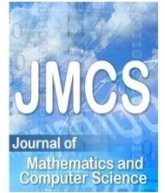




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Numerical Solution of Linear Emden Fowler Boundary Value Problem in Fuzzy Environment

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Abstract

In this paper a numerical method for solving Two Point Fuzzy Boundary Value Problems (TPFBVP) involving linear Emden Fowler equation is considered. The finite difference method (FDM) for solving TPFBVP is introduced and the proof of convergence of approximate solutions is brought in detail. Finally a numerical example is solved for illustrating the capability of method.

Keywords: Fuzzy numbers, fuzzy differential equations, two point fuzzy boundary value problems, Finite difference method.

1. Introduction

Nowadays, fuzzy differential equations (FDEs) [1, 2] are a popular topic studied by many researchers since it is utilized widely for the purpose of modeling problems in science and engineering. Most of the practical problems require the solution of a fuzzy differential equation (FDE) which satisfies fuzzy initial or boundary conditions, therefore a fuzzy initial or boundary problem should be solved. However, many fuzzy initial or boundary value problems could not be solved exactly; sometimes it is even impossible to find their analytical solutions. Thus, considering their approximate solutions is becoming more important. The two-point boundary value problem (TPBVP) occurs in a wide variety of problems in engineering [3, 4] and science, including the modeling of chemical reactions [5, 6], heat transfer [7,8], and diffusion [9,10], and the solution of optimal control problems [11,12]. Fuzzy two point boundary value problems (FTPFBVP) appears when the modeling of these problems cannot be sure is perfect and its nature is under uncertainty. Fuzzy ordinary differential equations are suitable mathematical models to model dynamical systems in which there exist uncertainties or vagueness. These models are used in various applications including population models [13], quantum optics gravity [14], and medicine [15, 16]. Fuzzy two point boundary value problems have been solved using FDM in [17] for linear problems with fuzzy boundary conditions, also a initial value method in [18] are applied to get the numerical solution for linear TPFBVP. The Finite Difference Method (FDM) is one of the simplest and of the oldest methods to linear TPFBVP

solve differential equations. It consists in approximating the differential operator by replacing the derivatives in the equation using differential quotients. The domains is partitioned in space and in time and approximations of the solution are computed at the space or time .To the best of our knowledge, this is the first attempt for solving linear fuzzy Emden Folwer. The structure of this paper is as follows. In section 2, some basic definitions and notations are given which will be used in other sections. In section 3, the defuzzification of TPFBVP is given in details. In section 4, structure of FDM is formulated for solving TPFBVP. In section 5, the definitions and thermos of consistency, stability and convergence related with FDM in section 4 are presented in details. In section 6, we introduced linear Emden Folwer in fuzzy environment with the proof the uniqueness of the difference approximation. In section 7 a numerical example is solved for illustrating of the method and finally, in section 8, we give the conclusion of this study.

2. Preliminaries

The definitions reviewed in this section are required in our work.

Definition 1 [19, 20]: Let \tilde{E} be the set of all upper semi-continuous normal convexfuzzy numbers with r -level bounded intervals such that:

$$[\mu]_r = \{t \in \mathbb{R} : \mu \geq r\}.$$

An arbitrary fuzzy number is represented by an ordered pair of functions

$$[\tilde{\mu}(t)]_r = [\underline{\mu}(t), \bar{\mu}(t)]_r \text{ for all } r \in [0,1] \text{ which satisfies:}$$

1. $\mu(t)$ is normal, i.e $\exists t_0 \in \mathbb{R}$ with $\mu(t_0) = 1$.
2. $\mu(t)$ is convex fuzzy set, i.e. $\mu(\lambda t + (1 - \lambda)s) \geq \min\{\mu(t), \mu(s)\}, \forall t, s \in \mathbb{R}, \lambda \in [0,1]$.
3. $[\tilde{\mu}(t)]_r \in \tilde{E}, \mu$ is upper semi continuous on \mathbb{R} .
4. $\{t \in \mathbb{R} : \mu(t) > 0\}$ is compact.
5. $\underline{\mu}(t)$ is a bounded left continuous non-decreasing function over $[0,1]$.
6. $\bar{\mu}(t)$ is a bounded left continuous non-increasing function over $[0,1]$.
7. $\underline{\mu}(t) \leq \bar{\mu}(t)$, for all $r \in [0, 1]$.

The r -level sets of a fuzzy number are much more effective as representation forms of fuzzy set than the above. Fuzzy sets can be defined by the families of their r -level sets based on the resolution identity theorem [21].

Definition 2.1 [22]: A mapping $\tilde{f}: T \rightarrow \tilde{E}$ (or $\tilde{P}(E)$) for some interval $T \subseteq \tilde{E}$ is called a fuzzy process or fuzzy function with crisp variable, and we denote r -level set by:

$$[\tilde{f}(t)]_r = [\underline{f}(t; r), \bar{f}(t; r)], t \in T, r \in [0,1]$$

where \tilde{E} be the set of all upper semi-continuous normal convexfuzzy numbers.

Definition 2.2 [23, 24]: Each function $f: X \rightarrow Y$ induces another function $\tilde{f}: F(X) \rightarrow F(Y)$ defined for each fuzzy interval U in X by:

$$\tilde{f}(U)(y) = \begin{cases} \text{Sup}_{x \in f^{-1}(y)} U(x), & \text{if } y \in \text{range}(f) \\ 0 & , \text{if } y \notin \text{range}(f) \end{cases}$$

This is called the Zadeh extension principle.

Definition 2.3 [25]: A fuzzy matrix of order $m \times s$ is defined $[\tilde{A}] = [\tilde{a}_{ij}, \mu_{\tilde{a}_{ij}}]$ as, where $\mu_{\tilde{a}_{ij}}$ is the membership function of the element \tilde{a}_{ij} in $[\tilde{A}]$, $\forall \tilde{a}_{ij} \in \tilde{E}$, for $i=1,2,\dots,m, j=1,2,\dots,s$. Thus for all $r \in [0,1]$

$$[\tilde{A}]_r = [\underline{A}, \overline{A}]_r, \text{ and } [\tilde{a}_{ij}]_r = [\underline{a}_{ij}, \overline{a}_{ij}]_r.$$

3. Defuzzification of FTPBVP

Consider the following second order FTPBVP

$$\begin{cases} \tilde{y}''(x) = \tilde{f}(x, \tilde{y}(x), \tilde{y}'(x)), x \in [x_0, x_n] \\ \tilde{y}(x_0) = \tilde{\alpha}, \tilde{y}(x_n) = \tilde{\beta} \end{cases} \quad (1)$$

