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A Fresh View on Numerical Correction and Optimization of Monte Carlo Algorithm and its Application for Fractional Differential Equation

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Abstract

In this paper we have used the Mote Carlo algorithm to obtain solution of some fractional differential equation. The fractional derivative is described in the Jumarie sense. The results obtained by this method have been compared with the exact solutions. Furthermore, comparisons indicate that is a very good agreement between the solutions of Monte Carlo algorithm and the exact solutions in terms of accuracy.

Keywords: Fractional Calculus**,** Monte Carlo algorithm**,** Jumarie derivative.

1. Introduction

The subject of fractional calculus (theory of integrals and derivatives of an arbitrary order) may be considered as an old and yet novel topic. It is an old topic because, the idea of this theory was planted over 300 years ago. Since that time the fractional calculus has drawn the attention of many researchers. In recent years with the rapid development of nonlinear problems, fractional calculus has played a significant role in many areas of science and engineering such as acoustics, control, continuum, damping law, edge detection, electromagnetism, hydrology, rheology, robotics, signalprocessing, thermal engineering, turbulence, viscoelasticity and many other problems [1]. For an interesting history and more scientific applications of fractional calculus, see [2].

One of the well-known stochastic methods which is preferable for solving high dimensional linear system of algebraic equations is the Monte Carlo method. Monte Carlo algorithms have some significant advantages. For example, they can approximate individual components of the solution without calculating the whole solution vector [3]. Also, for a large sparse linear system of algebraic equations, they are more efficient than direct or iterative numerical methods [4] and they are good candidates for parallelization because of the fact that many independent sample paths are used to estimate the solution [5].

In spite of all advantages, conventional Monte Carlo method converges slowly. Halton was the first one that proposed adaptive Monte Carlo methods in [6], which improve the convergence of Monte Carlo method exponentially. These methods have been used in different areas by the researchers [7].

2. Background on fractional derivatives

There are several definitions for fractional differential equations. These definitions include Grunwald-Letnikov, Riemann-Liouville, Caputo, Weyl, Marchaud, and Riesz fractional derivatives. Recently, a new modification ofRiemann–Liouville derivative is proposed by Jumarie [8]. It is well known that with the classical Riemann–Liouville definition of fractional derivative, the fractional derivative of a constant is not zero. The most useful alternative which has been proposed to cope with this feature is the so-called Caputo derivative [1]. With this definition, a fractional derivativewould be defined for differentiable function only. In order to deal with non-differentiable functions, Jumarie proposed a modification of the Riemann-Liouville definition which appears to provide a framework for a fractional calculus which is quite parallel with classical calculus [9]. This modification was successfully applied in the probability calculus, fractional Laplace problems, fractional variational equations and many other types of linear and nonlinear fractional differential equations [8-10].

2.1. Preliminaries and notations

This section deals with some preliminaries and notations regarding fractional calculus. For more details, see [1].

Definition 2.1.1. The Mittag-Leffler function $E_\alpha(z)$ with $\alpha \in R$ is defined by the following series representation, valid in the whole complex plane [9-11]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad z \in C. \tag{1}$$

Definition 2.1.2. The Riemann–Liouville integral operator of order α on the usual Lebesgue space $L_1[a,b]$ is defined as [1]

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(\varepsilon)(x-\varepsilon)^{\alpha-1}\, d\varepsilon, \quad \alpha > 0, x > 0. \tag{2}$$

Definition 2.1.3 (Jumarie Derivative). Let $f: R \rightarrow R$, $x \rightarrow f(x)$, denote a continuous (but not necessarily differentiable) function, and let $h > 0$ denote a constant discretization span. Define the forward operator FW by the equality [9, 10]

$$FWf(x)=f(x+h), \tag{3}$$

then the fractional difference of order α, $\alpha \in R$, $0 < \alpha < 1$, of f (x) is defined by the expression

$$\Delta^{\alpha} f(x) = (FW-1)^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[x+(\alpha-k)h], \quad (4)$$

and its fractional derivative is

$$f^{(\alpha)}(x) = D^{\alpha} f(x) = \frac{d^{\alpha} f(x)}{dx^{\alpha}} = \lim_{h\to 0} \frac{\Delta^{\alpha}[f(x)-f(0)]}{h^{\alpha}}. \quad (5)$$

As a direct consequence, some properties of Jumarie's derivative are given as follows.

