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Asymptotically Polynomial Type Solutions for Some 2-Dimensional Coupled Nonlinear ODEs

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Abstract

In this paper we have considered the following coupled system of nonlinear ordinary differential equations.

$$x_1^{n_1}(t)=f_1(t,x_2(t))$$

$$x_2^{n_2}(t)=f_2(t,x_1(t)) \qquad (1)$$

where f_1,f_2 are real valued functions on $[t_0,\infty)\times R$, $t\geq t_0>0$. We have given sufficient conditions on the nonlinear functions f_1,f_2, such that the solutions pair x_1,x_2 asymptotically behaves like a pair of real polynomials.

Keywords: Nonlinear Coupled Ordinary Differential Equations, Fixed-point Theorem, Assymptotically Polynomial like solutions

1. Introduction

There are many physical phenomena where systems of differential equations arise, in fact the fact that any two variables of the physical world are mutually dependent, makes us realize the importance and need for the study of these systems of differential equations. Some such studies can be found in [9]. The authors have worked on the existential analysis of certian generalized systems in [**Error! Reference source not found.**, **Error! Reference source not found.**, 6], in the recent past. Equally important is to find out the qualitative nature of solutions to these systems. In literature we find lot of work done on the qualitative behavior of solutions for ordinary differential equations. we can find a lot of work done in [3, 8] on the second order nonlinear differential equations. The same problem for higher order nonlinear differential equations was treated in [7]. We specifically mention [2], in which sufficient conditions for every solution of a n^{th} order nonlinear differential equation to be asymptotic to a real polynomial of at most degree $n-1$ at ∞. In this paper we have extended these type of results to 2-dimensional systems of nonlinear coupled ordinary differential equations. This kind of work would be helpful in analyzing the systems where the coefficient matrices are anti-diagonal.

In this paper we investigated the solutions of the coupled system (1), which behave asymptotically at ∞ like real polynomials in t. We have given sufficient conditions for the solution pair x_1,x_2 to behave like real polynomial pair of at most degree m_1,m_2 respectively, where $1\leq m_1\leq n_1-1$, $1\leq m_2\leq n_2-1$. We mention here that the nonlinear terms in the system are explicitly dependent on only one variable, this gives a scope for further findings where these nonlinear terms could be dependent on both the vaiables.

2. Main Result

We investigate the solutions of (1) which are defined for large t i.e on the interval $[T,\infty)$, where $T\geq t_0$ may depend on the solution.

Before we prove our main result, we give some preliminaries which we use in the proof.

Schauder's Fixed Point Theorem

Let E be a Banach Space and X any nonempty convex and closed subset of E. If S is a Continuous mapping into itself and SX is relatively compact, then the mapping S has at least one fixed point.

Definition

A set of real-valued functions H defined on $[T,\infty)$ is called equiconvergent at ∞ if all functions in H are convergent in R at the point ∞ and for every $\varepsilon>0$, there exists $T^{'}\geq T$ such that for all $h\in H$

$$\left|h(t)-lim_{s\to\infty}h(s)\right|<\varepsilon$$

for all $t\geq T_0$

Compactness Criterion

Let H be a equicontinuous and uniformly bounded subset of the Banach space B([T,∞))(Banach space of all continuous and bounded real valued functions on $[T,\infty)$). If H is equiconvergent at ∞, then it is also relatively compact.

Note: The Banach space $B([T,\infty))$ is endowed with the sup-norm $\|.\|$

$\|f\| = \sup_{t\geq T}|f(t)|$ for $h \in B([T,\infty))$

This compactness criterion is a corollary of the Ascoli-Arzella Theorem and is an adaptation of a lemma due to Avramescu [1].

Theorem 1

Let m_1,m_2 be integers with $1\le m_1\le n_1-1$, $1\le m_2\le n_2$ and assume that

$$|f_1(t,z)|\le p_1(t)g\left(\frac{|z|}{t^{m_1}}\right)+q_1(t)$$

$$|f_2(t,z)|\le p_2(t)g\left(\frac{|z|}{t^{m_2}}\right)+q_2(t) \tag{2}$$

for all $(t,z)\in[t_0,\infty)\times R$,

where p and q are nonnegative continuous real-valued functions on $[t_0,\infty)$ such that

$$\int_{t_0}^{\infty} t^{n_1-1}p_1(t)dt<\infty$$

$$\int_{t_0}^{\infty} t^{n_1-1}q_1(t)dt<\infty$$

$$\int_{t_0}^{\infty} t^{n_2-1}p_2(t)dt<\infty$$

$$\int_{t_0}^{\infty} t^{n_2-1}q_2(t)dt<\infty \tag{3}$$

and g is a nonnegative continuous real-valued function on $[0,\infty)$ which is not identically zero.