According to Section 2, It's not difficult for defuzzification of Eq. (1), where $\tilde{y}(x)$ is a fuzzy function of the crisp variable x and \tilde{f} is the fuzzy function of the crisp variable x and the fuzzy variable \tilde{y} . Here $\tilde{y}''(x)$ is the second order fuzzy derivative [26] of $\tilde{y}(x), \tilde{y}'(x)$, with $\tilde{\alpha}, \tilde{\beta}$ are the fuzzy numbers that refer to the fuzzy boundary conditions $\tilde{y}(x_0), \tilde{y}(x_n)$ of Eq.(1). We denote the fuzzy function $\tilde{y}(x)$ by $[\tilde{y}]_r = [\underline{y}, \overline{y}]_r$, for $x \in [x_0, x_n]$ and $r \in [0,1]$ it means that the r -level set of $\tilde{y}(x)$ can be defined as:

$$\begin{cases} [\tilde{y}(x)]_r = [\underline{y}(x; r), \overline{y}(x; r)] \\ [\tilde{y}'(x)]_r = [\underline{y}'(x; r), \overline{y}'(x; r)] \\ [\tilde{y}''(x)]_r = [\underline{y}''(x; r), \overline{y}''(x; r)] \end{cases} \quad (2)$$

$$\begin{cases} [\tilde{y}(x_0)]_r = [\underline{y}(x_0; r), \overline{y}(x_0; r)] = [\underline{\alpha}, \overline{\alpha}]_r \\ [\tilde{y}(x_n)]_r = [\underline{y}(x_n; r), \overline{y}(x_n; r)] = [\underline{\beta}, \overline{\beta}]_r \end{cases} \quad (3)$$

Now if we let $\mathcal{Y}(x) = y(x), y'(x)$, and for defuzzification we have

$$\tilde{\mathcal{Y}}(x; r) = [\underline{\mathcal{Y}}(x; r), \overline{\mathcal{Y}}(x; r)]$$

where

$$\underline{\mathcal{Y}}(x; r) = [\underline{y}(x; r), \underline{y}'(x; r), \underline{y}''(x; r)]$$

$$\overline{\mathcal{Y}}(x; r) = [\overline{y}(x; r), \overline{y}'(x; r), \overline{y}''(x; r)]$$

Also we can write

$$[\tilde{f}(x, \tilde{\mathcal{Y}})]_r = [f(x, \underline{\mathcal{Y}}; r), \overline{f}(x, \overline{\mathcal{Y}}; r)] \quad (4)$$

and by using the extension fuzzy principle we have

$$\begin{cases} \underline{f}(x, \tilde{\mathcal{Y}}; r) = \mathcal{F} [x, \underline{\mathcal{Y}}, \overline{\mathcal{Y}}]_r, \\ \overline{f}(x, \tilde{\mathcal{Y}}; r) = \mathcal{G} [x, \underline{\mathcal{Y}}, \overline{\mathcal{Y}}]_r \end{cases} \quad (5)$$

Since $\tilde{y}''(x) = f(x, \mathcal{Y}(x))$, we can define the following membership function

$$\begin{cases} \underline{f}(x, \tilde{\mathcal{Y}}(x; r); r) = \min\{\tilde{f}(x, \tilde{\mu}(r)) | \tilde{\mu}(r) \in \tilde{\mathcal{Y}}(t; r)\} \\ \overline{f}(x, \tilde{\mathcal{Y}}(x; r); r) = \max\{\tilde{f}(x, \tilde{\mu}(r)) | \tilde{\mu}(r) \in \tilde{\mathcal{Y}}(t; r)\} \end{cases} \quad (6)$$

where

$$\begin{cases} \underline{f}(x, \tilde{\mathcal{Y}}(x; r); r) = \mathcal{F}(x, \underline{\mathcal{Y}}(x; r), \overline{\mathcal{Y}}(x; r)) \\ \overline{f}(x, \tilde{\mathcal{Y}}(x; r); r) = \mathcal{G}(x, \underline{\mathcal{Y}}(x; r), \overline{\mathcal{Y}}(x; r)) \end{cases} \quad (7)$$

After this defuzzification can rewrite Eq. (1) for the lower and the upper bound of respectively for Eq. (1) and for all $r \in [0,1]$, we have

$$\begin{cases} \underline{y}''(x; r) = \mathcal{F}(x, \underline{\mathcal{Y}}(x; r), \overline{\mathcal{Y}}(x; r)), x \in [x_0, x_n] \\ \underline{y}(x_0; r) = [\underline{\alpha}]_r, \underline{y}(x_n; r) = [\underline{\beta}]_r \end{cases} \quad (8)$$

$$\begin{cases} \overline{y}''(x; r) = \mathcal{G}(x, \underline{\mathcal{Y}}(x; r), \overline{\mathcal{Y}}(x; r)), x \in [x_0, x_n] \\ \overline{y}(x_0; r) = [\overline{\alpha}]_r, \overline{y}(x_n; r) = [\overline{\beta}]_r \end{cases} \quad (9)$$

More details about the existence and uniqueness are in [17].

4. Finite Difference Scheme for Second Order Linear FTPBVP

It consists in approximating the differential operator by replacing the derivatives in the equation using differential quotients. The domain is partitioned in space and in time and approximations of the solution are computed at the space or time points. The error between the numerical solution and the exact solution is determined by the error that is committed by going from a differential operator to a difference operator. This error is called the called the truncation error [9]. The term truncation error reflects the fact that a finite part of a Taylor series is used in the approximation. We formulate this scheme in order to solve second order TPBVP in fuzzy environment .We consider fuzzy boundary value Eq. (1) can be written as:

$$\begin{cases} \tilde{y}''(x) = p(x)\tilde{y}'(x) + q(x)\tilde{y}(x) + \tilde{w}(x), x \in [x_0, x_n] \\ y(x_0) = \tilde{\alpha}, y(x_n) = \tilde{\beta} \end{cases} \quad (10)$$

Where $p(x)$, $q(x)$ and $w(x)$ are real continuous functions on $J = [x_0, x_n]$ with $\tilde{\alpha}, \tilde{\beta} \in \tilde{E}$, then according to Section 2 and for all $r \in [0,1]$, we have the same defuzzification of Eq. (1) of $\tilde{y}(x)$, $\tilde{w}(x)$, $\tilde{y}'(x)$, $\tilde{y}''(x)$, $\tilde{y}(x_0)$, $\tilde{y}(x_n)$.