Definition 2.1.4. Assume that function f (x) in Definition 2.1.3 is not a constant, then its fractional derivative of order α is defined by the following expression [9, 10]

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\varepsilon)^{-\alpha-1} f(\varepsilon) d\varepsilon, \ \ \alpha < 0. \quad (6)$$

For positive, one will set

$$f^{(\alpha)}(x) = (f^{(\alpha-1)}(x))' = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\varepsilon)^{-\alpha} \big(f(\varepsilon)-f(0)\big) d\varepsilon, 0 < \alpha < 1, \quad (7)$$

and

$$f^{(\alpha)}(x) = (f^{(\alpha-n)}(x))^{(n)}, n \le \alpha < n+1, n \ge 1. \quad (8)$$

Lemma 2.1.1. The integral with respect to $(dx)^{(\alpha)}$ is defined by [9, 10]

$$\int_0^x f(\varepsilon)(d\varepsilon)^{\alpha} = \alpha \int_0^x (x-\varepsilon)^{\alpha-1} f(\varepsilon) d\varepsilon, \ \ 0 < \alpha \le 1. \quad (9)$$

Lemma 2.1.2 [9, 10]

$$\frac{d^{\alpha}}{dx^{\alpha}} \int_0^{u(x)} f(\varepsilon)(d\varepsilon)^{\alpha} = \Gamma(\alpha+1) f\big(u(x)\big)(u'(x))^{\alpha}, 0 < \alpha \le 1. \quad (10)$$

Lemma 2.1.3. The fractional integration by part formula is defined by [9, 10]

$$\int_a^b u^{(\alpha)}(x) v(x) (dx)^{\alpha} = \Gamma(\alpha+1)[u(x)v(x)]_a^b - \int_a^b u(x) v^{(\alpha)}(x) (dx)^{\alpha}, 0 < \alpha \le 1. \quad (11)$$

Theorem 2.1.1. Assume that the continuous function f (x) has a fractional derivative of order α, then the following properties hold

$\frac{d^\alpha}{dx^\alpha} I^\alpha f(x) = f(x), \quad I^\alpha \frac{d^\alpha}{dx^\alpha} f(x) = f(x) - f(0), \quad 0 < \alpha \leq 1.$
(12)

Proof. We have

$$\frac{d^\alpha}{dx^\alpha} I^\alpha f(x) = \frac{d^\alpha}{dx^\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_0^x f(\varepsilon)(x-\varepsilon)^{\alpha-1} d\varepsilon\right) = \frac{d^\alpha}{dx^\alpha}\left(\frac{1}{\Gamma(\alpha)}\frac{1}{\alpha}\int_0^x f(\varepsilon)(d\varepsilon)^\alpha\right)$$

$=\frac{1}{\Gamma(\alpha+1)}\frac{d^\alpha}{dx^\alpha}\left(\int_0^x f(\varepsilon)(d\varepsilon)^\alpha\right) = \frac{1}{\Gamma(\alpha+1)}\Gamma(\alpha+1) f(x) = f(x).$
(13)

Moreover

$$I^\alpha \frac{d^\alpha}{dx^\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)}\left(\int_0^x \left(\frac{d^\alpha}{dx^\alpha} f(\varepsilon)\right)(d\varepsilon)^\alpha\right)$$

$= \frac{1}{\Gamma(\alpha+1)}\left(\Gamma(\alpha+1) f(\varepsilon)\right)_{\varepsilon=0}^{x} = f(x) - f(0).$
(14)

3. Monte Carlo methods

It is well known that Monte Carlo methods are more effective and more preferable than direct and iterative numerical methods for solving large sparse systems. In this section, we are going to solve the system of linear equations

CX=F,
(15)

Using conventional and adaptive Monte Carlo methods. Introducing A=$\{A_{i,j}\}_{i,j=1}^{n}$ =I-C, where I is an identity matrix, we have X = A X + F and therefore using recursive formula

$X^{k+1} = AX^k + F,$
(16)

We have an estimator for X under the assumption $max_i \sum_{j=1}^{n} A_{i,j} < 1$and the following Monte Carlo algorithms converge.