Let $c_{10},c_{11},\dots,c_{1m_1}$ and $c_{20},c_{21},\dots,c_{2n_1}$ be real numbers and T be a point with $T\ge t_0$, and suppose that there exists positive constants K_1,K_2 such that

$$\int_T^{\infty}\frac{(s-T)^{n_1-1}}{(n_1-1)!}p_1(s)ds\sup\left\{g(z):0\le z\le\frac{K_2}{T^{m_2}}+\sum_{i=0}^{m}\frac{|c_{2i}|}{T^{m_2-i}}\right\}$$
$$+\int_T^{\infty}\frac{(s-T)^{n_1-1}}{(n_1-1)!}q_1(s)ds\le K_1 \tag{4}$$

and

$$\int_T^\infty \frac{(s-T)^{n_2-1}}{(n_2-1)!} p_2(s)ds \sup\left\{ g(z) : 0 \le z \le \frac{K_1}{T^{m_1}} + \sum_{i=0}^{m} \frac{|c_{1i}|}{T^{m_1-i}} \right\}$$

$$+\int_T^\infty \frac{(s-T)^{n_2-1}}{(n_2-1)!} q_2(s)ds \le K_2 \qquad (5)$$

Then the system (1) has a solution pair $\{x_1, x_2\}$ on the interval $[T,\infty]$ such that

$$x_1 = c_{10} + c_{11}t + \ldots + c_{1m_1} t^{m_1}$$

$$x_2 = c_{20} + c_{21}t + \ldots + c_{2m_2} t^{m_2} \qquad (6)$$

Proof: By substituting

$$y_1(t) = x_1(t) - \left(c_{10} + c_{11}t + \ldots + c_{1m_1} t^{m_1}\right)$$

$$y_2(t) = x_2(t) - \left(c_{20} + c_{21}t + \ldots + c_{2m_2} t^{m_2}\right)$$

the system (1) gets transformed in to

$$y_1^{n_1}(t) = f_1\left(t, y_2(t) + \sum_{i=0}^{m_2} c_{2i} t^i\right)$$

$$y_2^{n_2}(t) = f_2\left(t, y_1(t) + \sum_{i=0}^{m_1} c_{1i} t^i\right) \qquad (7)$$

Therefore we can clearly see that it is sufficient to prove that the system (7) has a solution pair y_1, y_2 on the interval $[T,\infty]$ with

$$\lim_{t\to\infty} y_1^{\varrho_1}(t) = 0$$

$$\lim_{t\to\infty} y_2^{\varrho_2}(t) = 0 \qquad (8)$$

where $\varrho_1 = 0,1,\ldots,n_1 - 1$ and $\varrho_2 = 0,1,\ldots,n_2 - 1$

Now consider the Banach Space $E = B([T,\infty])$ with the sup-norm $|.|$, and define

$$Y_1 = \{y_1 \in E : \|y_1\| \le K_1\}$$

$$Y_2 = \{y_2 \in E : \|y_2\| \le K_2\}$$

Clearly Y_1, Y_2 are non-empty closed convex subsets of E. Let y_1 and y_2 be arbitrary functions in Y_1 and Y_2 respectively. Then for every $t \geq T$

$$\frac{\left|y_1(t) + \sum_{i=0}^{m_1} c_{1i} t^i\right|}{t^{m_1}} \le \frac{|y_1(t)|}{t^{m_1}} + \sum_{i=0}^{m_1} \frac{|c_{1i}|}{t^{m_1-i}} \le \frac{K_1}{T^{m_1}} + \sum_{i=0}^{m_1} \frac{|c_{1i}|}{T^{m_1-i}}$$

and

$$\frac{\left|y_2(t) + \sum_{i=0}^{m_2} c_{2i} t^i\right|}{t^{m_2}} \le \frac{|y_2(t)|}{t^{m_2}} + \sum_{i=0}^{m_2} \frac{|c_{2i}|}{t^{m_2-i}} \le \frac{K_2}{T^{m_2}} + \sum_{i=0}^{m_2} \frac{|c_{2i}|}{T^{m_2-i}}$$

