-W_n p\|\le\|x_n-p\|\le M_0$, from definition D we have $W_n x_n\in D$. Therefore by Step 3 we have $T_{\mu_n}W_n x_n\in F_\delta(T_t;D)$, so we have

$$x_{n+1}=\alpha_n[\gamma f(x_n)-AT_{\mu_n}W_n x_n]+\beta_n[x_n-T_{\mu_n}W_n x_n]+T_{\mu_n}W_n x_n$$

$$\in B_{\frac{\delta}{2}}+B_{\frac{\delta}{2}}+F_\delta(T_t;D)\subset B_\delta+F_\delta(T_t;D)\subset F_\epsilon(T_t;D)$$

For all $n \geq K_2$ in which $K_2 = max\{K_0, K_1\}$, so we have:

$$\| x_n - T_t x_n \| < \epsilon$$

Step 5. There exists a unique $x^* \in F$ such that

$$\limsup_{n\to\infty} \langle x_n - x^*, \gamma f(x^*) - Ax^* \rangle \leq 0$$

$P_F(I - A + \gamma f)$ is a contraction of H into itself because

$$\| P_F(I - A + \gamma f)x - P_F(I - A + \gamma f)y \| \leq \| (I - A + \gamma f)x - (I - A + \gamma f)y \|$$

$$\leq \| I - A \| \| x - y \| + \gamma\alpha \| x - y \| \leq (1 - (\bar{\gamma} - \gamma\alpha)) \| x - y \| ,$$

By Banach contraction principal, $P_F(I - A + \gamma f)$ has a unique fixed point $x^* \in F$. Then by Lemma 2.4, we have

$$\langle \gamma f(x^*) - Ax^*, y - x^* \rangle \leq 0 \quad \forall y \in F \tag{3.10}$$

We take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - x^*, \gamma f(x^*) - Ax^* \rangle = \lim_{k\to\infty} \langle x_{n_k} - x^*, \gamma f(x^*) - Ax^* \rangle \tag{3.11}$$

We may also assume that $x_{n_k} \rightharpoonup z$, using Step 4 and Lemma 2.3 we have $z \in Fix(\varphi)$. We will show that $z \in F$, from Lemma 2.7 it follows that $\bigcap_{i=1}^{\infty} Fix T_i = Fix(W)$. So we must show that $z \in Fix(W)$. Assume that $Wz \neq z$, since $z \in Fix(\varphi)$, and by assumption, $T_i Fix\,\varphi \subset Fix\,\varphi$ We have $T_i z \in Fix\,\varphi \quad \forall i \in N$, so $W_n z \in Fix\,\varphi$. Hence $T_{\mu_n} W_n z = W_n z$ using Opial property of Hilbert space, since $x_{n_k} \rightharpoonup z$, we have

$$\liminf_{k\to\infty} \| x_{n_k} - x \| > \liminf_{k\to\infty} \| x_{n_k} - z \| \quad \forall x \neq z$$

Since $Wz \neq z$ by using Step 2 and Lemma2.8, we have

$$\begin{aligned}
\liminf_{k\to\infty} \| x_{n_k} - z \| &< \liminf_{k\to\infty} \| x_{n_k} - Wz \| \\
&\leq \liminf_{k\to\infty} [\| x_{n_k} - T_{\mu_{n_k}} W_{n_k} x_{n_k} \| + \| T_{\mu_{n_k}} W_{n_k} x_{n_k} - T_{\mu_{n_k}} W_{n_k} z \| + \| T_{\mu_{n_k}} W_{n_k} z - Wz \|] \\
&\leq \liminf_{k\to\infty} [\| x_{n_k} - T_{\mu_{n_k}} W_{n_k} x_{n_k} \| + \| x_{n_k} - z \| + \| W_{n_k} z - Wz \|] \\
&= \liminf_{k\to\infty} \| x_{n_k} - z \|
\end{aligned}$$

This is a contradiction. Therefore $z \in F$. Noticing 2.10, 2.11 and $x_{n_k} \rightharpoonup z$ we have:

$$\limsup_{n\to\infty} \langle x_n - x^*, \gamma f(x^*) - Ax^* \rangle = \langle z - x^*, \gamma f(x^*) - Ax^* \rangle \leq 0.$$

