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On the inclusion graphs of S-acts

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Abstract

In this paper, we define the inclusion graph $\mathbb{Inc}(A)$ of an S-act A which is a graph whose vertices are non-trivial subacts of A and two distinct vertices B_1, B_2 are adjacent if $B_1 \subset B_2$ or $B_2 \subset B_1$. We investigate the relationship between the algebraic properties of an S-act A and the properties of the graph $\mathbb{Inc}(A)$. Some properties of $\mathbb{Inc}(A)$ including girth, diameter and connectivity are studied. We characterize some classes of graphs which are the inclusion graphs of S-acts. Finally, some results concerning the domination number of such graphs are given.

Keywords: S-Act, inclusion graph, diameter, girth, domination number.

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1. Introduction and preliminaries

The notion of an S-act over a monoid S is a fundamental concept in algebra, theoretical computer science and a variety of applications like automata theory and mathematical linguistics. Assigning graphs to algebraic structures is an approach to study algebraic properties via graph-theoretic properties. In this direction, many authors, e.g. [2, 3, 4, 7, 11, 12, 14], have been performed in connecting graph structures to various algebraic objects. Recently, inclusion graphs attached to rings, vector spaces and groups have been studied in [1, 8, 5]. Moreover, some works associating graphs to S-acts can be found in [6, 9, 13].

In this paper, we associate a graph $\mathbb{Inc}(A)$ to an S-act A, called the inclusion graph of A, whose vertices are non-trivial subacts of A in such a way that two distinct vertices B_1, B_2 are adjacent if $B_1 \subset B_2$ or $B_2 \subset B_1$. We investigate the relationship between the algebraic properties of an S-act A and the properties of the graph $\mathbb{Inc}(A)$. First we determine the girth and diameter of $\mathbb{Inc}(A)$. Then some classes of graphs which are the inclusion graphs of S-acts are characterized. Finally, we present some results dealing with the domination number of such graphs.

The following is a brief account of some basic definitions about S-acts and graphs.

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Throughout this paper, unless otherwise stated, S denotes a monoid with the identity 1. By a (left) S-act, we mean a non-empty set A on which S acts unitarily, that is, $(st)a = s(ta)$ and $1a = a$ for all $s, t \in S$ and $a \in A$. A (non-empty proper) subset B of A is called a (non-trivial) subact of A if $sb \in B$ for every $s \in S, b \in B$. The set of all non-trivial subacts of A is denoted by $\mathrm{Sub}(A)$. A non-empty subset I of S is said to be a left ideal of S if $st \in I$ for any $s \in S, t \in I$. Considering S as an S-act, any left ideal of S is a subact of S. An element $\theta \in A$ is said to be a zero element, if $s\theta = \theta$ for all $s \in S$. A simple S-act is the one with no non-trivial subact. A completely reducible S-act is one which is a disjoint union of simple subacts. For more information about S-acts and related notions, the reader is referred to [10].

Let G be a (simple) graph with a vertex set $V(G)$. By order of G, we mean the cardinality of $V(G)$ which is simply denoted by $|G|$. For any $u, v \in V(G)$, a u,v-path (or $u - v$) is a path with starting vertex u and ending vertex v. The distance between two vertices u, v, denoted by $d(u, v)$, is defined as the length of the shortest path joining u and v if it exists, and otherwise, $d(u, v) = \infty$. The diameter of G, denoted by $\mathrm{diam}(G)$, is the largest distance between pairs of vertices of G. The number of vertices adjacent to a vertex v is called the degree of v and denoted by $\deg(v)$. The girth of a graph is the length of its shortest cycle, and a graph with no cycle has infinite girth. A null graph is a graph with no edges. A graph is connected if there is a path between every two distinct vertices. A complete graph is a graph in which every pair of distinct vertices are adjacent. We denote the complete graph with n vertices by $K_n, n \in \mathbb{N}$. A path and a cycle of length n are denoted by P_n and C_n, respectively. Two graphs G_1, G_2 are isomorphic if and only if there exists a bijection from $V(G_1)$ to $V(G_2)$ preserving the adjacency and non-adjacency. For undefined terms and concepts about graphs, one may consult [15].

2. Main results

In this section we first determine the girth of the graph $\mathbb{Inc}(A)$ for an S-act A. Then we characterize those cycles which are inclusion graphs of some S-acts. Moreover, we study connectivity and diameter for the inclusion graphs. Finally, the domination number of such graphs is briefly studied.

