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Improved bounds for the spectral norms of r-circulant matrices with k-Fibonacci and k-Lucasnumbers

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Abstract

We are concerned with the spectral norms of r-circulant matrices with the k-Fibonacci and k-Lucas numbers. By using Abel transformation and some identities, weobtain some new lower bounds for the spectral norms of r-circulant matrices. Havingcompared some known results, the obtained bounds are more precise.

Keywords:k-Fibonacci number,k-Lucas number, r-circulant matrix, Spectral norm.

1. Introduction

For any integer number$k \geqslant 1$, the k-Fibonacci and$k$-Lucas sequences $\{F_{k,n}\}$and$\{L_{k,n}\}$are defined by the following recursive relations

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \quad (1.1)$$

with conditions $F_{k,0} = 0$,$F_{k,1} = 1$, and

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1}$$

with $L_{k,0} = 2$,$L_{k,1} = k$.

In particular, if $k = 1$ the classical Fibonacci and Lucas sequences, denote by abbreviation $\{F_n\}$and $\{L_n\}$, are obtained. There are a large number of properties on the k-Fibonacci and k-Lucas sequences are discussed [1–3]. Here some of theproperties that we will need later are summarized below [2].

- $F_{k,i-1}F_{k,i+1} - F_{k,i}^2 = (-1)^i$;
- $L_{k,i}^2 = (k^2+4)F_{k,i}^2 + 4(-1)^i$;

- $L_{k,i}L_{k,i+1} = (4+k^2)F_{k,i}F_{k,i+1} + 2k(-1)^i$.

There are many works concerning estimates for spectral norms of special matrices, which have connections with numerical analysis. For example, Zhou [4]investigated the explicit formulae of spectral norms for g-circulant matrices. Solak [5] established some bounds for the circulant matrix $[F_{(\mathrm{mod}(j-i,n))}]_{i,j=1}^n$and $[L_{(\mathrm{mod}(j-i,n))}]_{i,j=1}^n$on the spectral and Euclidean norms. Bani-Domi and Kittaneh[6] established two general norm equalities for circulant and skew circulant operator matrices.We refer the readers to references [7, 8].

Recently, ShenandCen[9]giventheboundsofthespectralnormsof r -circulantmatrices $C_r(F_0, F_1, \cdots, F_{n-1})andC_r(L_0, L_1, \cdots, L_{n-1})$. In [10] they generalized the topic and found the upper and lower bounds for the norms of r-circulantmatrices with the$k$-Fibonacci and k-Lucas numbers. In the present paper, we proceed to study the topic in [10]. More precisely, we consider the bounds for the spectralnormsofmatrices $A = C_r(F_{k,0}, F_{k,1}, \cdots, F_{k,n-1})$ and $B = C_r(L_{k,0}, L_{k,1}, \cdots, L_{k,n-1})$, andobtainsomeimprovedlowerboundsrecurringtoAbeltransformationand some identities. The main results in this paper also generalize and improve theresults in [5,9].

2. Preliminaries

In this section, we present some known lemmas and results that will be used in thefollowing study.

Definition 2.1. For $c_0, c_1, \cdots, c_{n-1} \in \mathbb{C}$, the r-circulant matrix C, denoted by$C = C_r(c_0, c_1, \cdots, c_{n-1})$, is of the form

$$\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & c_0 \end{pmatrix}.$$

It is obvious that the matrix$C_r$turns into a classical circulant matrix for$r = 1$.

For any matrix $A = [a_{ij}]$of order$n$, it is well-known that the Frobenius (orEuclidean) norm of matrix $A$is

$$\|A\|_F = \left[\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right]^{\frac{1}{2}}$$

and the spectral norm of$A$is given by

$$\|A\|_2 = \sqrt{\max_{1\leqslant i\leqslant n} \lambda_i(A^H A)},$$

where A^H istheconjugate transposeof A and $\lambda_i(A^H A)$ istheeigenvalue of $A^H A$ .The following inequality holds:

$$\frac{1}{\sqrt{n}}\|A\|_F \leqslant \|A\|_2 \leqslant \|A\|_F.$$

The following lemmas are the main results in [10].