So

$$g\left(\frac{\left|y_1(t) + \sum_{i=0}^{m_1} c_{1i} t^i\right|}{t^{m_1}}\right) \le \tau_1$$

$$g\left(\frac{\left|y_2(t) + \sum_{i=0}^{m_2} c_{2i} t^i\right|}{t^{m_2}}\right) \le \tau_2$$

where τ_1, τ_2 are defined as

$$\theta_1 = \theta_1(c_{10}, c_{11}, ..., c_{1m_1} : T : K_1) = \sup\left\{g(z) : 0 \le z \le \frac{K_1}{T^{m_1}} + \sum_{i=0}^{m_1} \frac{|c_{1i}|}{T^{m_1-1}}\right\}$$

$$\theta_2 = \theta_2(c_{20}, c_{21}, ..., c_{2m_2} : T : K_2) = \sup\left\{g(z) : 0 \le z \le \frac{K_2}{T^{m_2}} + \sum_{i=0}^{m_2} \frac{|c_{2i}|}{T^{m_2-1}}\right\}$$

From (2) we get

$$\left| f_1\left(t, y_2(t) + \sum_{i=0}^{m_2} c_{2i} t^i \right) \right| \leq \tau_2 p_1(t) + q_1(t)$$

$$\left| f_2\left(t, y_1(t) + \sum_{i=0}^{m_1} c_{1i} t^i \right) \right| \leq \tau_1 p_2(t) + q_2(t) \qquad (9)$$

for every $t \geq T$

Thus, from (3) we can conclude that

$$\int_T^\infty \frac{(s-T)^{n_1-1}}{(n_1-1)!} f_1\left(s, y_2(s) + \sum_{i=0}^{m_2} c_{2i} s^i \right) ds$$

$$\int_T^\infty \frac{(s-T)^{n_2-1}}{(n_2-1)!} f_2\left(s, y_1(s) + \sum_{i=0}^{m_1} c_{1i} s^i \right) ds$$

exist in R.

Now, by using (9) for every $t \geq T$ we get that

$$\left| \int_t^\infty \frac{(s-T)^{n_1-1}}{(n_1-1)!} f_1\left(s, y_2(s) + \sum_{i=0}^{m_2} c_{2i} s^i \right) ds \right|$$

$$\leq \tau_2 \int_T^\infty \frac{(s-T)^{n_1-1}}{(n_1-1)!} p_1(s) ds + \int_T^\infty \frac{(s-T)^{n_1-1}}{(n_1-1)!} q_1(s) ds$$

$$\left| \int_t^\infty \frac{(s-T)^{n_2-1}}{(n_2-1)!} f_2\left(s, y_1(s) + \sum_{i=0}^{m_1} c_{1i} s^i \right) ds \right|$$

$$\leq \tau_1 \int_T^\infty \frac{(s-T)^{n_2-1}}{(n_2-1)!} p_2(s) ds + \int_T^\infty \frac{(s-T)^{n_2-1}}{(n_2-1)!} q_2(s) ds$$

From (4) and (5) we get

$$\left| \int_t^\infty \frac{(s-T)^{n_1-1}}{(n_1-1)!} f_1\left(s, y_2(s) + \sum_{i=0}^{m_2} c_{2i} s^i \right) ds \right| \leq K_1$$

$$\left| \int_t^\infty \frac{(s-T)^{n_2-1}}{(n_2-1)!} f_2\left(s, y_1(s) + \sum_{i=0}^{m_1} c_{1i} s^i\right) ds \right| \le K_2 \tag{10}$$

for every $t \ge T$. As this is true for any pair y_1, y_2, we now define mappings S_1 and S_2 on Y_1 and Y_2 respectively as

$$\left(S_1 y_1\right)(t) = (-1)^{n_1} \int_t^\infty \frac{(s-T)^{n_1-1}}{(n_1-1)!} f_1\left(s, y_2(s) + \sum_{i=0}^{m_2} c_{2i} s^i\right) ds$$

with

$$y_2(s) = \int_s^\infty \frac{(r-T)^{n_2-1}}{(n_2-1)!} f_2\left(r, y_1(r) + \sum_{i=0}^{m_1} c_{1i} r^i\right) dr$$

and

$$\left(S_2 y_2\right)(t) = (-1)^{n_2} \int_t^\infty \frac{(s-T)^{n_2-1}}{(n_2-1)!} f_2\left(s, y_1(s) + \sum_{i=0}^{m_1} c_{1i} s^i\right) ds$$

with

$$y_1(s) = \int_s^\infty \frac{(r-T)^{n_1-1}}{(n_1-1)!} f_1\left(r, y_2(r) + \sum_{i=0}^{m_2} c_{2i} r^i\right) dr$$

for every $t \ge T$.