Note that the inclusion graph for a simple S-act is undefined because it has no vertex. So we consider non-simple S-acts when dealing with their inclusion graphs throughout the paper.

Remark 2.1. It is clear that if A and B are isomorphic S-acts, then their graphs $\mathbb{Inc}(A)$ and $\mathbb{Inc}(B)$ are equivalent. The converse is not true in general. To see this, take the monoid $S = \{1, s\}$ where $s^2 = 1$. Consider two S-acts $A = \{a, b, c\}$ with trivial action and $B = \{a, b, c, d\}$ presented by the following action table:

	a	b	c	d
1	a	b	c	d
s	a	b	d	c

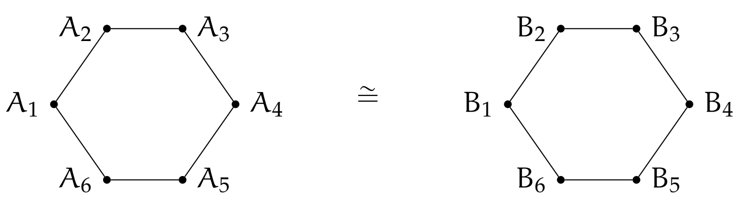
The non-trivial subacts of A and B are

$$A_1 = \{a\}, \quad A_2 = \{a, b\}, \quad A_3 = \{b\}, \quad A_4 = \{b, c\}, \quad A_5 = \{c\}, \quad A_6 = \{a, c\},$$

and

$$B_1 = \{a\}, \quad B_2 = \{a, b\}, \quad B_3 = \{b\}, \quad B_4 = \{b, c, d\}, \quad B_5 = \{c, d\}, \quad B_6 = \{a, c, d\},$$

respectively. Then $\mathbb{Inc}(A) \cong \mathbb{Inc}(B) \cong C_6$ whereas A and B are not isomorphic S-acts:



$\mathbb{Inc}(A)$ $\mathbb{Inc}(B)$

It is natural to ask whether a graph is isomorphic to the inclusion graph of an S-act. Here we consider complete graphs and cycles and characterize those ones satisfying this property.

We say that an S-act A is uniserial, if all of its subacts are totally ordered by inclusion, or equivalently, for any two (cyclic) subacts B and C of A, either $B \subseteq C$ or $C \subseteq B$. This generalizes the well-known notion of a uniserial module extensively studied in the literature.

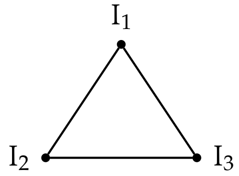
Clearly, for each S-act A, the graph $\mathbb{Inc}(A)$ is complete if and only if A is a uniserial S-act. The following example shows that every complete graph is the inclusion graph of a (uniserial) S-act. As we shall see, this is not the case for cycles in general.

Example 2.2.

(i) Consider the monogenic semigroup $S = \{s, s^2, s^3, \dots, s^{n+1}\}$, $s^{n+2} = s^{n+1}, n \in \mathbb{N}$. Then all distinct non-trivial left ideals of S form the chain

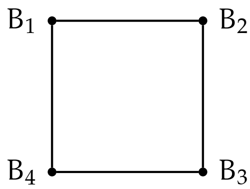
$$\langle s^n \rangle \subset \langle s^{n-1} \rangle \subset \langle s^{n-2} \rangle \subset \cdots \subset \langle s \rangle,$$

where $\langle s^k \rangle = \{s^i \mid k+1 \leqslant i \leqslant n+1\}$, for every $1 \leqslant k \leqslant n$. So S is a uniserial S-act and clearly the graph $\mathbb{Inc}(S)$ is isomorphic to the complete graph K_n. In particular, the inclusion graph of the monogenic semigroup $S = \{s, s^2, s^3, s^4\}, s^5 = s^4$, is isomorphic to the cycle C_3 with the vertices $I_1 = \{s^4\}$, $I_2 = \{s^3, s^4\}$, $I_3 = \{s^2, s^3, s^4\}$:



(ii) The non-trivial left ideals of the semigroup $S = (\mathbb{N}, +)$ are exactly the sets $n + \mathbb{N} = \{n + k \mid k \in \mathbb{N}\}$ where $n \in \mathbb{N}$. Further, $m + \mathbb{N} \subset n + \mathbb{N}$ if and only if $m > n$, for every $m, n \in \mathbb{N}$. Then S is a uniserial S-act and the graph $\mathbb{Inc}(S)$ is complete with countably infinite vertices.