Lemma 2.1.[10]Let $A = C_r(F_{k,0}, F_{k,1}, \cdots, F_{k,n-1})$ be an r-circulant matrix.

(1)If $|r| > 1$, then

$$\sqrt{\frac{F_{k,n-1}F_{k,n}}{k}} \leqslant \|A\|_2 \leqslant \frac{|r| - |r|^n(F_{k,n} + |r|F_{k,n-1})}{1 - k|r| - |r|^2}.$$

(2) If |r| < 1, then

$$|r|\sqrt{\frac{F_{k,n-1}F_{k,n}}{k}} \leqslant \|A\|_2 \leqslant \frac{F_{k,n-1} + F_{k,n} - 1}{k}.$$

Lemma 2.2.[10]Let $B = C_r(L_{k,0}, L_{k,1}, \cdots, L_{k,n-1})$ be an r-circulant matrix.

(1) If $|r| \geqslant 1$, then

$$\sqrt{\left(k + \frac{4}{k}\right)F_{k,n-1}F_{k,n} + 2(1 + (-1)^{n-1})} \leqslant \|B\|_2 \leqslant \delta_1(k, r).$$

(2) If $|r| < 1$, then

$$|r|\sqrt{\left(k + \frac{4}{k}\right)F_{k,n-1}F_{k,n} + 2(1 + (-1)^{n-1})} \leqslant \|B\|_2 \leqslant \delta_2(k).$$

where $\delta_1(k, r)$ and $\delta_2(k)$ are given by

$$\delta_1(k, r) = \frac{2 - k|r| - |r|^n[(k + 2|r|)F_{k,n} + (2 - k|r|)F_{k,n-1}]}{1 - k|r| - |r|^2},$$

$$\delta_2(k) = \frac{(k + 2)F_{k,n} + (2 - k)F_{k,n-1} + k - 2}{k}.$$

3. Main results

In this section, we assume that $\delta_1(k, r)$ and $\delta_2(k)$ are defined in (2.6) and (2.7),respectively. We start this section by giving the following lemma.

Lemma 3.1. Suppose that $\{F_{k,i}\}$ is k-Fibonacci sequence with $F_{k,0} = 0, F_{k,1} = 1$, then the following identities hold:

(1) $\sum_{i=0}^{n} F_{k,i}^2 = \frac{1}{k}F_{k,n}F_{k,n+1}$;

(2) $$\sum_{i=0}^{n} F_{k,i}F_{k,i+1} = \begin{cases} \dfrac{F_{k,n+1}^2}{k}, & n \text{ is odd}, \\ \dfrac{F_{k,n+1}^2 - 1}{k}, & n \text{ is even}. \end{cases}$$

Proof.(1) According to the recurrence relation (1.1), we have

$$k\sum_{i=0}^{n}F_{k,i}^2=\sum_{i=0}^{n}F_{k,i}(F_{k,i+1}-F_{k,i-1})=\sum_{i=0}^{n}F_{k,i}F_{k,i+1}-\sum_{i=0}^{n}F_{k,i-1}F_{k,i}$$

$$=\sum_{i=0}^{n}F_{k,i}F_{k,i+1}-\sum_{i=-1}^{n-1}F_{k,i}F_{k,i+1}=F_{k,n}F_{k,n+1}.$$

(2) From the recurrence relation (1.1)

$$\begin{aligned}kF_{k,i}F_{k,i+1}&=\left(kF_{k,i}F_{k,i+1}+F_{k,i}^2\right)-F_{k,i}^2\\&=F_{k,i}(F_{k,i}+kF_{k,i+1})-F_{k,i}^2\\&=F_{k,i}F_{k,i+2}-F_{k,i}^2.\end{aligned}$$