4.1 Approximation of the First Order Derivative

In this section we derive a finite difference formula for the fuzzy derivative $[\tilde{y}'(x)]_r$ with its accuracy analysis. Suppose the function $[\tilde{y}(x)]_r \in C^2(\tilde{E})$ continuous for all $x \in J$. Using Taylor series for any $h>0$ we have the forward difference formula:

$$\tilde{y}(x+h;r) = \tilde{y}(x;r) + h\tilde{y}'(x;r) + \frac{h^2}{2}\tilde{y}''(x;r) + \frac{h^3}{6}\tilde{y}'''(x;r) + \frac{h^4}{24}\tilde{y}^{(4)}(\zeta^+;r) \quad (11)$$

where $\zeta^+ \in [x+h, x]$, and backward difference formula:

$$\tilde{y}(x-h;r) = \tilde{y}(x;r) - h\tilde{y}'(x;r) + \frac{h^2}{2}\tilde{y}''(x;r) - \frac{h^3}{6}\tilde{y}'''(x;r) + \frac{h^4}{24}\tilde{y}^{(4)}(\zeta^-;r) \quad (12)$$

where $\zeta^- \in [x-h, x]$. By subtracting (3.4) from (3.5) we have fuzzy central difference formula :

$$\tilde{y}'(x;r) = \frac{\tilde{y}(x+h;r) - \tilde{y}(x-h;r)}{2h} + O(h^2) \quad (13)$$

where $O(h^2) = -\frac{h^3}{3}\tilde{y}'''(\zeta;r)$, and $\zeta \in [x-h, x+h]$. Hence, for every $h \in [0, h_0]$, we have the following bound on the approximation truncation error:

$$\left| \frac{\tilde{y}(x+h;r) - \tilde{y}(x-h;r)}{2h} \right| \leq \tilde{C}h^2$$

where

$$\underline{C} = \sup_{\zeta \in [x-h, x+h]} \left\{ \min \left(\frac{h^3}{3} \underline{y}'''(\zeta; \mu), \frac{h^3}{3} \overline{y}'''(\zeta; \mu) \mid \mu \in [\underline{y}(\zeta), \overline{y}(\zeta)]_r \right) \right\}$$

$$\overline{C} = \sup_{\zeta \in [x-h, x+h]} \left\{ \max \left(\frac{h^3}{3} \underline{y}'''(\zeta; \mu), \frac{h^3}{3} \overline{y}'''(\zeta; \mu) \mid \mu \in [\underline{y}(\zeta), \overline{y}(\zeta)]_r \right) \right\}$$

Here μ represent the membership function of the FTPBVP (10) for all $r \in [0,1]$ and h is the step size given by $h = \frac{x_n - x_0}{N}$, where N is the number of iterative.

4.2 Approximation of the Second Order Derivative

In this section we derive a finite difference formula for $\tilde{y}''(x;r)$ with its accuracy analysis. Suppose the function $[\tilde{y}(x)]_r \in C^4(\tilde{E})$ continuous for all $x \in J$. Using the Taylor series for any $h > 0$ and by adding formula (3.4) from (3.5) we have central fuzzy difference formula:

$$\tilde{y}''(x;r) = \frac{\tilde{y}(x+h;r) - 2\tilde{y}(x;r) + \tilde{y}(x-h;r)}{h^2} + O(h^2) \quad (14)$$

where $O(h^2) = -\frac{h^4}{12}\tilde{y}^{(4)}(\zeta;r)$, $\zeta \in [x-h, x+h]$. Hence, for every $h \in [0, h_0]$, we have the following bound on the approximation truncation error:

$$\left| \frac{\tilde{y}(x+h;r) - 2\tilde{y}(x;r) + \tilde{y}(x-h;r)}{h^2} \right| \leq \tilde{C}h^2$$

where

$$\underline{C} = \sup_{\zeta \in [x-h, x+h]} \left\{ \min \left(\frac{h^4}{12} \underline{y}^{(4)}(\zeta; \mu), \frac{h^4}{12} \overline{y}^{(4)}(\zeta; \mu) \mid \mu \in [\underline{y}(\zeta), \overline{y}(\zeta)]_r \right) \right\}$$

$$\overline{C} = \sup_{\zeta \in [x-h, x+h]} \left\{ \max \left(\frac{h^4}{12} \underline{y}^{(4)}(\zeta; \mu), \frac{h^4}{12} \overline{y}^{(4)}(\zeta; \mu) \mid \mu \in [\underline{y}(\zeta), \overline{y}(\zeta)]_r \right) \right\}$$

Here μ represent the membership function of the FTPBVP (10) for all $r \in [0,1]$ and h is the step size given by $h = \frac{x_n - x_0}{N}$, where N is the number of iterations. In our work we set

$\tilde{y}(x + h; r) = [\tilde{y}_{i+1}]_r$, $\tilde{y}(x - h; r) = [\tilde{y}_{i-1}]_r$, and $\tilde{y}(x; r) = [\tilde{y}_i]_r$ for $i=1, 2 \dots n$ for arbitrary positive integer n we subdivide the interval $J = [x_0, x_n]$ as $x_0 < x_1 < \dots < x_{n-1} < x_n$. Let $\pi = \{x_i\}_{i=0}^{n+1}$ denote a uniform partition of the interval J such that $x_i = x_0 + ih$, $i=0,1,\dots,n+1$, with $h = x_n - x_0/n + 1$, and for all $r \in [0,1]$ we can rewrite the fuzzy central difference formula as:

$$\begin{cases} [\tilde{y}'_i]_r = \frac{[\tilde{y}_{i+1}]_r - [\tilde{y}_{i-1}]_r}{2h} \\ [\tilde{y}''_i]_r = \frac{[\tilde{y}_{i+1}]_r - 2[\tilde{y}_i]_r + [\tilde{y}_{i-1}]_r}{h^2} \end{cases} \tag{15}$$

The Uniqueness of the Difference Approximation of Eq. (10) is given in [15]

5. Consistency, Stability and Convergence

To study the accuracy and the computability of the difference approximation in fuzzy environment, such that $\{\tilde{y}_{i+1}\}_{i=0}^n$, we introduce the concepts of consistency, stability and convergence of finite difference methods. The basic result proved in this section is that, for a consistent method, stability implies convergence.

Definition 5.1 (Consistency): Let

$$[\tilde{\mathcal{J}}_{i,\pi}]_r = \tilde{L}_h \tilde{v}(x_i; r) - \tilde{L} \tilde{v}(x_i; r), i = 1, 2, \dots, n$$

where \tilde{v} is smooth and continuous fuzzy function on \tilde{E} , $\forall x \in J$, $\tilde{L}h$ is the difference operator and \tilde{L} is the linear operator of Eq. (10). Then the difference problem (15) is consistent with the differential problem (10) if:

$$[\tilde{\mathcal{J}}_{i,\pi}(\tilde{v})]_r \rightarrow 0 \text{ as } h \rightarrow 0$$

Where $[\tilde{\mathcal{J}}_{i,\pi}(\tilde{v})]_r$ refer to local truncation (or local discretization) errors.

