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# On Some New Generalized Difference Sequence Spaces Defined By A Modulus Function

**Tanweer Jalal**

Department of Mathematics, National Institute of Technology, Hazratbal, Srinagar-190006
*tjalal@rediffmail.com*

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***Abstract***

In this paper we introduces the new generalized difference sequences spaces $\left[\hat{V},\lambda,f,p\right]_0\left(\Delta_u^r,E\right)$, $\left[\hat{V},\lambda,f,p\right]_1\left(\Delta_u^r,E\right)$ ,$\left[\hat{V},\lambda,f,p\right]_\infty\left(\Delta_u^r,E\right)$ , $\hat{S}_\lambda\left(\Delta_u^r,E\right)$ and $\hat{S}_{\lambda_0}\left(\Delta_u^r,E\right)$ (where $E$ is any Banach space) which arise from the notion of generalized de la Vallée-Poussin means and the concept of modulus function. We also give some inclusion relations between these spaces.

**Keywords:** Difference sequence spaces, modulus function, paranorm, statistical convergence.

## 1. Introduction

Let ω be that set of all sequences real or complex numbers and $l_\infty$, c and $c_0$ be respectively the Banach spaces of bounded, convergent and null sequences x = ( $x_k$ ) with the usual norm $\|x\| = \sup_k |x_k|$, where k ϵ N = 1.2.., the set of positive integers. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$.

The generalized de la Vall$e'$e-Poussin means of a sequence x is defined by

$$t_n(x) = \frac{1}{\lambda_n}\sum_{k\in I_n} x_k \quad , \quad \text{where } I_n = [n-\lambda_n+1,\ n] \text{ for n = 1,2, . . .}$$

A sequence $x = (x_k)$ is said to be $(V,\lambda)$-summable to a number $l$ [ 8 ] if $t_n(x)\to l$ as $n\to\infty$. If $\lambda_n$= n , then $(V,\lambda)$—summability and strong $(V,\lambda)$—summability are reduced to $(C,1)-$ summability and $[C,1]$ –summability

Kizmaz [ 7 ] defined the difference sequence spaces

$$X(\Delta) = \{x = (x_k) : \Delta x \in X\}$$

for $X = l_\infty, c \; and \; c_0$ where $\Delta x = (x_k - x_{k+1})$. Then Et and Colak [ 3 ] generalized the above sequence spaces as:

$$X(\Delta^r) = \{x = (x_k) : \Delta^r x \in X\},$$

for $X = l_\infty, c \; and \; c_0$ where $r \in N$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^r x = (\Delta^r x_k - \Delta^r x_{k+1})$ and so $\Delta^r x_k = \sum_{v=0}^{k} \binom{r}{v} x_{k+v}$.

Later on the difference sequence spaces have been studied by Malkowsky and Parashar [12], Et and Basarir [ 2 ] and others.

The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value. Let X be a linear space. A function $p : X \to R$ is called a paranorm, if

$(P.1)$ $p(0) \geq 0$

$(P.2)$ $p(x) \geq 0$ for all $x \in X$

$(P.3)$ $p(-x) = p(x)$ for all $x \in X$

$(P.4)$ $p(x+y) = p(x) + p(y)$ for all $x, y \in X$ ( triangle inequality )

$(P.5)$ If $(\lambda_n)$ is a senquence of scalers with $\lambda_n \to \lambda \ (n\to\infty)$ and $(x_n)$ is a sequence of vectors with $p(x_n - x) \to 0 \ (n\to\infty)$, then with $p(\lambda_n x_n - \lambda x) \to 0 \ (n\to\infty)$, (continuity of multiplication of vectors).

A paranorm $p$ for which $p(x) = 0$ implies x = 0 is called total. It is well known that the metric of any linear metric space is given by some total paranorm ([16],Theorem 10.4.2 Page 183).

Following Ruckle [14] and Maddox [11], a modulus function $f$ is a function from $[0,\infty)$ to $[0,\infty)$ such that

( i ) $f(x)=0$ if and only if $x=0$, ( ii ) $f(x+y)\le f(x)+f(y)$ for all $x,y\ge 0$,

( iii ) $f$ is increasing, ( iv ) $f$ is continuous from right at $x=0$.