It follows from (1.3) that

$$F_{k,i}F_{k,i+2}-F_{k,i+1}^2=(-1)^{i+1}.$$

Substituting (3.2) into (3.1), we obtain

$$kF_{k,i}F_{k,i+1}=F_{k,i+1}^2-F_{k,i}^2+(-1)^{i+1}.$$

Evaluate summation from $0$to n, we have

$$\begin{aligned}k\sum_{i=0}^{n}F_{k,i}F_{k,i+1}&=\sum_{i=0}^{n}F_{k,i+1}^2-\sum_{i=0}^{n}F_{k,i}^2+\sum_{i=0}^{n}(-1)^{i+1}\\&=F_{k,n+1}^2+\sum_{i=0}^{n}(-1)^{i+1}.\end{aligned}$$

It follows from

$$\sum_{i=0}^{n-1}(-1)^{i+1}=\begin{cases}0, & n \text{ is odd},\\-1, & n \text{ is even}\end{cases}$$

that conclusion (2) holds. This concludes the proof of the lemma.□

The following lemma can be found in [11].

Lemma 3.2.(Abel transformation) Suppose that $\{a_i\}$ and $\{b_i\}$ are two sequences, $S_i=a_1+a_2+\cdots+a_i(i=1,2,\cdots)$, then

$$\sum_{i=1}^{n}a_ib_i=S_nb_n-\sum_{i=1}^{n-1}(b_{i+1}-b_i)S_i.$$

The following theorem provides a precise estimate for the spectral norms of r -circulantmatrix $C_r(F_{k,0},F_{k,1},\cdots,F_{k,n-1})$.

Theorem 3.1. Suppose that $A = C_r(F_{k,0}, F_{k,1}, \cdots, F_{k,n-1})$ is an r-circulant matrix. Let

$$\delta(k,r) = \frac{1}{k^2}\left[nk|r|^2 F_{k,n-1}F_{k,n} + (1-|r|^2)F_{k,n}^2\right].$$

Then the following conclusions hold:

(1) $|r| \geqslant 1$

(i) If n is odd, then

$$\sqrt{\frac{\delta(k,r)}{n} + \frac{|r|^2-1}{nk^2}} \leqslant \|A\|_2 \leqslant \frac{|r| - |r|^n(F_{k,n} + |r|F_{k,n-1})}{1-k|r|-|r|^2};$$

(ii) If n is even, then

$$\sqrt{\frac{\delta(k,r)}{n}} \leqslant \|A\|_2 \leqslant \frac{|r| - |r|^n(F_{k,n} + |r|F_{k,n-1})}{1-k|r|-|r|^2}.$$

(2) $|r| < 1$

(i) If n is odd, then

$$\sqrt{\frac{\delta(k,r)}{n} + \frac{|r|^2-1}{nk^2}} \leqslant \|A\|_2 \leqslant \frac{F_{k,n} + F_{k,n-1} - 1}{k};$$

(ii) If n is even, then

$$\sqrt{\frac{\delta(k,r)}{n}} \leqslant \|A\|_2 \leqslant \frac{F_{k,n} + F_{k,n-1} - 1}{k}.$$

Proof. According to Lemma 2.1, we need only to prove the left parts "$\leqslant$". The matrix A is of the form

$$A = \begin{pmatrix} F_{k,0} & F_{k,1} & F_{k,2} & \cdots & F_{k,n-1} \\ rF_{k,n-1} & F_{k,0} & F_{k,1} & \cdots & F_{k,n-2} \\ rF_{k,n-2} & rF_{k,n-1} & F_{k,0} & \cdots & F_{k,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rF_{k,1} & rF_{k,2} & rF_{k,3} & \cdots & F_{k,0} \end{pmatrix}.$$

Therefore

$$\|A\|_F^2 = \sum_{i=0}^{n-1}(n-i)F_{k,i}^2 + |r|^2\sum_{i=1}^{n-1} i\cdot F_{k,i}^2 = n\sum_{i=0}^{n-1}F_{k,i}^2 + (|r|^2-1)\sum_{i=1}^{n-1} i\cdot F_{k,i}^2.$$