Definition 5.2: The difference problem (15) is locally p^{th} order accurate if sufficiently smooth data, there exists a positive constant $[\tilde{\mathcal{C}}]_r = [\underline{\mathcal{C}}, \overline{\mathcal{C}}]_r$, independent of h , such that:

$$\begin{aligned} \sup_{1 \leq i \leq n} \left\{ \min \left| [\underline{\mathcal{J}}_{i,\pi}(\underline{v})]_r, [\overline{\mathcal{J}}_{i,\pi}(\overline{v})]_r \right| \right\} &\leq [\underline{\mathcal{C}}]_r h^p \\ \sup_{1 \leq i \leq n} \left\{ \max \left| [\underline{\mathcal{J}}_{i,\pi}(\underline{v})]_r, [\overline{\mathcal{J}}_{i,\pi}(\overline{v})]_r \right| \right\} &\leq [\overline{\mathcal{C}}]_r h^p \end{aligned}$$

The following lemma demonstrates that the difference problem (15) is consistent with (10) and is locally second order accurate.

Lemma 5.1: If $\tilde{v} \in C^4(J)$ then

$$[\tilde{\mathcal{J}}_{i,\pi}(\tilde{v})]_r = \frac{-h^2}{12} \left(\tilde{v}^{(4)}(\tau_i; r) - 2p(x_i) \tilde{v}'''(\theta_i; r) \right)$$

$$\forall r \in [0,1], i = 1, 2, \dots, n$$

τ_i, θ_i lie in (x_{i-1}, x_{i+1})

Proof:

According to definition (5.2) one can write

$$[\tilde{J}_{i,\pi}(\tilde{v})]_r = \left[\frac{\tilde{v}(x_{i+1};r) - 2\tilde{v}(x_i;r) + \tilde{v}(x_{i-1};r)}{h^2} - \tilde{v}''(x_i;r) \right] + p(x_i) \left[\frac{\tilde{v}(x_{i+1};r) - \tilde{v}(x_{i-1};r)}{2h} - \tilde{v}'(x_i;r) \right] \tag{16}$$

It is easy to show using Taylor's theorem that:

$$\frac{\tilde{v}(x_{i+1};r) - \tilde{v}(x_{i-1};r)}{2h} - \tilde{v}'(x_i;r) = \frac{h^2}{3} \tilde{v}'''(\theta_i;r) \tag{17}$$

$$\theta_i \in (x_{i-1}, x_{i+1})$$

$$\frac{\tilde{v}(x_{i+1};r) - 2\tilde{v}(x_i;r) + \tilde{v}(x_{i-1};r)}{h^2} - \tilde{v}''(x_i;r) = \frac{h^2}{12} \tilde{v}^{(4)}(\tau_i;r) \tag{18}$$

$$\tau_i \in (x_{i-1}, x_{i+1})$$

The desired result now follows on substituting (17) and (18) in (16).

Definition 5.3 (Stability): The linear difference operator \tilde{L}_h is stable if, for sufficiently small h , there exists a constant K , independent of h , such that

$$|[\underline{v}_i]_r| \leq K \left\{ \sup \left\{ \min \left((|[\underline{v}_0]_r|, |[\underline{v}_{n+1}]_r|), |[\bar{v}_0]_r|, |[\bar{v}_{n+1}]_r| \right) \right\} \right\} + \sup_{1 \leq j \leq n} \left\{ \min \left([L_h \underline{v}_j]_r, [\bar{L}_h \bar{v}_j]_r \right) \right\}$$

$$|[\bar{v}_i]_r| \leq K \left\{ \sup \left\{ \max \left((|[\underline{v}_0]_r|, |[\underline{v}_{n+1}]_r|), |[\bar{v}_0]_r|, |[\bar{v}_{n+1}]_r| \right) \right\} \right\} + \sup_{1 \leq j \leq n} \left\{ \max \left([L_h \underline{v}_j]_r, [\bar{L}_h \bar{v}_j]_r \right) \right\}$$

$$\forall r \in [0,1], i = 0,1, \dots, n + 1.$$

We now prove that, for h sufficiently small, the difference operator \tilde{L}_h of (16) is stable.

Theorem 5.1: If the functions p and q satisfy (), then the difference operator \tilde{L}_h of Eq. (4.1) is stable for, $h < \min \left(\frac{2}{p^*}, \frac{p^*}{q^*} \right)$ with $K = \{1, 1/q^*\}$

Proof: If

$$|[\underline{v}_i]_r| \leq \sup_{1 \leq j \leq n} \left\{ \min \left(|[\underline{v}_j]_r|, |[\bar{v}_j]_r| \right) \right\}$$

$$|[\bar{v}_i]_r| \leq \sup_{1 \leq j \leq n} \left\{ \min \left(|[\underline{v}_j]_r|, |[\bar{v}_j]_r| \right) \right\}$$

$$\forall r \in [0,1], i = 0,1, \dots, n + 1.$$

then from (4.1) we obtain

$$d_i[\underline{v}_i]_r \leq (|e_i| + |c_i|) |[\underline{v}_i]_r| + h^2 \sup_{1 \leq j \leq n} \left\{ \min \left(|[\underline{v}_j]_r|, |[\bar{v}_j]_r| \right) \right\}$$

$$d_i[\bar{v}_i]_r \leq (|e_i| + |c_i|) \left| [\bar{v}_j]_r \right| + h^2 \sup_{1 \leq j \leq n} \left\{ \max \left(\left| [L_h v_j]_r \right|, \left| [\bar{L}_h \bar{v}_j]_r \right| \right) \right\}$$

then

$$d_i[\tilde{v}_i]_r = -e_i[\tilde{v}_{i+1}]_r - c_i[\tilde{v}_{i-1}]_r + h^2[\tilde{L}_h \tilde{v}_{i+1}]_r$$

Thus

if $h < \min(2/p^*, p^*/q^*)$, then

$$d_i = |c_i| + |e_i| + h^2 q_i$$

and it follows that

$$h^2 q_i \left| [v_i]_r \right| \leq h^2 \sup_{1 \leq j \leq n} \left\{ \min \left(\left| [L_h v_j]_r \right|, \left| [\bar{L}_h \bar{v}_j]_r \right| \right) \right\}$$

$$h^2 q_i \left| [\bar{v}_i]_r \right| \leq h^2 \sup_{1 \leq j \leq n} \left\{ \max \left(\left| [L_h v_j]_r \right|, \left| [\bar{L}_h \bar{v}_j]_r \right| \right) \right\}$$

or

$$\left| [v_i]_r \right| \leq \frac{1}{q^*} \sup_{1 \leq j \leq n} \left\{ \min \left(\left| [L_h v_j]_r \right|, \left| [\bar{L}_h \bar{v}_j]_r \right| \right) \right\}$$

$$\left| [\bar{v}_i]_r \right| \leq \frac{1}{q^*} \sup_{1 \leq j \leq n} \left\{ \max \left(\left| [L_h v_j]_r \right|, \left| [\bar{L}_h \bar{v}_j]_r \right| \right) \right\}$$