(iii) The cycle C_4 is the inclusion graph of no S-act. Indeed, suppose that C_4 is the inclusion graph of an S-act A and B_1, B_2, B_3 and B_4 are all non-trivial subacts of A as the following:



With no loss of generality, assume that $B_1 \subset B_2$. Then $B_3 \subset B_2$, $B_3 \subset B_4$, $B_1 \subset B_4$. It is easily seen that $B_1 \cup B_3 \neq A, B_i$ for all $i \in \{1, 2, 3, 4\}$ which is a contradiction.

(iv) Let $A = \{a, b\}$ be an S-act with trivial action. Then $B_1 = \{a\}$ and $B_2 = \{b\}$ are only non-trivial subacts of A which are not adjacent and so $\text{girth}(\mathbb{Inc}(A)) = \infty$.

Theorem 2.3. *For each S-act* A, $\text{girth}(\mathbb{Inc}(A)) \in \{3, 6, \infty\}$.

Proof. First we show that for each $n > 6$, $\text{girth}(\mathbb{Inc}(A)) \neq n$. On the contrary, let $B_1 - B_2 - \cdots - B_n - B_1$ be the shortest cycle of order n. If $B_1 \cup B_4 \neq A$, then $B_1 - B_1 \cup B_4 - B_4$ is a path with shorter length between B_1 and B_4 which is a contradiction. So $B_1 \cup B_4 = A$, and by the same way, $B_1 \cap B_4 = \emptyset$, $B_1 \cup B_5 = A$ and $B_1 \cap B_5 = \emptyset$. Hence, $B_4 = B_5$ which is a contradiction. It remains to show that $\text{girth}(\mathbb{Inc}(A)) \neq 4, 5$. Let $B_1 - B_2 - B_3 - B_4 - B_1$, where $B_1 \subset B_2$, be the shortest cycle in $\mathbb{Inc}(A)$. Then $B_3 \subset B_2$, $B_3 \subset B_4$ and

$B_1 \subset B_4$. It is easily seen that $B_1 \cup B_3 \neq A, B_i$ for all $i \in \{1,2,3,4\}$. Then $B_1 - B_1 \cup B_3 - B_2 - B_1$ forms a cycle of order 3 which is a contradiction. If $B_1 - B_2 - B_3 - B_4 - B_5 - B_1$, where $B_1 \subset B_2$, is the shortest cycle in $\mathbb{Inc}(A)$, then $B_3 \subset B_2$, $B_3 \subset B_4$, $B_5 \subset B_4$ and $B_5 \subset B_1$. This implies that B_5 is adjacent to B_2 which is a contradiction. □

In light of Remark 2.1, Example 2.2 and Theorem 2.3, those cycles which are the inclusion graphs are fully characterized.

Corollary 2.4. *The cycle C_n is the inclusion graph of an S-act if and only if $n = 3$ or $n = 6$.*

In the following, we study the connectivity and diameter of the inclusion graphs.

Theorem 2.5. *Let A be an S-act. Then $\mathbb{Inc}(A)$ is disconnected if and only if it is a null graph with $|\mathbb{Inc}(A)| = 2$. Moreover, if $\mathbb{Inc}(A)$ is connected, then $\operatorname{diam}(\mathbb{Inc}(A)) \leqslant 3$.*

Proof. Suppose that $|\mathbb{Inc}(A)| \geqslant 3$. We show that there exists a path between B_1, B_2 for every two distinct non-trivial subacts B_1, B_2 of A. Let B_1 and B_2 be non-adjacent. If $B_1 \cap B_2 \neq \emptyset$ or $B_1 \cup B_2 \neq A$, then there exists a B_1, B_2-path. Now let $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = A$. Since $|\mathbb{Inc}(A)| \geqslant 3$, A contains a non-trivial subact B_3 with $B_3 \neq B_1, B_2$. If $B_1 \cap B_3 = \emptyset$ and $B_1 \cup B_3 = A$, then $B_2 = B_3$ which is a contradiction. So either $B_1 \cap B_3 \neq \emptyset$ or $B_1 \cup B_3 \neq A$. In the same way, either $B_2 \cap B_3 \neq \emptyset$ or $B_2 \cup B_3 \neq A$. We consider the following cases:

Case 1. Let $B_1 \cap B_3 \neq \emptyset$ and $B_2 \cap B_3 \neq \emptyset$. Note that $B_1 \cap B_3 \neq B_2$, $B_2 \cap B_3 \neq B_1$. Then

$$B_1 - B_1 \cap B_3 - B_3 - B_2 \cap B_3 - B_2,$$

is a B_1, B_2-path provided that $B_1 \cap B_3 \neq B_1, B_3$ and $B_2 \cap B_3 \neq B_2, B_3$. Otherwise, we get a path with shorter length between B_1, B_2. Hence, $d(B_1, B_2) \leqslant 4$.

Case 2. Let $B_1 \cap B_3 \neq \emptyset$ and $B_2 \cup B_3 \neq A$. We have $B_1 \cap B_3 \neq B_2$, $B_2 \cup B_3 \neq B_1$. Then

$$B_1 - B_1 \cap B_3 - B_3 - B_2 \cup B_3 - B_2,$$

is a B_1, B_2-path provided that $B_1 \cap B_3 \neq B_1, B_3$ and $B_2 \cup B_3 \neq B_2, B_3$. Otherwise, we get a path with shorter length between B_1, B_2. Hence, $d(B_1, B_2) \leqslant 4$.

Other cases have the same proof.

The converse is obvious. For the second part, first note that the above proof implicitly states that if $\mathbb{Inc}(A)$ is connected, then $\operatorname{diam}(\mathbb{Inc}(A)) \leqslant 4$. We claim that 4 is impossible for the diameter. Assume on the contrary that $\mathbb{Inc}(A)$ is a connected inclusion graph of an S-act A with $\operatorname{diam}(\mathbb{Inc}(A)) = 4$. Then there exist two distinct vertices B_1, B_5 in $\mathbb{Inc}(A)$ for which $B_1 - B_2 - B_3 - B_4 - B_5$ is the shortest B_1, B_5-path. It is clear to see that $B_1 \cup B_4 = A, B_1 \cap B_4 = \emptyset$, $B_1 \cup B_5 = A$ and $B_1 \cap B_5 = \emptyset$. Thus we get $B_4 = B_5$ which is a contradiction. □

Corollary 2.6. *Let A be an S-act with two zero elements and $|A| \geqslant 3$. Then $\mathbb{Inc}(A)$ is connected.*

Proof. If θ_1 and θ_2 are two zero elements of A, then the sets $\{\theta_1\}, \{\theta_2\}$ and $\{\theta_1, \theta_2\}$ are distinct non-trivial subacts of A. Hence, by Theorem 2.5, $\mathbb{Inc}(A)$ is connected. □

In what follows, we study the connectivity of the inclusion graphs of cyclic, free and cofree S-acts. Let us first recall some definitions from [10].

By a cyclic S-act, we mean an S-act A generated by an element $a \in A$, that is, $A = Sa$ where

$$Sa = \{sa \mid s \in S\}.$$

An S-act A is called free if it has a basis X, i.e., each element $a \in A$ is uniquely represented as $a = sx$ for some $s \in S$ and $x \in X$. In this case, $A \cong \coprod_{x \in X} S$. The dual categorical notion of free is the cofree S-act which is isomorphic to an S-act of the form X^S, the set of all maps from S to a non-empty set X, with the action given by $(sf)(t) = f(ts)$ for $s, t \in S$ and $f \in X^S$. The set X is called a cobasis for A.

Proposition 2.7. *Let A be an S-act. Then the following assertions hold:*

(i) *If A is cyclic, then $\mathbb{I}\mathrm{nc}(A)$ is connected. In particular, $\mathbb{I}\mathrm{nc}(S)$ is connected.*

(ii) *If A is a free S-act with a basis X where $|X| > 2$, then $\mathbb{I}\mathrm{nc}(A)$ is connected.*

(ii) *If A is a cofree S-act and $|A| \geqslant 3$, then $\mathbb{I}\mathrm{nc}(A)$ is connected.*

Proof.

(i) Consider a cyclic S-act A with disconnected inclusion graph. Using Theorem 2.5, A has only two non-trivial subacts, say B and C, such that $B \cup C = A$ and $B \cap C = \emptyset$. Clearly, B and C are simple subacts of A so that A is completely reducible. Note that a cyclic S-act is completely reducible if and only if it is simple (see [10, Lemma I.5.32]). This implies that A is simple which is a contradiction.