Using Abel transformation, we have

$$\sum_{i=1}^{n-1} i\cdot F_{k,i}^2 = (n-1)\sum_{i=1}^{n-1}F_{k,i}^2 - \sum_{i=1}^{n-2}\sum_{j=1}^{i}F_{k,j}^2.$$

Combining (3.4) and (3.5) gives

$$\|A\|_F^2=[(n-1)|r|^2+1]\sum_{i=0}^{n-1}F_{k,i}^2+(1-|r|^2)\sum_{i=1}^{n-2}\sum_{j=1}^{i}F_{k,j}^2.$$

By Lemma 3.1(1), one can obtain that

$$\|A\|_F^2=\frac{(n-1)|r|^2+1}{k}F_{k,n-1}F_{k,n}+\frac{(1-|r|^2)}{k}\sum_{i=1}^{n-2}F_{k,i}F_{k,i+1}$$

$$=\frac{1}{k}\Big[n|r|^2F_{k,n-1}F_{k,n}+(1-|r|^2)\sum_{i=0}^{n-1}F_{k,i}F_{k,i+1}\Big].$$

If n is odd, by direct calculation, together with Lemma 3.1(2), we can obtain that

$$\|A\|_F^2=\frac{1}{k}\left[n|r|^2F_{k,n-1}F_{k,n}+(1-|r|^2)\cdot\frac{F_{k,n}^2-1}{k}\right]$$

$$=\frac{1}{k^2}\left[nk|r|^2F_{k,n-1}F_{k,n}+(1-|r|^2)(F_{k,n}^2-1)\right]$$

$$=\delta(k,r)+\frac{|r|^2-1}{k^2}.$$

It follows from (2.1) that

$$\|A\|_2\geqslant\sqrt{\frac{\delta(k,r)}{n}+\frac{|r|^2-1}{nk^2}}.$$

Similarly, if n is even, then

$$\|A\|_F^2=\frac{1}{k}\left[n|r|^2F_{k,n-1}F_{k,n}+(1-|r|^2)\cdot\frac{F_{k,n}^2}{k}\right]$$

$$=\frac{1}{k^2}\left[nk|r|^2F_{k,n-1}F_{k,n}+(1-|r|^2)F_{k,n}^2\right]=\delta(k,r).$$

According to (2.1), we have

$$\|A\|_2\geqslant\sqrt{\frac{\delta(k,r)}{n}}.$$

Therefore the conclusions hold.□

Remark 3.1. Here we make a comparison to the lower bounds between Theorem3.1 and Lemma 2.1. We shall illustrate that our results in Theorem 3.1 are strongerthan Lemma 2.1. For $|r|\geqslant 1$, it can be verified easily that

$$\begin{cases} \sqrt{\frac{\delta(k,r)}{n}+\frac{|r|^2-1}{nk^2}} \geqslant \sqrt{\frac{F_{k,n-1}F_{k,n}}{k}}, & n \text{ is odd}, \\ \sqrt{\frac{\delta(k,r)}{n}} \geqslant \sqrt{\frac{F_{k,n-1}F_{k,n}}{k}}, & n \text{ is even}, \end{cases}$$

and if $|r| < 1$, we deduce that

$$\sqrt{\frac{\delta(k,r)}{n}} > \sqrt{\frac{\delta(k,r)}{n}+\frac{|r|^2-1}{nk^2}} \geqslant |r|\sqrt{\frac{F_{k,n-1}F_{k,n}}{k}},$$

which show that the lower bounds in Theorem 3.1 are more precise than (2.2) and(2.3).

If we let $k = 1$ in Theorem 3.1, then we easily derive the following corollarywhich improve the results in [9].