Thus if $\max x_0 \leq i \leq x_{n+1}$ $[\tilde{v}_{i+1}]_r$ occurs for $1 \leq j \leq n$, then

$$\sup_{0 \leq i \leq n+1} \left\{ \min \left(\left| [v_i]_r \right|, \left| [\bar{v}_i]_r \right| \right) \right\} \leq \frac{1}{q^*} \sup_{1 \leq j \leq n} \left\{ \min \left(\left| [L_h v_j]_r \right|, \left| [\bar{L}_h \bar{v}_j]_r \right| \right) \right\}$$

$$\sup_{1 \leq i \leq n+1} \left\{ \max \left(\left| [v_i]_r \right|, \left| [\bar{v}_i]_r \right| \right) \right\} \leq \frac{1}{q^*} \sup_{1 \leq j \leq n} \left\{ \max \left(\left| [L_h v_j]_r \right|, \left| [\bar{L}_h \bar{v}_j]_r \right| \right) \right\}$$

and clearly

$$\sup_{0 \leq i \leq n+1} \left\{ \min \left(\left| [v_i]_r \right|, \left| [\bar{v}_i]_r \right| \right) \right\} \leq K \left\{ \sup \left\{ \min \left(\left(\left| [v_0]_r \right|, \left| [v_{n+1}]_r \right| \right), \left(\left| [\bar{v}_0]_r \right|, \left| [\bar{v}_{n+1}]_r \right| \right) \right) \right\} + \sup_{1 \leq j \leq n} \left\{ \min \left(\left| [L_h v_j]_r \right|, \left| [\bar{L}_h \bar{v}_j]_r \right| \right) \right\} \right\}$$

$$\sup_{1 \leq i \leq n+1} \left\{ \max \left(\left| [v_i]_r \right|, \left| [\bar{v}_i]_r \right| \right) \right\} \leq K \left\{ \sup \left\{ \max \left(\left(\left| [v_0]_r \right|, \left| [v_{n+1}]_r \right| \right), \left(\left| [\bar{v}_0]_r \right|, \left| [\bar{v}_{n+1}]_r \right| \right) \right) \right\} + \sup_{1 \leq j \leq n} \left\{ \max \left(\left| [L_h v_j]_r \right|, \left| [\bar{L}_h \bar{v}_j]_r \right| \right) \right\} \right\}$$

Thus if $\max x_0 \leq i \leq x_{n+1}$ $[\tilde{v}_i]_r = \sup\{[\tilde{v}_0]_r, [\tilde{v}_{n+1}]_r\}$ such that

$$\left| [v_i]_r \right| = \sup \left\{ \min \left(\left(\left| [v_0]_r \right|, \left| [v_{n+1}]_r \right| \right), \left(\left| [\bar{v}_0]_r \right|, \left| [\bar{v}_{n+1}]_r \right| \right) \right) \right\}$$

$$|[\tilde{v}_i]_r| = \sup \left\{ \max \left(\left(|[\underline{v}_0]_r|, |[\underline{v}_{n+1}]_r| \right), \left(|[\bar{v}_0]_r|, |[\bar{v}_{n+1}]_r| \right) \right) \right\}$$

then the above equation follows immediately. An immediate consequence of stability is the uniqueness (and hence existence since the problem is linear) of the difference approximation $\{[\tilde{y}_i]_r\}_{i=0}^n$, for, if they were two solutions, their difference $\{[\tilde{v}_i]_r\}_{i=0}^n$, say, would satisfy:

$$|[\tilde{L}_h \tilde{v}_j]_r| = 0, j = 1, 2, \dots, n$$

$$[\tilde{v}_0]_r = [\tilde{v}_{n+1}]_r = 0$$

Stability then implies that $[\tilde{v}_i]_r = 0$ for $i = 0, 1, 2, \dots, n + 1$.

Definition 5.4(Convergence): Let $\tilde{Y}(x; r)$ be the exact solution of the boundary value Eq. (10), and $\{[\tilde{y}_i]_r\}_{i=0}^n$ be the difference approximation defined by Eq. (15). The difference approximation converges to $\tilde{Y}(x; r)$ if

$$\max_{1 \leq j \leq n} |\tilde{Y}(x_j; r) - \tilde{y}(x_j; r)| \rightarrow 0$$

or

$$[\underline{\mathbb{E}}]_r = \sup_{1 \leq j \leq n} \left\{ \min \left(\left| \underline{Y}(x_j; r) - \underline{y}(x_j; r) \right|, \left| \bar{Y}(x_j; r) - \bar{y}(x_j; r) \right| \right) \right\} \rightarrow 0$$

$$[\bar{\mathbb{E}}]_r = \sup_{1 \leq j \leq n} \left\{ \max \left(\left| \underline{Y}(x_j; r) - \underline{y}(x_j; r) \right|, \left| \bar{Y}(x_j; r) - \bar{y}(x_j; r) \right| \right) \right\} \rightarrow 0$$

$\forall r \in [0, 1]$,

as $h \rightarrow 0$. The difference $\tilde{Y}(x_j; r) - \tilde{y}(x_j; r)$ is the global truncation (or discretization) error $[\tilde{\mathbb{E}}]_r = [\underline{\mathbb{E}}, \bar{\mathbb{E}}]_r$ at the point $x_j, j = 1, 2, \dots, n$

Definition 5.9: The difference approximation $\{[\tilde{y}_i]_r\}_{i=0}^n$ is a p^{th} approximation to the solution $\tilde{Y}(x; r)$ if for h sufficiently small; there exists a constant $[\tilde{\mathcal{C}}]_r$ independent of h such that

$$\max_{1 \leq j \leq n} |\tilde{Y}(x_j; r) - \tilde{y}(x_j; r)| \leq [\tilde{\mathcal{C}}]_r h^p$$

The basic result connecting consistency, stability and convergence is given in the following theorem.