The following inequality will be used throughout this paper. Let $p=(p_k)$ be a sequence of positive real numbers with $0<p_k\le \sup_k p_k=H$ and let $D=\max(1,2^{H-1})$. For $a_k, b_k \in C$ the set of complex numbers and for all $k\in N$, we have ( see [ 10 ])

$$|a_k+b_k|^{p_k}\le D\left\{|a_k|^{p_k}+|b_k|^{p_k}\right\} \tag{1}$$

## 2. Some New Sequence Spaces Defined By Modulus Function

In this section we prove some results involving the sequence spaces $\left[\hat{V},\lambda,f,p\right]_0(\Delta_u^r,E)$, $\left[\hat{V},\lambda,f,p\right]_1(\Delta_u^r,E)$ and $\left[\hat{V},\lambda,f,p\right]_\infty(\Delta_u^r,E)$.

**Definition 2.1** : Let E be a Banach space. We define $\omega(E)$ to be the vector space of all E-valued sequences that is $\omega(E)=\{x=(x_k):x_k\in E\}$. Let $f$ be a modulus function and $p=(p_k)$ be any sequence of strictly positive real numbers. We define the following sequence sets

$$\left[\hat{V},\lambda,f,p\right]_1(\Delta_u^r,E)=\left\{x\in\omega(E):\lim_n\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta_u^r x_{k+m}-l\right\|\right)\right]^{p_k}=0,\ \textit{for some } l, \textit{uniformly in } m\right\},$$

$$\left[\hat{V},\lambda,f,p\right]_0(\Delta_u^r,E)=\left\{x\in\omega(E):\lim_n\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta_u^r x_{k+m}\right\|\right)\right]^{p_k}=0, \textit{uniformly in } m\right\},$$

$$\left[\hat{V},\lambda,f,p\right]_\infty(\Delta_u^r,E)=\left\{x\in\omega(E):\sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta_u^r x_k\right\|\right)\right]^{p_k}<\infty\right\}.$$

If $x\in\left[\hat{V},\lambda,f,p\right]_1(\Delta_u^r,E)$ then we write as $x_k\to l\left[\hat{V},\lambda,f,p\right]_1(\Delta_u^r,E)$ and $l$ will be called $\lambda_E-\textit{difference}$ limit of $x$ with respect to the modulus $f$.

Throughout the paper Z will denote any of the notation 0 , 1, $\infty$. In case $f(x)=x, p_k = 1$ for all $k \in N$ we shall write $\left[\hat{V}, \lambda\right]_Z (\Delta_u^r, E)$, and $\left[\hat{V}, \lambda, f\right]_Z (\Delta_u^r, E)$ instead of $\left[\hat{V}, \lambda, f, p\right]_Z (\Delta_u^r, E)$ respectively.

**Theorem 2.2.** Let the sequence $(p_k)$ be bounded. Then the sequence spaces $\left[\hat{V}, \lambda, f, p\right]_z (\Delta_u^r, E)$ are linear.

**Proof :** We shall prove it for $\left[\hat{V}, \lambda, f, p\right]_0 (\Delta_u^r, E)$. The others can be proved in the same manner.

Let $x, y \in \left[\hat{V}, \lambda, f, p\right]_0 (\Delta_u^r, E)$ and $\beta, \mu \in C$. Then there exists positive numbers $M_\beta$ *and* $N_\mu$ such that $|\beta| \le M_\beta$ *and* $|\mu| \le N_\mu$. Since $f$ is sub additive and $\Delta_u^r$ is linear we have

$$\frac{1}{\lambda_n} \sum_{k \notin I_n} \left[ f\left( \left\| \Delta_u^r (\beta x_{k+m} + \mu y_{k+m}) \right\| \right) \right]^{p_k}$$

$$\le \frac{1}{\lambda_n} \sum_{k \notin I_n} \left[ f\left( |\beta| \left\| \Delta_u^r x_{k+m} \right\| \right) + f\left( |\mu| \left\| \Delta_u^r y_{k+m} \right\| \right) \right]^{p_k}$$

$$\le D (M_\beta)^H \frac{1}{\lambda_n} \sum_{k \notin I_n} \left[ f\left( \left\| \Delta_u^r x_{k+m} \right\| \right) \right]^{p_k} + D \frac{1}{\lambda_n} (N_\mu)^H \sum_{k \in I_n} \left[ f\left( \left\| \Delta_u^r y_{k+m} \right\| \right) \right]^{p_k} \to 0 \quad \text{as}$$