(ii) It follows from hypothesis that the number of non-trivial subacts of A is greater than 2. Hence, Theorem 2.5 gives the assertion.

(iii) Using the assumption, A can be considered as the S-act X^S for a cobasis X where $|X| > 1$. Since every constant map in A is a zero element and there exist exactly $|X|$ constant maps in A, A contains at least two zero elements and hence $\mathbb{I}\mathrm{nc}(A)$ is connected by Corollary 2.6. □

A non-trivial subact M of an S-act A is called minimal, if $B \subseteq M$ for some subact B of A implies that $B = M$. We denote the set of all minimal subacts of A by $\mathrm{Min}(A)$.

Remark 2.8. Let A be an S-act. If $\deg(M) < \infty$ for a minimal subact M of A, then the number of minimal subacts of A is finite. Indeed, if $M_1, M_2, M_3, \cdots$ be infinite minimal subacts of A other than M, then the infinite strict ascending chain

$$M \subset M \cup M_1 \subset M \cup M_1 \cup M_2 \subset \cdots,$$

gives that $\deg(M) = \infty$ which is a contradiction. Further, if $\mathbb{I}\mathrm{nc}(A)$ is complete, then A contains at most one minimal subact.

Theorem 2.9. *Let A be an S-act and $\mathbb{I}\mathrm{nc}(A)$ have no cycle. Then $\mathbb{I}\mathrm{nc}(A)$ is a null graph (with one or two vertices) or P_i where $i \in \{1, 2, 3, 4\}$.*

Proof. It follows from the assumption that A has a minimal subact. If M_1, M_2, M_3 are three distinct minimal subacts of A, then

$$M_1 - M_1 \cup M_2 - M_2 - M_2 \cup M_3 - M_3 - M_3 \cup M_1 - M_1,$$

is a cycle which is a contradiction. So $|\mathrm{Min}(A)| \leqslant 2$. The following cases may occur.

Case 1. Let A have only one minimal subact, say M. Then every subact of A contains M. We claim that $|\mathbb{I}\mathrm{nc}(A)| \leqslant 3$. On the contrary, let B_1, B_2, B_3 be another distinct non-trivial subacts of A. Since $\mathbb{I}\mathrm{nc}(A)$ has no cycle, $B_1 \cup B_3 = B_2 \cup B_3 = A$ and $B_1 \cap B_3 = B_2 \cap B_3 = M$ whence $B_1 = B_2$ which is a contradiction. Thus the graph $\mathbb{I}\mathrm{nc}(A)$ is one of the graphs: one-vertice graph, or the paths P_1 or P_2.

Case 2. Let A have two distinct minimal subacts, say M_1, M_2. If $M_1 \cup M_2 = A$, then A has no another non-trivial subact and $\mathbb{I}\mathrm{nc}(A)$ is a null graph with two distinct vertices. If $M_1 \cup M_2 \neq A$, then $\mathbb{I}\mathrm{nc}(A)$ contains at least the three vertices $M_1, M_2, M_1 \cup M_2$. we claim that $|\mathbb{I}\mathrm{nc}(A)| \leqslant 5$. Assume contrarily that B_1, B_2, B_3 are another distinct non-trivial subacts of A. We show that each B_i contains only one minimal. Otherwise, $M_1 \cup M_2 \subset B_i$ and then $M_1 - M_1 \cup M_2 - B_i - M_1$ is a cycle which is a contradiction. Moreover, if B_i and B_j intersect in a minimal subact as M_1, then $B_i \cup B_j \neq A$ because $M_2 \nsubseteq B_i \cup B_j$ and in this case the cycle $M_1 - B_i - B_i \cup B_j - M_1$ yields a contradiction. Therefore, each B_i contains only one minimal subact and each minimal subact is contained in only one B_i. This contradicts the number of B_i's. So, in addition to $M_1, M_2, M_1 \cup M_2$, $\mathbb{I}\mathrm{nc}(A)$ contains at most two another vertices. It is straightforward to see that $\mathbb{I}\mathrm{nc}(A)$ is one of the paths P_2 or P_3 or P_4. □

Here we study the domination number of the inclusion graphs and determine them for the graphs of some S-acts.