Theorem 5.2: Suppose $\tilde{Y}(x; r) \in C^4(J)$, and $h < \frac{2}{p^*}$, Then the difference solution $\{[\tilde{y}_i]_r\}_{i=0}^n$ of (2.1) is convergent to the $\tilde{Y}(x; r)$ solution of Eq. (15) Moreover,

$$\max_{1 \leq j \leq n} |\tilde{Y}(x_j; r) - \tilde{y}(x_j; r)| \leq [\tilde{\mathcal{C}}]_r h^2$$

where

$$\underline{\mathcal{C}} = \sup_{\zeta \in [x-h, x+h]} \left\{ \min \left(\frac{h^4}{12} \underline{y}^{(4)}(\zeta; \mu), \frac{h^4}{12} \bar{y}^{(4)}(\zeta; \mu) \mid \mu \in [\underline{y}(\zeta), \bar{y}(\zeta)]_r \right) \right\}$$

$$\bar{\mathcal{C}} = \sup_{\zeta \in [x-h, x+h]} \left\{ \max \left(\frac{h^4}{12} \underline{y}^{(4)}(\zeta; \mu), \frac{h^4}{12} \bar{y}^{(4)}(\zeta; \mu) \mid \mu \in [\underline{y}(\zeta), \bar{y}(\zeta)]_r \right) \right\}$$

Proof

Under the given conditions, the difference problem (15) is consistent with the boundary value problem (10) and the operator $[\tilde{L}_h]_r$ is stable.

Since

$$[\tilde{L}_h]_r (\tilde{Y}(x_j; r) - \tilde{y}(x_j; r)) = \tilde{W}(x_j; r) - \tilde{y}(x_j; r) = [\tilde{L}_h]_r \tilde{Y}(x_j; r) - [\tilde{L}_h]_r \tilde{y}(x_j; r) = -[\tilde{\mathcal{T}}_{i,\pi}(\tilde{y})]_r$$

and

$$\tilde{Y}(x_0; r) - \tilde{y}(x_0; r) = \tilde{Y}(x_{n+1}; r) - \tilde{y}(x_{n+1}; r) = 0$$

the stability of $[\tilde{L}_h]_r$ implies that

$$|\tilde{Y}(x_j; r) - \tilde{y}(x_j; r)| \leq \frac{1}{q^*} \max_{1 \leq j \leq n} [\tilde{\mathcal{T}}_{i,\pi}(\tilde{y})]_r$$

such that

$$|\underline{Y}(x_j; r) - \underline{y}(x_j; r)| \leq \frac{1}{q^*} \sup_{1 \leq j \leq n} \left\{ \min \left([\underline{\mathcal{T}}_{i,\pi}(\underline{y})]_r, [\overline{\mathcal{T}}_{i,\pi}(\overline{y})]_r \right) \right\}$$

$$|\overline{Y}(x_j; r) - \overline{y}(x_j; r)| \leq \frac{1}{q^*} \sup_{1 \leq j \leq n} \left\{ \max \left([\underline{\mathcal{T}}_{i,\pi}(\underline{y})]_r, [\overline{\mathcal{T}}_{i,\pi}(\overline{y})]_r \right) \right\}$$

The desired result follows from the Lemma in [14]. It follows from this theorem that $\{\tilde{y}_i\}_r^n$ is a second-order approximation to the solution $\tilde{Y}(x; r)$ of Eq. (10).

6. FTPBVP Linear Emden Fowler Equation

Consider the following linear Emden Fowler FTPBVP

$$\begin{cases} \tilde{y}''(x) = \tilde{p}(x)\tilde{y}'(x) + q(x)\tilde{y}(x) + \tilde{w}(x), x \in [x_0, x_n] \\ y(x_0) = \tilde{\alpha}, y(x_n) = \tilde{\beta} \end{cases} \quad (17)$$

Where $q(x)$ are real continuous functions on $J = [x_0, x_n]$ with $\tilde{\alpha}, \tilde{\beta} \in \tilde{E}$. Here $\tilde{p}(x) = \frac{\tilde{\eta}}{x}$, and $\tilde{w}(x)$ are continuous fuzzy function, $\tilde{\eta}$ is fuzzy number such that $\tilde{\eta} \in \tilde{E}$.

6.1 The Uniqueness of the Difference Approximation

According to Eq. (17) $\tilde{p}(x) \in \tilde{E}$ such that $\tilde{p}(x; r) = [\underline{p}(x; r), \overline{p}(x; r)]$, If their exist positive constant

$[\underline{p}^*, \overline{p}^*]_r, q^*$, and $[\underline{p}_*, \overline{p}_*]_r, q_*$ such that

$$0 \leq [\underline{p}_*]_r \leq |\underline{p}(x; r)| \leq [\underline{p}^*]_r, 0 \leq [\overline{p}_*]_r \leq |\overline{p}(x; r)| \leq [\overline{p}^*]_r, 0 \leq q_* \leq |q(x)| \leq q^*$$

(18)

Substituting and simplify formula (3.8) in Eq. (6.1) we have

$$\left\{ \begin{array}{l} h^2 [\underline{L}_i \underline{y}_i]_r : - \left(1 + \frac{h}{2} [\underline{p}_i]_r\right) [\underline{y}_{i-1}]_r + (2 + h^2 q_i) [\underline{y}_i]_r - \left(1 - \frac{h}{2} p_i\right) [\underline{y}_{i+1}]_r = h^2 [\underline{w}_i]_r \\ \underline{y}(x_0; r) = [\underline{\alpha}]_r, \underline{y}(x_n; r) = [\underline{\beta}]_r \\ h^2 [\overline{L}_i \overline{y}_i]_r : - \left(1 + \frac{h}{2} [\overline{p}_i]_r\right) [\overline{y}_{i-1}]_r + (2 + h^2 q_i) [\overline{y}_i]_r - \left(1 - \frac{h}{2} p_i\right) [\overline{y}_{i+1}]_r = h^2 [\overline{w}_i]_r \\ \overline{y}(x_0; r) = [\overline{\alpha}]_r, \overline{y}(x_n; r) = [\overline{\beta}]_r \\ i = 1, 2, \dots, n \end{array} \right. \tag{19}$$

Now from the above equations we can construct the following linear system in matrix form $AY=B$ as follows:

$$[\underline{Y}]_r = \begin{bmatrix} [\underline{y}_0]_r \\ [\underline{y}_1]_r \\ \vdots \\ [\underline{y}_{n-1}]_r \\ [\underline{y}_n]_r \end{bmatrix}, [\overline{Y}]_r = \begin{bmatrix} [\overline{y}_0]_r \\ [\overline{y}_1]_r \\ \vdots \\ [\overline{y}_{n-1}]_r \\ [\overline{y}_n]_r \end{bmatrix}, [\underline{A}]_r = \begin{bmatrix} d_1 & [\underline{e}_1]_r & & & \\ [\underline{c}_2]_r & \cdot & \cdot & & \\ & \vdots & \vdots & \cdot & \\ & \cdot & \cdot & \cdot & [\underline{e}_{n-1}]_r \\ & & [\underline{c}_n]_r & d_n & \end{bmatrix}$$