Clearly we can see that S_1 maps Y_1 into itself and are valid. Now we shall show that these mappings have fixed points using the Schauder's fixed point theorem. We will do this for S_1 and similar proof follows for S_2 also, which we exclude.

Since $S_1 Y_1 \subseteq Y_1$ and Y_1 is closed, convex, S_1 is uniformly bounded. As $t \to \infty$, $S_1 Y_1 \to 0$. So $S_1 Y_1$ is convergent in R. Moreover for some $t \ge T' > T$, we have

$$\left|S_1 y_1(t) - 0\right| = \left| \int_t^\infty \frac{(s-T)^{n_1-1}}{(n_1-1)!} f_1\left(s, y_2(s) + \sum_{i=0}^{m_2} c_{2i} s^i\right) ds \right|$$

$$\leq \int_t^\infty \frac{(s-T)^{n_1-1}}{(n_1-1)!}\left|f_1\left(s,y_2(s)+\sum_{i=0}^{m_2}c_{2i}s^i\right)\right|ds$$

$$\leq \tau_2\int_t^\infty \frac{(s-t)^{n_1-1}}{(n_1-1)!}p_1(s)ds+\int_t^\infty \frac{(s-t)^{n_1-1}}{(n_1-1)!}q_1(s)ds$$

$$\leq K_1-\tau_2\int_T^{T'} \frac{(s-t)^{n_1-1}}{(n_1-1)!}p_1(s)ds+\int_T^{T'} \frac{(s-t)^{n_1-1}}{(n_1-1)!}q_1(s)ds \tag{11}$$

So, by using (3) and suitably choosing T', we can easily see that S_1Y is equiconvergent at ∞.

Now by using (9) for any $y_1\in Y_1$ and for every t_1,t_2 with $T\leq t_1<t_2$, we get

$$\left|\int_{t_2}^\infty \frac{(s-t_2)^{n_1-1}}{(n_1-1)!}f_1\left(s,y_2(s)+\sum_{i=0}^{m_2}c_{2i}s^i\right)ds\right.$$

$$\left.-\int_{t_1}^\infty \frac{(s-t_1)^{n_1-1}}{(n_1-1)!}f_1\left(s,y_2(s)+\sum_{i=0}^{m_2}c_{2i}s^i\right)ds\right|$$

$$=\left|\int_{t_2}^\infty\left[\int_r^\infty \frac{(s-r)^{n_1-2}}{(n_1-2)!}f_1\left(s,y_2(s)+\sum_{i=0}^{m_2}c_{2i}s^i\right)ds\right]dr\right.$$

$$\left.-\int_{t_1}^\infty\left[\int_r^\infty \frac{(s-r)^{n_1-2}}{(n_1-2)!}f_1\left(s,y_2(s)+\sum_{i=0}^{m_2}c_{2i}s^i\right)ds\right]dr\right|$$

$$=\left|-\int_{t_1}^{t_2}\left[\int_r^\infty \frac{(s-r)^{n_1-2}}{(n_1-2)!}f_1\left(s,y_2(s)+\sum_{i=0}^{m_2}c_{2i}s^i\right)ds\right]dr\right|$$

$$\leq\int_{t_1}^{t_2}\left[\int_r^\infty \frac{(s-r)^{n_1-2}}{(n_1-2)!}\left|f_1\left(s,y_2(s)+\sum_{i=0}^{m_2}c_{2i}s^i\right)\right|ds\right]dr$$

$$\leq \tau_2 \int_{t_1}^{t_2} \left[\int_r^{\infty} \frac{(s-r)^{n_1-2}}{(n_1-2)!} p_1(s)ds \right] dr + \int_{t_1}^{t_2} \left[\int_r^{\infty} \frac{(s-r)^{n_1-2}}{(n_1-2)!} q_1(s)ds \right] dr$$

By using condition (3) we can always have a bound on the right hand side of the inequality, so $S_1 Y$ is equicontinuous. So by the given compactness criterion $S_1 Y$ is relatively compact.

