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Singular Values of One Parameter Family $\lambda\frac{b^z-1}{z}$

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Abstract

The singular values of one parameter family of entire functions $f_\lambda(z)=\lambda\frac{b^z-1}{z}$ and $f_\lambda(0)=\lambda\ln b$, $\lambda\in\mathbb{R}\setminus\{0\}$, $z\in\mathbb{C}$, $b>0$, $b\neq1$ are investigated. It is shown that all the critical values of $f_\lambda(z)$ belong to the right half plane for $0<b<1$ and the left half plane for $b>1$. It is described that the function $f_\lambda(z)$ has infinitely many singular values. It is also found that all these singular values are bounded and lie inside the open disk centered at origin and having radius $|\lambda\ln b|$.

Keywords: Critical values, Singular values.

1. Introduction

It is known that if singular values exist in transcendental functions, then it is very crucial to determine the dynamical behavior of these functions. The dynamics of one parameter family λe^z, that has only one singular value, is studied in detail [1, 2, 3]. This exponential family is simpler than other families of functions which involving exponential maps [4, 5, 6, 7] and having more than one or infinitely many singular values. The singular values and dynamics of family $\lambda\frac{e^z-1}{z}$ are studied by Kapoor and Prasad [8]. The singular values of one parameter families of functions are investigated in [9, 10, 11].

In this work, the singular values of one parameter family of function $\frac{b^z-1}{z}$ for $b>0$, $b\neq1$ which is a generalized family of function $\frac{e^z-1}{z}$, is considered. Suppose, for this purpose, the following one parameter family of functions

$$T=\left\{ f_\lambda(z)=\lambda\frac{b^z-1}{z} \text{ and } f_\lambda(0)=\lambda\ln b:\lambda\in\mathbb{R}\setminus\{0\}, z\in\mathbb{C}, b>0, b\neq 1\right\}$$

A point z^* is said to be a critical point of $f(z)$ if $f'(z^*)=0$. The value $f(z^*)$ corresponding to a critical point z^* is called a critical value of $f(z)$. A point $w\in\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ is said to be an asymptotic value for $f(z)$, if there exists a continuous curve $\gamma:[0,\infty)\to\hat{\mathbb{C}}$ satisfying $\lim_{t\to\infty}\gamma(t)=\infty$ and $\lim_{t\to\infty} f(\gamma(t))=w$. A singular value of f is defined to be either a critical value or an asymptotic value of f. A function f is called critically bounded or it is said to be a function of bounded type if the set of all singular values of f is bounded, otherwise unbounded-type. The importance of singular values in the dynamics of a transcendental functions can be seen in [12, 13, 14].

This paper is organized as follows: It is shown that, in Theorem 2.1, the function $f_\lambda\in T$ has no zeros in the left half plane and $f_\lambda(z)$ maps the right half plane inside the open disk centered at origin and having radius $|\lambda\ln b|$ for $0<b<1$. It is also found that, in Theorem 2.2, the function $f_\lambda\in T$ has no zeros in the right half plane and $f_\lambda(z)$ maps the left half plane inside the open disk centered at origin and having radius $|\lambda|\ln b$ for $b>1$. In Theorem 2.3, it is proved that the function $f_\lambda\in T$ has infinitely many singular values. It is seen that all the singular values are bounded and lie inside the open disk centered at origin and having radius $|\lambda\ln b|$ in Theorem 2.4.

2. Infinitely Many Bounded Singular Values of $f_\lambda\in T$

Let us denote the left half plane and the right half plane by $H^-=\{z\in\hat{\mathbb{C}}:\mathrm{Re}(z)<0\}$ and $H^+=\{z\in\hat{\mathbb{C}}:\mathrm{Re}(z)>0\}$ respectively. The function $f_\lambda\in T$ has no zeros in the left half plane and $f_\lambda(z)$ maps the right half plane inside the open disk for $0<b<1$, are found in the following theorem:

Theorem 2.1. Let $f_\lambda\in T$ for $0<b<1$. Then,

(a) $f'_\lambda(z)$ has no zeros in the left half plane H^-.