$$[\overline{A}]_r = \begin{bmatrix} d_1 & [\overline{e}_1]_r & & & \\ [\overline{c}_2]_r & \cdot & \cdot & & \\ & \vdots & \vdots & \cdot & \\ & \cdot & \cdot & \cdot & [\overline{e}_{n-1}]_r \\ & & [\overline{c}_n]_r & d_n & \end{bmatrix}$$

$$[\underline{B}]_r = h^2 \begin{bmatrix} [\underline{w}_0]_r \\ [\underline{w}_1]_r \\ \vdots \\ [\underline{w}_{n-1}]_r \\ [\underline{w}_n]_r \end{bmatrix} - \begin{bmatrix} [\underline{c}_2 \underline{\alpha}]_r \\ 0 \\ \vdots \\ 0 \\ [\underline{e}_n \underline{\beta}]_r \end{bmatrix}, [\overline{B}]_r = h^2 \begin{bmatrix} [\overline{w}_0]_r \\ [\overline{w}_1]_r \\ \vdots \\ [\overline{w}_{n-1}]_r \\ [\overline{w}_n]_r \end{bmatrix} - \begin{bmatrix} [\overline{c}_2 \overline{\alpha}]_r \\ 0 \\ \vdots \\ 0 \\ [\overline{e}_n \overline{\beta}]_r \end{bmatrix}$$

where $[\underline{A}, \overline{A}]_r$ are $m \times s$ fuzzy matrices such that for all $r \in [0,1]$ we have

$$[\underline{c}_i]_r = - \left(1 + \frac{h}{2} [\underline{p}_i]_r\right), [\overline{c}_i]_r = - \left(1 + \frac{h}{2} [\overline{p}_i]_r\right), d_i = (2 + h^2 q_i), \text{ and}$$

$[\underline{e}_i]_r = - \left(1 - \frac{h}{2} [\underline{p}_i]_r\right), [\overline{e}_i]_r = - \left(1 - \frac{h}{2} [\overline{p}_i]_r\right)$ for $i = 1, 2, \dots, n$. We prove that there is a unique $\{\tilde{y}_{i+1}\}_{i=0}^n$ by showing that the tri-diagonal matrix $[\tilde{A}]_r$ is strictly diagonally dominant and hence nonsingular.

Theorem 6.1: Suppose that the functions $\tilde{p}(x; r)$ and $q(x)$ satisfy (18) and the step size,

$$h_1 < \inf \left\{ \min \left\{ \left(2 / [\underline{p}]_r, [\underline{p}]_r / q^* \right), \left(2 / [\overline{p}]_r, [\overline{p}]_r / q^* \right) \right\} \right\}$$

$$h_2 < \inf \left\{ \max \left\{ \left(2 / [\underline{p}]_r, [\underline{p}]_r / q^* \right), \left(2 / [\overline{p}]_r, [\overline{p}]_r / q^* \right) \right\} \right\}$$

then the matrix $[\tilde{A}]_r$ is strictly diagonally dominant and hence nonsingular.

Proof

If

$$h_1 < \inf \left\{ \min \left\{ \left(2 / [\underline{p}]_r, [\underline{p}]_r / q^* \right), \left(2 / [\overline{p}]_r, [\overline{p}]_r / q^* \right) \right\} \right\}$$

, then

$$|[\underline{c}_i]_r| = \left(1 + \frac{h}{2} [\underline{p}_i]_r \right), \quad |[\underline{e}_i]_r| = \left(1 - \frac{h}{2} [\underline{p}_i]_r \right)$$

and

$$|[\underline{c}_i]_r| + |[\underline{e}_i]_r| = 2 < d_i \quad i = 1, 2, \dots, n$$

then $|[\underline{e}_1]_r| < d_1, |[\underline{c}_n]_r| < d_n$.

Similarly if

$$h_2 < \inf \left\{ \max \left\{ \left(2 / [\underline{p}]_r, [\underline{p}]_r / q^* \right), \left(2 / [\overline{p}]_r, [\overline{p}]_r / q^* \right) \right\} \right\}$$

, then

$$|[\overline{c}_i]_r| = \left(1 + \frac{h}{2} [\underline{p}_i]_r \right), \quad |[\overline{e}_i]_r| = \left(1 - \frac{h}{2} [\underline{p}_i]_r \right)$$

and

$$|[\overline{c}_i]_r| + |[\overline{e}_i]_r| = 2 < d_i \quad i = 1, 2, \dots, n$$

then $|[\overline{e}_1]_r| < d_1, |[\overline{c}_n]_r| < d_n$.

Since $h_1 = h_2 = h$, we conclude that

$$|[\tilde{c}_i]_r| + |[\tilde{e}_i]_r| = 2 < d_i \quad i = 1, 2, \dots, n$$

then $|[\tilde{e}_1]_r| < d_1, |[\tilde{c}_n]_r| < d_n$.

which completes the proof. Thus Eq. (17) has unique solution.

6.2 Numerical Examples

Consider the following liner Emden- Fowler FTPBVP

$$\tilde{y}''(x) - \frac{\tilde{\eta}}{x} \tilde{y}'(x) = \frac{2}{x^2} \tilde{y}(x) , x \in [1,2] \tag{20}$$

$$y(1) = \tilde{\sigma}, y(2) = \tilde{\sigma}$$

Where $\tilde{\sigma}$ is a triangular fuzzy number [18] having r -level set $[0.8+0.2r, 1.2-0.2r]$ and $\tilde{\eta} = [r + 1, 3 - r]$ for all $r \in [0,1]$.

Using Maple16 package to obtain the exact solution of this problem such that:

$$\underline{Y}(x; r) = \left(\left(\left(\left(1 - 2 \frac{1}{2} r + 1 + \frac{1}{2} \sqrt{r^2 + 4r - 4} \right) t \frac{1}{2} r + 1 - \frac{1}{2} \sqrt{r^2 + 4r - 4} \right. \right. \right. \\ \left. \left. \left. + t \frac{1}{2} r + 1 + \frac{1}{2} \sqrt{r^2 + 4r - 4} \left(2 \frac{1}{2} r + 1 - \frac{1}{2} \sqrt{r^2 + 4r - 4} - 1 \right) \right) \right) \right) \\ (4 + r) \Big/ \left(-5 2 \frac{1}{2} r + 1 + \frac{1}{2} \sqrt{r^2 + 4r - 4} \right. \\ \left. + 5 2 \frac{1}{2} r + 1 - \frac{1}{2} \sqrt{r^2 + 4r - 4} \right)$$

$$\bar{Y}(x; r) = - \left(\left(\left(\left(1 - 2 \frac{2 - \frac{1}{2} r + \frac{1}{2} \sqrt{8 - 8r + r^2}} \right) t \frac{2 - \frac{1}{2} r - \frac{1}{2} \sqrt{8 - 8r + r^2}} \right. \right. \right. \\ \left. \left. \left. + t \frac{2 - \frac{1}{2} r + \frac{1}{2} \sqrt{8 - 8r + r^2}} \left(2 \frac{2 - \frac{1}{2} r - \frac{1}{2} \sqrt{8 - 8r + r^2}} - 1 \right) \right) \right) \right) \\ (-6 + r) \Big/ \left(-5 2 \frac{2 - \frac{1}{2} r + \frac{1}{2} \sqrt{8 - 8r + r^2}} \right. \\ \left. + 5 2 \frac{2 - \frac{1}{2} r - \frac{1}{2} \sqrt{8 - 8r + r^2}} \right)$$

$\forall r \in [0,1]$.