$n \to \infty$, uniformly in $m$. This proves that $\left[\hat{V}, \lambda, f, p\right]_0 (\Delta_u^r, E)$ is a linear space.

**Theorem 2.3.** Let $f$ be a modulus function, then

$$\left[\hat{V}, \lambda, f, p\right]_0 (\Delta_u^r, E) \subset \left[\hat{V}, \lambda, f, p\right]_1 (\Delta_u^r, E) \subset \left[\hat{V}, \lambda, f, p\right]_\infty (\Delta_u^r, E).$$

**Proof:** The first inclusion is obvious. We establish the second inclusion.

Let $x \in \left[\hat{V}, \lambda, f, p\right]_1 (\Delta_u^r, E)$. By definition of $f$ we have for all $m \in N$ ,

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta_u^r x_{k+m}\right\|\right)\right]^{p_k} = \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta_u^r x_{k+m} - l + l\right\|\right)\right]^{p_k}$$

$$\leq D\,\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta_u^r x_{k+m} - l\right\|\right)\right]^{p_k} + D\,\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| l\right\|\right)\right]^{p_k}.$$

There exists a positive integer $K_l$ such that $\|l\| \leq K_l$. Hence we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta_u^r x_{k+m}\right\|\right)\right]^{p_k} \leq \frac{D}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta_u^r x_{k+m} - l\right\|\right)\right]^{p_k} + \frac{D}{\lambda_n}\left[K_l\, f(1)\right]^{H}\lambda_n.$$

Since $x \in \left[\hat{V}, \lambda, f, p\right]_0 \left(\Delta_u^r, E\right)$ we have $x \in \left[\hat{V}, \lambda, f, p\right]_\infty \left(\Delta_u^r, E\right)$ and this completes the proof.

**Theorem 2.4.** $\left[\hat{V}, \lambda, f, p\right]_0 \left(\Delta_u^r, E\right)$ is a paranormed (need not be total paranormed ) space with

$$g_\Delta(x) = \sup_{m,n}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta_u^r x_{k+m}\right\|\right)\right]^{p_k}\right)^{1/M} \quad \textit{where}\ \ M = \max\left(1, \sup_k p_k\right).$$

**Proof:** From Theorem 2. 3 , for each $x \in \left[\hat{V}, \lambda, f, p\right]_0 \left(\Delta_u^r, E\right)$, $g_\Delta(x)$ exists . Clearly $g_\Delta(x) = g_\Delta(-x)$. Furthermore $g_\Delta(x) = 0$ implies $\Delta^r x_k = 0 \Rightarrow x = 0$. Since $f(x) = 0$, we get $g_\Delta(x) = 0$ *for* $x = 0$. Since $p_k / M \leq 1$ *and* $M > 1$, using the Minkowski's inequality and definition of $f$ , for each n , we have

$$g_\Delta(x+y) = \sup_{m,n}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta_u^r x_{k+m} + \Delta_u^r y_{k+m}\right\|\right)\right]^{p_k}\right)^{1/M}$$

$$\leq \sup_{m,n}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta_u^r x_{k+m}\right\|\right) + f\left(\left\| \Delta_u^r y_{k+m}\right\|\right)\right]^{p_k}\right)^{1/M}$$

$$\leq \sup_{m,n}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta_u^r x_{k+m}\right\|\right)\right]^{p_k}\right)^{1/M} + \sup_{m,n}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta_u^r y_{k+m}\right\|\right)\right]^{p_k}\right)^{1/M}$$