(b) $f_\lambda(z)$ maps the right half plane H^+ inside the open disk centered at origin and having radius $|\lambda\ln b|$.

Proof. *(a)* Since $f'_\lambda(z)=\lambda\frac{(z\ln b-1)b^z+1}{z^2}=0$, which implies $b^{-z}=1-z\ln b$. Then, the real and imaginary parts of this equation are

$$b^{-x}\cos(y\ln b)=1-x\ln b \tag{1}$$

$$b^{-x}\sin(y\ln b)=y\ln b \tag{2}$$

Thus, it shows that the function $f'_\lambda(z)$ has no zeros in H^- for $0<b<1$.

(b) Suppose that the line segment γ is defined by $\gamma(t)=tz$, $t\in[0,1]$. Further, let the function $h(z)=b^z$ for an arbitrary fixed $z\in\mathbb{C}$. Then

$$\int_\gamma h(z)dz=\int_0^1 h(\gamma(t))\gamma'(t)\,dt=z\int_0^1 b^{tz}dt=\frac{1}{\ln b}\left(b^z-1\right)$$

Since $M\equiv\max_{t\in[0,1]}\left|h(\gamma(t))\right|=\max_{t\in[0,1]}\left|b^{tz}\right|<1$ for $z\in H^+$, then

$$\left|b^z-1\right|=\left|\ln b\int_\gamma h(z)dz\right|\le M\,|z|\,|\ln b|<|z|\,|\ln b|$$

$$\left|\frac{b^z-1}{z}\right|<|\ln b|\quad \text{for all}\quad z\in H^+.$$

Hence,

$$\left|f_\lambda(z)\right|=\left|\lambda\frac{b^z-1}{z}\right|<|\lambda\ln b|\quad \text{for all}\quad z\in H^+.$$

Therefore, $f_\lambda(z)$ maps H^+ inside the open disk centered at origin and having radius $|\lambda\ln b|$. The proof of the theorem is completed for $0<b<1$.

In the following theorem, it is shown that the function $f_\lambda\in T$ has no zeros in the right half plane and $f_\lambda(z)$) maps the left half plane inside the open disk for $b>1$:

Theorem 2.2. Let $f_\lambda\in T$ for $b>1$. Then,

(i) $f_\lambda'(z)$ has no zeros in the right half plane H^+.

(ii) $f_\lambda(z)$ maps the left half plane H^- inside the open disk centered at origin and having radius $|\lambda|\ln b$.

Proof. For $b>1$, the proof of the theorem may be obtained similarly as Theorem 2.1.

The following theorem describes that the function $f_\lambda\in T$ has infinitely many singular values:

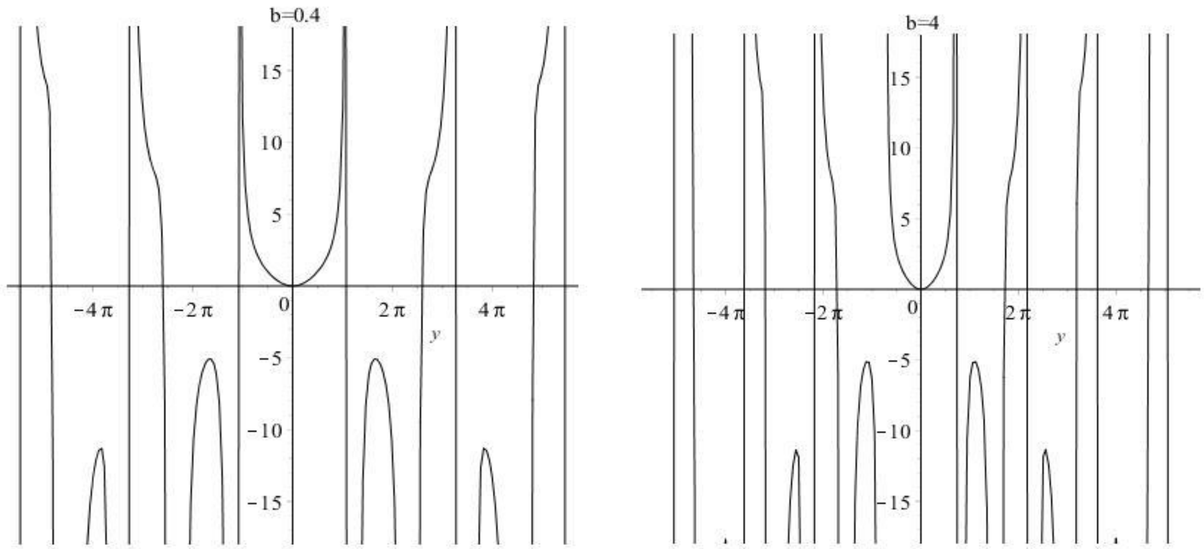
Theorem 2.3. Let $f_\lambda\in T$. Then, the function $f_\lambda(z)$ possesses infinitely many singular values.