As in the previous section it's easy to apply finite difference formula (15) on Eq. (20). For simplicity we only need one point of $1 \leq x_i \leq 2$ to shows the exact and the numerical results for the approximate difference solution for the lower and upper bounds of Eq. (20) with step size $h=1/300$ and $x=1.7$ are given in the following table below:

Table 1: Difference approximate solution $\underline{y}(1.7; r)$ and $\bar{y}(1.7; r)$ of Eq. (20) at $h=1/100$

r	$\underline{y}(1.7; r)$	$\bar{y}(1.7; r)$
0	0.882872593700	1.325395523000
0.25	0.938586144800	1.270487846000
0.5	0.994138712700	1.215446825000
0.75	1.049615481000	1.160280731000
1	1.104999989000	1.104999989000

From Table (1) one can see that the numerical results are satisfies the convex triangular fuzzy number, and for plotting these solutions with exact one we need to plot for each point x such that $0 \leq r \leq 1$. For simplest illustration we plot the numerical and exact solution when $x=1.7$ as shows in the following figure:

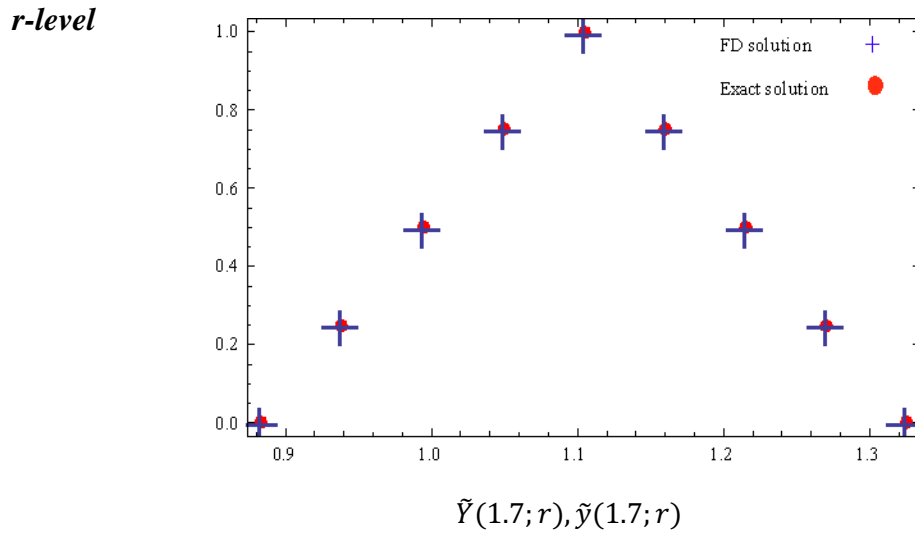


Figure 1: Exact and FD solution $\tilde{y}(1.7; r)$ of Eq. (20) when $h=1/300$

For simplest illustration of finite difference method in fuzzy environment of Eq. (20) we solved this problem at $r = 0.5$ with step size $h = 1/300$ for $1 \leq x_i \leq 2, i=0, 2 \dots n$ as shows in next figures and table.:

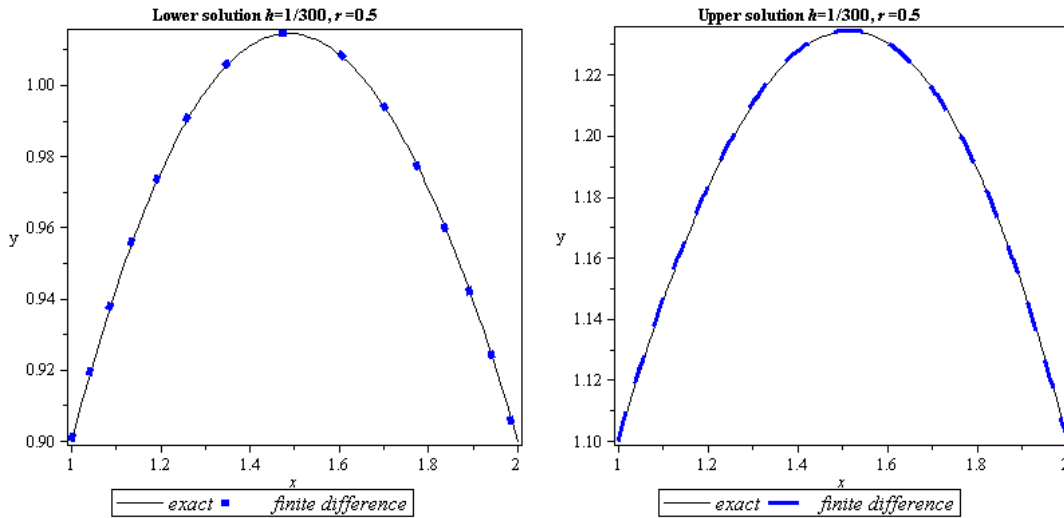


Figure 2: Difference approximate solution $\underline{y}(x; 0.5)$ and $\bar{y}(x; 0.5)$

Eq. (20) when $h=1/300$

The next table shows the absolute errors $[\tilde{E}_h]_r = [\underline{E}_h, \bar{E}_h]_r$ for Eq. (20) for all $x \in [0,1]$ as such that

Table 2: Absolute error of Eq. (20) at $h=1/300$ and $r=0.5$.

x	$[E_{1/300}]_{0.5}$	$[\bar{E}_{1/300}]_{0.5}$
1	0	0
1.2	2.30918×10^{-7}	1.39529×10^{-7}
1.4	3.27059×10^{-7}	1.96330×10^{-7}
1.6	1.81888×10^{-7}	3.84815×10^{-8}
1.8	9.09440×10^{-7}	3.57208×10^{-8}
2	0	0

7. Conclusions

In this paper, Finite Difference Method (FDM) has been successfully introduced and applied to solve Two Point Fuzzy Boundary Value Problems involving linear Emden Folwer equation. Numerical examples including linear and nonlinear fuzzy initial value problems show the efficiency of implemented numerical method. The convergence and uniqueness of the difference approximation have been presented and proved.

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