Step6. The sequence $\{x_n\}$ converges strongly to x^*. By Lemma 2.6 we have

$$\begin{aligned}
\| x_{n+1} - x^* \|^2 &= \| \alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(T_{\mu_n} W_n x_n - x^*) \|^2 \\
&\leq [\beta_n \| x_n - x^* \| + (1-\beta_n) - \alpha_n \bar{\gamma}) \| x_n - x^* \|]^2 + 2\alpha_n \langle \gamma f(x_n) - Ax^*, x_{n+1} - x^* \rangle \\
&= (1-\alpha_n \bar{\gamma})^2 \| x_n - x^* \|^2 + 2\alpha_n \langle \gamma f(x_n) - Ax^*, x_{n+1} - x^* \rangle
\end{aligned} \tag{3.12}$$

On the other hands,

$$\langle \gamma f(x_n)-\gamma f(x^*), x_{n+1}-x^*\rangle \le \gamma\alpha\| x_n - x^*\| \, \| x_{n+1}-x^*\|$$
$$\le \gamma\alpha\| x_n - x^*\| \sqrt{(1-\alpha_n\bar{\gamma})^2\| x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n)-Ax^*, x_{n+1}-x^*\rangle}$$
$$\le \gamma\alpha(1-\alpha_n\bar{\gamma})\| x_n - x^*\|^2 + \gamma\alpha\| x_n - x^*\| \sqrt{2\langle \gamma f(x_n)-Ax^*, x_{n+1}-x^*\rangle}\sqrt{\alpha_n}$$

Since $\{x_n\}$ is bounded, we can take a constant $G_0 > 0$ such that

$$\gamma\alpha\| x_n - x^*\| \sqrt{2\langle \gamma f(x_n)-Ax^*, x_{n+1}-x^*\rangle}\sqrt{\alpha_n} < G_0, \forall n \in N$$

So from the above and 2.12 we reach the following

$$\| x_{n+1}-x^*\|^2$$
$$\le (1-\alpha_n\bar{\gamma})^2\| x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n)-\gamma f(x^*), x_{n+1}-x^*\rangle + 2\alpha_n\langle \gamma f(x^*)-Ax^*, x_{n+1}-x^*\rangle$$

$$\le (1-\alpha_n\bar{\gamma})^2\| x_n - x^*\|^2 + 2\alpha_n\gamma\alpha(1-\alpha_n\bar{\gamma})\| x_n - x^*\|^2 + 2\alpha_n G_0\sqrt{\alpha_n}$$
$$+2\alpha_n\langle \gamma f(x^*)-Ax^*, x_{n+1}-x^*\rangle$$
$$= (1-(2(\bar{\gamma}-\gamma\alpha)+\alpha_n\bar{\gamma}(2\gamma\alpha-\bar{\gamma}))\alpha_n))\| x_n - x^*\|^2 + 2\alpha_n G_0\sqrt{\alpha_n} + 2\alpha_n\langle \gamma f(x^*)-Ax^*, x_{n+1}-x^*\rangle$$

It then follows that

$$\| x_{n+1}-x^*\|^2 \le (1-\gamma_n)\| x_n - x^*\|^2 + \alpha_n\delta_n. \qquad (3.13)$$

Where

$$\gamma_n = (2(\bar{\gamma}-\gamma\alpha)+\alpha_n\bar{\gamma}(2\gamma\alpha-\bar{\gamma}))\alpha_n,$$
$$\delta_n = 2G_0\sqrt{\alpha_n} + 2\langle \gamma f(x^*)-Ax^*, x_{n+1}-x^*\rangle$$

By Step 4, we get $\limsup_{n\to\infty}\delta_n \le 0$. Since $\alpha_n \to 0$ and $0 < \bar{\gamma}-\gamma\alpha$, we may assume, with no loss of generality, that $\alpha_n < \| A\|^{-1}$ and

$$0 < 2(\bar{\gamma}-\gamma\alpha)+\alpha_n\bar{\gamma}(2\gamma\alpha-\bar{\gamma}) < 1$$

This together with applying Lemma 2.9 to 2.13 concludes that

$\lim_{n\to\infty}\| x_n - x^*\| = 0$.

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