Corollary 3.1. Let $A = C_r(F_0, F_1, \cdots, F_{n-1})$ be an r-circulant matrix, and F_i bethe i-thFibonacci number. Let

$$\delta(1,r) = n|r|^2F_{n-1}F_n + (1-|r|^2)F_n^2.$$

(1) $|r| \geqslant 1$, then

$$\begin{cases} \sqrt{\frac{\delta(1,r)+|r|^2-1}{n}} \leqslant \|A\|_2 \leqslant \frac{|r|-|r|^n(F_n+|r|F_{n-1})}{1-|r|-|r|^2}, & n \text{ odd}, \\ \sqrt{\frac{\delta(1,r)}{n}} \leqslant \|A\|_2 \leqslant \frac{|r|-|r|^n(F_n+|r|F_{n-1})}{1-|r|-|r|^2}, & n \text{ even}. \end{cases}$$

(2) $|r| > 1$, then

$$\begin{cases} \sqrt{\frac{\delta(1,r)+|r|^2-1}{n}} \leqslant \|A\|_2 \leqslant F_{n+1}-1, & n \text{ odd}, \\ \sqrt{\frac{\delta(1,r)}{n}} \leqslant \|A\|_2 \leqslant F_{n+1}-1, & n \text{ even}. \end{cases}$$

Let $r = 1$ in Theorem 3.1, then we have the following result.

Corollary 3.2. Let $A = C_1(F_{k,0}, F_{k,1}, \cdots, F_{k,n-1})$ be a circulant matrix. Then

$$\sqrt{\frac{F_{k,n-1}F_{k,n}}{k}} \leqslant \|A\|_2 \leqslant \frac{F_{k,n}+F_{k,n-1}-1}{k}.$$

The following corollary improve the lower bounds in [5].

Corollary 3.3. Let $A = C_1(F_0, F_1, \cdots, F_{n-1})$ be a circulant matrix. Then

$$\sqrt{F_{n-1}F_n} \leqslant \|A\|_2 \leqslant F_{n+1}-1.$$

Now we consider the spectral norms of r-circulantmatrices $B = C_r(L_{k,0}, L_{k,1}, \cdots, L_{k,n-1})$. The following lemma are needed.

Lemma 3.3. Suppose that $\{L_{k,i}\}$ is a k-Lucas sequence with $L_{k,0}=2$, $L_{k,1}=k$, then

(1) $\sum_{i=0}^{n} L_{k,i}^2 = \frac{1}{k} L_{k,n} L_{k,n+1} + 2$.

(2) $\sum_{i=0}^{n} L_{k,i}^2 = \begin{cases} \left(k+\frac{4}{k}\right) F_{k,n} F_{k,n+1}, & n \text{ is odd}, \\ \left(k+\frac{4}{k}\right) F_{k,n} F_{k,n+1} + 4, & n \text{ is even}. \end{cases}$

(3) $\sum_{i=0}^{n} L_{k,i} L_{k,i+1} = \begin{cases} \left(k+\frac{4}{k}\right) F_{k,n+1}^2, & n \text{ is odd}, \\ \left(k+\frac{4}{k}\right) F_{k,n+1}^2 + k - \frac{4}{k}, & n \text{ is even}. \end{cases}$

Proof. (1) According to the recurrence relation (1.2), we have

$$k\sum_{i=0}^{n} L_{k,i}^2 = \sum_{i=0}^{n} L_{k,i}(L_{k,i+1}-L_{k,i-1}) = \sum_{i=0}^{n} L_{k,i}L_{k,i+1} - \sum_{i=0}^{n} L_{k,i-1}L_{k,i}$$

$$= \sum_{i=0}^{n} L_{k,i}L_{k,i+1} - \sum_{i=-1}^{n-1} L_{k,i}L_{k,i+1} = L_{k,n}L_{k,n+1} + 2k.$$

(2) It follows from (1.5) and (1).

(3) In the light of (1.5), we have

$$\sum_{i=0}^{n} L_{k,i}L_{k,i+1} = (4+k^2)\sum_{i=0}^{n} F_{k,i}F_{k,i+1} + 2k\sum_{i=0}^{n}(-1)^i.$$