Let G be a graph. The (open) neighborhood $N(x)$ of a vertex $x \in V(G)$ is the set of vertices which are adjacent to x. For a subset T of vertices, we put

$$N(T) = \bigcup_{x \in T} N(x), \quad N[T] = N(T) \cup T.$$

A set of vertices T in G is a dominating set, if $N[T] = V(G)$. The domination number of G is the minimum cardinality of a dominating set of G and is denoted as $\gamma(G)$.

An S-act A is said to be Artinian, if every descending chain of subacts of A terminates. It can be easily seen that every non-empty subact of an Artinian S-act contains a minimal subact.

Proposition 2.10. *Let A be an S-act. Then $\gamma(\mathbb{Inc}(A)) \leqslant 2$ provided that each of the following assertions hold:*

(i) *A contains a minimal subact;*

(ii) *A contains a zero element;*

(iii) $|\mathrm{Sub}(A)| < \infty$*;*

(iv) $|A| < \infty$*;*

(v) *A has trivial action;*

(vi) *A is Artinian.*

Proof.

(i) Let M be a minimal subact of A and $W := \{B \in \mathrm{Sub}(A) \mid M \nsubseteq B\}$. If $W = \emptyset$, then for every non-trivial subact B of A, $M \subseteq B$ and so $\{M\}$ is a dominating set. If $W \neq \emptyset$, then $\{M, \bigcup_{B \in W} B\}$ forms a dominating set in $\mathbb{Inc}(A)$. Hence, $\gamma(\mathbb{Inc}(A)) \leqslant 2$.

(ii) Using (i), $\{z\}$ is a minimal subact of A where z is a zero element.

The assertions (iii),(iv),(v) and (vi) are consequences of (i). □

Proposition 2.11. *The following assertions hold:*

(i) *Let A be the coproduct of a family $\{A_i \mid i \in I\}$ of S-acts with $|I| > 1$ and $\gamma(\mathbb{Inc}(A_j)) = 1$ for some $j \in I$. Then $\gamma(\mathbb{Inc}(A)) = 2$.*

(ii) *If F is a free S-act with a non-singleton basis and $\gamma(\mathbb{Inc}(S)) = 1$, then $\gamma(\mathbb{Inc}(F)) = 2$.*

Proof.

(i) Suppose that $\{T\}$ is a dominating set of $\mathbb{Inc}(A_j)$. Let $B = \coprod_{i \in I} B_i$ be a non-trivial subact of A where B_i's are (possibly empty) subacts of A_i. If $B_j \subseteq T$, then $B \subseteq \coprod_{i \in I} U_i$ where $U_j = T, U_i = A_i$ for all $i \neq j$ and if $T \subseteq B_j$, then $T \subseteq B$. Thus $\{\coprod_{i \in I} U_i, T\}$ is a dominating set of $\mathbb{Inc}(A)$ so that $\gamma(\mathbb{Inc}(A)) \leqslant 2$. Now we show that $\gamma(\mathbb{Inc}(A)) \neq 1$. On the contrary, let $\{B = \coprod_{i \in I} B_i\}$ be a dominating set of $\mathbb{Inc}(A)$ and $s \in I$. Then one of the subacts $\coprod_{i \neq s} A_i$ or A_s is non-adjacent to B in $\mathbb{Inc}(A)$ which is a contradiction.

(ii) follows from (i). □

Example 2.12. Consider the monoid $S = \{1, s\}$ where s is an idempotent element and the S-act $A = \{a, b, c\}$ with the action defined by $1c = c, sc = a$ and a, b are fixed elements. Then all non-trivial subacts of A are the sets $\{a\}, \{b\}, \{a, b\}$ and $\{a, c\}$. It is clear that $\{\{a\}, \{b\}\}$ is a dominating set in $\mathbb{Inc}(A)$ and $\gamma(\mathbb{Inc}(A)) = 2$.

An independent set in a graph is a set of pairwise non-adjacent vertices. The independence number of G, written as $\alpha(G)$, is the maximum size of an independent set.

Remark 2.13. In [13], it has shown that the independence number of the intersection graph of an S-act A equals the number of minimal subacts of A. But this is not the case for the inclusion graphs. For instance, let $A = \{a, b, c, d\}$ be an S-act with trivial action. Then $\alpha(\mathbb{Inc}(A)) = 6$ whereas $|\mathrm{Min}(A)| = 4$.

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