Now we will show that the mapping S_1 is continuous. Let y_{1v} be an arbitrary sequence in Y_1, converging to y_1 under the norm defined before. From (9) we have

$$\left| f_1 \left(t, y_{1v}(t) + \sum_{i=0}^{m_2} c_{2i} t^i \right) \right| \leq \tau_2 p_1(t) + q_1(t)$$

for every $t \geq T$ and for all v

Now, by applying the Lebesgue dominated convergence theorem we get

$$lim_{v\to\infty} \int_t^{\infty} \frac{(s-t)^{n_1-1}}{(n_1-1)!} f_1 \left(s, y_{1v}(s) + \sum_{i=0}^{m_1} c_{2i} s^i \right) ds$$

$$= \int_t^{\infty} \frac{(s-t)^{n_1-1}}{(n_1-1)!} f_1 \left(s, y_1(s) + \sum_{i=0}^{m_1} c_{2i} s^i \right) ds$$

So we have shown the pointwise convergence i.e

$$\lim_{v\to\infty} \left(S_1 y_{1v} \right)(t) = \left(S_1 y_1 \right)(t)$$

Now, consider an arbitrary subsequence u_μ of $S_1 y_{1v}$. Since $S_1 Y$ is relatively compact, there exists a subsequence v_λ of u_μ and a v in E such that v_λ converges uniformly to v. So

$$\lim_{v\to\infty} \left(S_1 y_{1v} \right)(t) = \left(S_1 y_1 \right)(t) = v$$

for all $t \geq T$ under the sup-norm. So S_1 is continuous.

Thus we have shown that S_1 satisfies all the assumptions of Schauder's theorem, So S_1 has a fixed point $y_1 \in Y$ such that $S_1 y_1 = y_1$. That implies

$$y_1(t)=(-1)^{n_1}\int_t^{\infty}\frac{(s-t)^{n_1-1}}{(n_1-1)!}f_1\left(s,y_2(s)+\sum_{i=0}^{m_1}c_{2i}s^i\right)ds$$

So we can see that

$$y_1^{n_1}(t)=f_1\left(t,y_2(t)+\sum_{i=0}^{m_2}c_{2i}t^i\right)$$

for all $t\geq T$.

similarly we can show for the other function y_2 also. Consequently y_1,y_2 satisfy the transformed system (7). It is also easy to verify that y_1,y_2 satisfy the condition (8) This completes the proof of the theorem. □

Example: Consider the following system of equations

$$x_1^n(t)=a(t)\left|x_2(t)\right|^{\gamma} sgn\, x_2(t)$$

$$x_2^n(t)=b(t)\left|x_1(t)\right|^{\gamma} sgn\, x_1(t) \qquad (12)$$

where a and b are continuous real valued functions on $[0,\infty]$ and γ is a positive real number. Let m be an integer with $1\leq m\leq n-1$, and assume that

$$\int_0^{\infty} t^{n-1+m\gamma}\left|a(t)\right|dt<\infty$$

$$\int_0^{\infty} t^{n-1+m\gamma}\left|b(t)\right|dt<\infty \qquad (13)$$

and let $c_0,c_1,c_2,.....,c_m$ and $d_0,d_1,d_2,......,d_m$ be real numbers and T be a point with $T\geq 0$ and suppose that there exist positive constants K_1 and K_2 such that

$$[\int_T^{\infty}\frac{(s-T)^{n-1}}{(n-1)!}s^{m\gamma}\left|a(s)\right|ds](\frac{K_1}{T^m}+\sum_{i=0}^{m}\frac{\left|c_i\right|}{T^{m-1}})^{\gamma}\leq K_1$$

$$[\int_T^{\infty}\frac{(s-T)^{n-1}}{(n-1)!}s^{m\gamma}\left|b(s)\right|ds](\frac{K_2}{T^m}+\sum_{i=0}^{m}\frac{\left|d_i\right|}{T^{m-1}})^{\gamma}\leq K_2 \qquad (14)$$

Then by invoking the main result that we have proved earlier (12) has a pair of solutions x_1 and x_2 which asymptotically behave like m^{th} degree polynomials.

Remark: We note here that these results are simple and hold true only for a special case of coupled systems. The nature of conditions (2) needs to be discussed and analyzed so that more general non-linearities can be brought under the view of these results. The possibility of such an extension needs to be highlighted.

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