If n is odd, it follows from Lemma 3.1(2), together with $\sum_{i=0}^{n}(-1)^i = 0$ that

$$\sum_{i=0}^{n} L_{k,i}L_{k,i+1} = \left(k+\frac{4}{k}\right)F_{k,n+1}^2.$$

Similarly, if n is even, then

$$\sum_{i=0}^{n} L_{k,i}L_{k,i+1} = \frac{4+k^2}{k}(F_{k,n+1}-1) + 2k$$

$$= \left(k+\frac{4}{k}\right)F_{k,n+1}^2 + k - \frac{4}{k}.$$

This completes the proof.□

Theorem 3.2. Suppose that $B = C_r(L_{k,0}, L_{k,1}, \cdots, L_{k,n-1})$ be r-circulant matrix. Let

$$\mu(k,r)=\left(k+\frac{4}{k}\right)\left[((n-1)|r|^2+1)F_{k,n-1}F_{k,n}+\frac{1-|r|^2}{k}F_{k,n-1}^2\right].$$

Then the following conclusions hold:

(1) $|r|\geqslant 1$

(i) If n is odd, then

$$\sqrt{\frac{\mu(k,r)+2(1-|r|^2)}{n}+2(1+|r|^2)}\leqslant\|B\|_2\leqslant\delta_1(k,r).$$

(ii) If n is even, then

$$\sqrt{\frac{\mu(k,r)+(1-|r|^2)(2n-1-4/k^2)}{n}}\leqslant\|B\|_2\leqslant\delta_1(k,r).$$

(2) $|r|<1$

(i) If n is odd, then

$$\sqrt{\frac{\mu(k,r)+2(1-|r|^2)}{n}+2(1+|r|^2)}\leqslant\|B\|_2\leqslant\delta_2(k).$$

(ii) If n is even, then

$$\sqrt{\frac{\mu(k,r)+(1-|r|^2)(2n-1-4/k^2)}{n}}\leqslant\|B\|_2\leqslant\delta_2(k).$$

Proof. It is clear that

$$\|B\|_F^2=n\sum_{i=0}^{n-1}L_{k,i}^2+(|r|^2-1)\sum_{i=1}^{n-1}i\cdot L_{k,i}^2.$$

By applying Abel transformation, together with Lemma 3.3(1), we have

$$\sum_{i=1}^{n-1}i\cdot L_{k,i}^2=(n-1)\sum_{i=1}^{n-1}L_{k,i}^2-\sum_{i=1}^{n-2}\sum_{j=1}^{i}L_{k,j}^2$$

$$=(n-1)\sum_{i=1}^{n-1}L_{k,i}^2-\frac{1}{k}\sum_{i=1}^{n-2}(L_{k,i}L_{k,i+1}-2k).$$

Substituting (3.7) into (3.6), we obtain

$$\|B\|_F^2=[(n-1)|r|^2+1]\sum_{i=0}^{n-1}L_{k,i}^2+\frac{1-|r|^2}{k}\sum_{i=1}^{n-2}L_{k,i}L_{k,i+1}+2n(1-|r|^2).$$

If n is odd, by direct calculation, together with Lemma 3.3, we derive that

$$\|B\|_F^2=\Big(k+\frac{4}{k}\Big)\Big[((n-1)|r|^2+1)F_{k,n-1}F_{k,n}+\frac{1-|r|^2}{k}F_{k,n-1}^2\Big]+2(n+n|r|^2+1-|r|^2).$$

Similarly, if n is even, then

$$\|B\|_F^2=\Big(k+\frac{4}{k}\Big)[(n-1)|r|^2+1]F_{k,n-1}F_{k,n}+\frac{1-|r|^2}{k}\Big(k+\frac{4}{k}\Big)(F_{k,n-1}^2-1)+2n(1-|r|^2)$$
$$=\Big(k+\frac{4}{k}\Big)\Big[((n-1)|r|^2+1)F_{k,n-1}F_{k,n}+\frac{1-|r|^2}{k}F_{k,n-1}^2\Big]+(1-|r|^2)\Big(2n-1-\frac{4}{k^2}\Big).$$