$$\leq g_\Delta(x) + g_\Delta(y).$$

Hence $g_{\Delta}(x)$ is sub additive. Finally, to check the continuity of multiplication, let us take any complex number $\beta$. By definition of $f$ we have

$$g_{\Delta}(\beta x) = \sup_{m,n}\left( \frac{1}{\lambda_n} \sum_{k\in I_n} \left[ f\left( \left\| \Delta_u^r(\beta x_{k+m}) \right\| \right)^{p_k} \right] \right)^{1/M} \leq K_{\beta}^{\frac{H}{M}} g_{\Delta}(x)$$

where $K_{\beta}$ is a positive integer such that $|\beta|.< K_{\beta}$. Now, let $\beta \to 0$ for any fixed $x$ with

$g_{\Delta}(x) \neq 0$. By definition of $f$ for $|\beta| < 1$, we have

$$\frac{1}{\lambda_n} \sum_{k\in i_n} \left[ f\left( \left\| \beta \Delta_u^r x_{k+m} \right\| \right) \right]^{p_k} < \varepsilon, \text{ for } n > n_0(\varepsilon). \qquad (2)$$

Also, for all n with $1 \leq n \leq n_0$, and for all m , taking $\beta$ small enough , since $f$ is continuous we have

$$\frac{1}{\lambda_n} \sum_{k\in I_n} \left[ f\left( \left\| \beta \Delta_u^r x_{k+m} \right\| \right) \right]^{p_k} < \varepsilon . \qquad (3)$$

( 2 ) and ( 3 ) together imply that $g_{\Delta}(\beta x) \to 0 \text{ as } \beta \to 0$.

**Theorem 2. 5 .** If $r \geq 1$, then the inclusion $\left[ \hat{V}, \lambda, f \right]_Z \left( \Delta_u^{r-1}, E \right) \subset \left[ V, \lambda, f \right]_Z \left( \Delta_u^r, E \right)$ is strict . In general $\left[ \hat{V}, \lambda, f \right]_Z \left( \Delta_u^i, E \right) \subset \left[ \hat{V}, \lambda, f \right]_Z \left( \Delta_u^r, E \right)$ for all $i = 1, 2, ..., r-1$ and the inclusion is strict.

**Proof :** We give the proof for $Z = \infty$ only . For $Z = 0$ *and* $Z = 1$, the proof is similar.

Let $x \in \left[ \hat{V}, \lambda, f, \right]_{\infty} \left( \Delta_u^r, E \right)$. Then we have

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k\in I_n} \left[ f\left( \left\| \Delta_u^r x_{k+m} \right\| \right) \right]^{p_k} < \infty .$$

By definition of $f$ , we have

$$\frac{1}{\lambda_n} \sum_{k\in I_n} \left[ f\left( \left\| \Delta_u^r x_{k+m} \right\| \right) \right] \leq$$

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\|\Delta_u^{r-1}x_{k+m}\|\right)\right]+\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\|\Delta_u^{r}x_{k+m+1}\|\right)\right]<\infty .$$

*Thus* $\left[\hat{V},\lambda,f\right]_\infty\left(\Delta_u^{r-1},E\right)\subset\left[\hat{V},\lambda,f\right]_\infty\left(\Delta_u^{r},E\right)$. Proceeding in this way one can show that $\left[\hat{V},\lambda,f\right]_\infty\left(\Delta_u^{i},E\right)\subset\left[\hat{V},\lambda,f\right]_\infty\left(\Delta_u^{r},E\right)$ for $i=1,2,3,...,r-1$.

*Let* $E=C$ *, and* $\lambda_n=n$ for each $n\in N$ .Then the sequence $x=\left(k^r\right)$, belongs to $\left[\hat{V},\lambda,f\right]_\infty\left(\Delta_u^{r},E\right)$, but does not belong to $\left[\hat{V},\lambda,f\right]_\infty\left(\Delta_u^{r-1},E\right)$, for $f(x)=x$.