Proof. For critical points, $f_\lambda'(z)=0$. It gives the equation $(z\ln b-1)b^z+1=0$. After simplifying, using the real and imaginary parts of this equation

$$\frac{y\ln b}{\sin(y\ln b)}-b^{y\cot(y\ln b)-\frac{1}{\ln b}}=0 \tag{3}$$

$$x=\frac{1}{\ln b}-y\cot(y\ln b) \tag{4}$$

It is seen that, from Figure 1(a) for $0<b<1$ and Figure 2(b) for $b>1$, the Eq. (3) has infinitely many solutions since intersections are increasing on horizontal axis for expanding interval.



(a) For $b = 0.4$ (b) For $b = 4$

Figure 1: Graphs of $\dfrac{y \ln b}{\sin(y \ln b)} - b^{y \cot(y \ln b) - \frac{1}{\ln b}}$

Let $\{y_k\}_{k=-\infty, k\neq 0}^{+\infty}$ be the solutions of Eq. (3). Then, from Eq. (4), $x_k = \dfrac{1}{\ln b} - y_k \cot(y_k \ln b)$ for k nonzero integer. For $z_k = x_k + iy_k$, the critical values $f_\lambda(z_k) = \lambda \dfrac{b^{z_k} - 1}{z_k}$ are distinct for different k. It shows that the function $f_\lambda(z)$ has infinitely many critical values for $0 < b < 1$ and $b > 1$.

Since $f_\lambda(z) \to \infty$ as $z \to \infty$ along both positive and negative real axes for $0 < b < 1$ and $b > 1$ respectively, then the point 0 is a finite asymptotic value of $f_\lambda \in T$.

Therefore, it conclude that the function $f_\lambda \in T$ possesses infinitely many singular values for $0 < b < 1$ and $b > 1$. This proves the theorem.

In the following theorem, it is proved that $f_\lambda \in T$ has bounded singular values and lie inside the open disk:

Theorem 2.3. Let $f_\lambda \in T$. Then, all the singular values of $f_\lambda(z)$ are bounded and lie inside the open disk centered at origin and having radius $|\lambda \ln b|$.

Proof. For $0 < b < 1$, by Theorem 2.1(a), the function $f'_\lambda(z)$ has no zeros in the left half plane H^-. Hence, all the critical points lie in the right half plane H^+. By using Theorem 2.1(b), the function $f_\lambda(z)$ maps H^+ inside the open disk centered at origin and having radius $|\lambda \ln b|$. It follows that all the critical values of $f_\lambda(z)$ are lying inside the open disk centered at origin and having radius $|\lambda \ln b|$ for $0 < b < 1$.

Similarly, for $b > 1$, using Theorem 2.2 (i) and (ii), it is easily deduce that all the critical values of $f_\lambda(z)$ are lying inside the open disk centered at origin and having radius $|\lambda| \ln b$.

Since $f_\lambda(z)$ has only one asymptotic value 0, so all the singular values of $f_\lambda \in T$ are bounded and lie inside the open disk centered at origin and having radius $|\lambda \ln b|$.

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