By (2.1) and Lemma 2.2 we get the conclusions hold.□

Remark 3.2. Now we illustrate that our results in Theorem 3.2 are more betterthan Lemma 2.2. For $|r|\geqslant 1$, it can be verified easily that

$$\begin{cases}\sqrt{\dfrac{\mu(k,r)+2(1-|r|^2)}{n}+2(1+|r|^2)}\geqslant\sqrt{\Big(k+\dfrac{4}{k}\Big)F_{k,n-1}F_{k,n}+4}, & n \text{ odd},\\ \sqrt{\dfrac{\mu(k,r)+(1-|r|^2)(2n-1-4/k^2)}{n}}\geqslant\sqrt{\Big(k+\dfrac{4}{k}\Big)F_{k,n-1}F_{k,n}}, & n \text{ even}.\end{cases}$$

and if $|r|<1$, we deduce that

$$\begin{cases}\sqrt{\dfrac{\mu(k,r)+2(1-|r|^2)}{n}+2(1+|r|^2)}\geqslant|r|\sqrt{\Big(k+\dfrac{4}{k}\Big)F_{k,n-1}F_{k,n}+4}, & n \text{ odd},\\ \sqrt{\dfrac{\mu(k,r)+(1-|r|^2)(2n-1-4/k^2)}{n}}>|r|\sqrt{\Big(k+\dfrac{4}{k}\Big)F_{k,n-1}F_{k,n}}, & n \text{ even}.\end{cases}$$

which show that the lower bounds in Theorem 3.2 are more precise than (2.4) and(2.5).

If we let $k=1$ in Theorem 3.2, then we derive the following corollary whichgives more better lower bounds than [9].

Corollary 3.4. Suppose that $B=C_r(L_0,L_1,\cdots,L_{n-1})$ is an r-circulant matrix,and L_i is the i-th Fibonacci number. Let

$$\mu(1,r)=5\big[((n-1)|r|^2+1)F_{n-1}F_n+(1-|r|^2)F_{n-1}^2\big].$$

(1) $|r|\geqslant 1$, then

$$\begin{cases}\sqrt{\dfrac{\mu(1,r)+2(1-|r|^2)}{n}+2(1+|r|^2)}\leqslant\|B\|_2\leqslant\delta_1(1,r), & n \text{ odd},\\ \sqrt{\dfrac{\mu(1,r)+(1-|r|^2)(2n-1-4/k^2)}{n}}\leqslant\|B\|_2\leqslant\delta_1(1,r), & n \text{ even}.\end{cases}$$

(2) $|r|<1$, then

$$\begin{cases} \sqrt{\frac{\mu(1,r)+2(1-|r|^2)}{n}+2(1+|r|^2)} \leqslant \|B\|_2 \leqslant 2F_n+F_{n+1}-1, & n \text{ odd},\\ \sqrt{\frac{\mu(1,r)+(1-|r|^2)(2n-1-4/k^2)}{n}} \leqslant \|B\|_2 \leqslant 2F_n+F_{n+1}-1, & n \text{ even}.\end{cases}$$

If we let$r=1$in Theorem 3.2, then we have the following result.

Corollary 3.5. Let$B=C_1(L_{k,0},L_{k,1},\cdots,L_{k,n-1})$be a circulant matrix. Then

$$\begin{cases} \sqrt{\left(k+\frac{4}{k}\right)F_{k,n-1}F_{k,n}+4} \leqslant \|B\|_2 \leqslant \delta_2(k), & n \text{ odd},\\ \sqrt{\left(k+\frac{4}{k}\right)F_{k,n-1}F_{k,n}} \leqslant \|B\|_2 \leqslant \delta_2(k), & n \text{ even}.\end{cases}$$

The following corollary improve the result in [5].

Corollary 3.6. Let$B=C_1(L_0,L_1,\cdots,L_{n-1})$be a circulant matrix. Then

$$\begin{cases} \sqrt{5F_{k,n-1}F_{k,n}+4} \leqslant \|B\|_2 \leqslant 2F_n+F_{n+1}-1, & n \text{ odd},\\ \sqrt{5F_{k,n-1}F_{k,n}} \leqslant \|B\|_2 \leqslant 2F_n+F_{n+1}-1, & n \text{ even}.\end{cases}$$

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