(If $x=\left(k^r\right)$, *then* $\Delta_u^r x_k=(-1)^r r!$ *and* $\Delta_u^{r-1}x_k\ (-1)^{r+1}r!\left(k+\frac{(r-1)}{2}\right)$ for all $k\in N$).

The proof of the following result is a routine work.

**Theorem 2.6.** $\left[\hat{V},\lambda,f,p\right]_1\left(\Delta_u^{r-1},E\right)\subset\left[\hat{V},\lambda,f\ p\right]_0\left(\Delta_u^{r},E\right)$.

**Theorem 2. 7.** Let $f$, $f_1,f_2$ be modulus functions. Then we have

(i) $\left[\hat{V},\lambda,f_1,p\right]_Z\left(\Delta_u^{r},E\right)\subset\left[\hat{V},\lambda,f\circ f_1,p\right]_Z\left(\Delta_u^{r},E\right)$,

(ii) $\left[\hat{V},\lambda,f_1,p\right]_Z\left(\Delta_u^{r},E\right)\cap\left[\hat{V},\lambda,f_2,p\right]_Z\left(\Delta_u^{r},E\right)\subset\left[\hat{V},\lambda,f_1+f_2,p\right]_Z\left(\Delta_u^{r},E\right)$.

**Proof:** We shall only prove ( i ) .Let $\varepsilon>0$ and choose $\delta$ *with* $0<\delta<1$ such that $f(t)<\varepsilon$ *for* $0\le t\le\delta$.Write $y_k=f_1\left(\|\Delta_u^r x_{k+m}\|\right)$ and consider

$$\sum_{k\in I_n}\left[f\left(y_{k+m}\right)\right]^{p_k}=\sum_1\left[f\left(y_{k+m}\right)\right]^{p_k}+\sum_2\left[f\left(y_{k+m}\right)\right]^{p_k}$$

Where the first summation is over $y_k\le\delta$ and the second summation is over $y_k>\delta$. Since $f$ is continuous , we have

$$\sum_1\left[f\left(y_{k+m}\right)\right]^{p_k}<\lambda_n\,\varepsilon^H \tag{4}$$

and for $y_k > \delta$, we use the fact that $y_k < \frac{y_k}{\delta} \le 1 + \frac{y_k}{\delta}$. By definition of $f$ we have for $y_k > \delta$, $f(y_k) < 2 f(1) \frac{y_k}{\delta}$. Hence

$$\frac{1}{\lambda_n} \sum_2 \left[ f\left( y_{k+m} \right) \right]^{p_k} \le \max \left( 1, \left( 2 f(1) \delta^{-1} \right)^H \right) \frac{1}{\lambda_n} \sum_{k \in I_n} y_{k+m} . \qquad (5)$$

From (4) and (5) , we obtain $\left[ \hat{V}, \lambda, f, p \right]_0 \left( \Delta_u^r, E \right) \subset \left[ \hat{V}, \lambda, f, p \right]_0 \left( \Delta_u^r, E \right)$.

The proof of (ii) follows from the following inequality :

$$\left\lfloor \left( f_1 + f_2 \right) \left( \left\| \Delta_u^r x_{k+m} \right\| \right) \right\rfloor^{p_k} \le D \left\lfloor f_1 \left( \left\| \Delta_u^r x_{k+m} \right\| \right) \right\rfloor^{p_k} + D \left\lfloor f_2 \left( \left\| \Delta_u^r x_{k+m} \right\| \right) \right\rfloor^{p_k} .$$

The following result is a consequence of Theorem 2.7. (i).

**Theorem 2. 8.** Let $f$ be a modulus function. Then

$$\left[ \hat{V}, \lambda, p \right]_Z \left( \Delta_u^r, E \right) \subset \left[ \hat{V}, \lambda, f, p \right]_Z \left( \Delta_u^r, E \right).$$

## 3. $\hat{S}_\lambda \left( \Delta_u^r, E \right)$ -Statistical Convergence

In this section we establish a relation between the sets $\hat{S}_\lambda \left( \Delta_u^r, E \right)$ and $\left[ \hat{V}, \lambda, f, p \right]_1 \left( \Delta_u^r, E \right)$ The idea of statistical convergence was introduced by Fast $[5]$, and latter studied by various authors like Fridy $[6]$, Kolk[8], Mursaleen [13], Et and Nurry [4], Savas [15], Arani et al. [1] and many others.

**Definition 2.9.** A sequence $x = (x_k)$ is said to be $\lambda_E^r - statistically\ convergent$ to a number $l$ if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left\| \Delta_u^r \; x_{k+m} - l \right\| \ge \varepsilon \right\} \right| = 0, \text{ uniformly in m.}$$

In this case we write $\hat{S}_\lambda \left( \Delta_u^r, E \right) - \lim x = l \ or \ x_k \to l \ \hat{S}_\lambda \left( \Delta_u^r, E \right)$. If $\lambda_n = n$, and $l = 0$ we shall write $\hat{S} \left( \Delta_u^r, E \right)$ $and$ $\hat{S}_{\lambda_0} \left( \Delta_u^r, E \right)$ instead of $\hat{S}_\lambda \left( \Delta_u^r, E \right)$.

We establish a relation between the sets $\hat{S}_\lambda\left(\Delta_u^r, E\right)$ and $\left[\hat{V}, \lambda, f, p\right]_1\left(\Delta_u^r, E\right)$.

**Theorem 2. 10.** The inclusion $\hat{S}_\lambda\left(\Delta_u^r, E\right) \subset \left[\hat{V}, \lambda, f, p\right]_1\left(\Delta_u^r, E\right)$ holds if and only if $f$ is bounded.

**Proof:** We assume that $f$ is bounded and $x \in \hat{S}_\lambda\left(\Delta_u^r, E\right)$. Then there exists a constant $M$ such that $f(x) \le M$ for all $x \ge 0$. Let s $\in > 0$ be given. We choose $\eta$ *and* $\delta > 0$ such that $M\delta + f(\eta) < \varepsilon$. Since $x \in \hat{S}_\lambda\left(\Delta_u^r, E\right)$, there are $l \in C$ *and* $n_0 = n_0(\eta, \gamma) \in N$

such that $\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left\|\Delta_u^r x_{k+m} - l\right\| \ge \eta\right\}\right| < \delta$, for all $n \ge n_0$ and for all $m$.

Therefore $\frac{1}{\lambda_n}\sum_{k\in I_n} f\left(\left\|\Delta_u^r x_{k+m} - l\right\|\right)^{p_k} =$

$$\frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ \left|\Delta_u^r x_{k+m} - l\right| \ge \eta}} f\left(\left\|\Delta_u^r x_{k+m} - l\right\|\right)^{p_k} + \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ \left|\Delta_u^r x_{k+m} - l\right| < \eta}} f\left(\left\|\Delta_u^r x_{k+m} - l\right\|\right)^{p_k}$$

$$\le M\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left\|\Delta_u^r x_{k+m} - l\right\| \ge \eta\right\}\right| + f(\eta)$$

$$< M\delta + f(\eta) < \varepsilon \text{ for all } n \ge n_0 \text{ and for all } m.$$

Hence $x \in \left[\hat{V}, \lambda, f, p\right]_1\left(\Delta^r, E\right)$.

Conversely we assume that $f$ is unbounded. Then there exists a positive sequence $(t_k)$ of positive numbers with $f(t_k) = k^2$, for $k = 1, 2, \ldots$. If we choose

$$\Delta_u^r x_i = \begin{cases} t_k & , i = k^2, i = 1, 2, \ldots\ldots \\ 0 & , \quad otherwise. \end{cases}$$

Then we have

$\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left\|\Delta_u^r x_{k+m}\right\|\geq\varepsilon\right\}\right|\leq\frac{\sqrt{\lambda_{n-1}}}{\lambda_n}$ for all $n \text{ and } m$, and so $x\in\hat{S}_\lambda\left(\Delta_u^r,E\right)$ but $x\notin\left[\hat{V},\lambda f,p\right]_1\left(\Delta_u^r,E\right)$ for $E=C$.

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