3.1. Formal Monte Carlo method

The base of the formal Monte Carlo method is to express each component of the solution vector as the expectation of some random variable. To estimate the inner product of two vector h and x^{k+1}obtained from Eq. (2), for an arbitrary integer k>0, simulate Z independent random paths$i_0^{(s)} \to i_1^{(s)} \to \cdots \to i_k^{(s)}, s = 1, \ldots, Z$ of Marcov chain with initial distribution p=$(p_1, p_2, \ldots, p_n)$ and transition matrix P with the following properties:

$p_i > 0 \; if \; h_i \neq 0,$

$p_{i,j} > 0 \; if \; A_{i,j} \neq 0, i, j = 1, \ldots, n.$
(17)

Define $\delta_k^{(s)}(h) = \frac{h_{i0}}{p_{i_0}^{(s)}} \sum_{m=0}^{k} w_m^{(s)} f_{i_m}^{(s)}$ *for s=1,…,Z, where*

$$w_m^{(s)} = w_{m-1}^{(s)} \frac{A_{i_{m-1}^{(s)} i_m^{(s)}}}{P_{i_{m-1}^{(s)} i_m^{(s)}}}, \; w_0^{(s)} \equiv 1 . \qquad (18)$$

We calculate

$$\theta_k = \frac{1}{Z} \sum_{s=1}^{Z} \delta_k^{(s)}(h),$$
(19)

which is an unbiased estimator of the inner product$< h, x^{(k+1)} >$.

3.2. Conformable Monte Carlo algorithm

Consider $F^0 = F, \theta_k^{(0)} = 0, F^{(d)} = F^{(d-1)} - C\theta_k^{(d-1)}, d = 1, \ldots, r,$ where r is the number of stages and $\theta_k^{(d)}$ is the approximate solution of

$C\Delta^d x = F^d,$

using described conventional Monte Carlo method which random paths are generated through a fixed transition matrix P. Then

$\varphi_k^{(d)}(h) = \varphi_k^{(d-1)}(h) + \theta_k^{(d)}, \; \varphi_k^0(h) = 0,$
(20)

is the approximated solution of linear system (1). It is shown in [6] that

$$\lim_{r\to\infty} F^{(r)} = 0, \lim_{r\to\infty} \theta_k^{(r)} = 0, \lim_{r\to\infty} \varphi_k^{(r)}(h) = x_j, \qquad (21)$$

Where x_j is the j-th component of the exact solution to (1). Note that if r = 1, we have the formal Monte Carlo method

4. Results analysis

In this section, two numerical results are presented to support our theatrical analysis.

Example1. Consider a fractional Poisson equation with the following conditions

$$\frac{\partial^\alpha f}{\partial x^\alpha}+\frac{\partial^\beta f}{\partial y^\beta}=(x^2+y^2)e^{\lambda xy}, 0<\alpha,\beta\le 2, \lambda>0, \quad (22)$$

Where

$$f(0,y)=1, f(2,y)=e^{2y}, f(x,0)=1, f(x,1)=e^x, 0\le x\le 2, 0\le y\le 1, \quad (23)$$

It is to be noted that for $\alpha=\beta=2, \lambda=1$, has the exact solution e^{xy}.

In Table1, the Comprising of our scheme with the exact solution is presented by using Conformable Monte Carlo algorithm:

Table 1: Comprising of the suggested scheme with the exact solution

Node	(0.25,0.25)	(0.25,0.75)	(0.75,0.75)	(0.9,1)
Exact solution	1.06449	1.20623	1.75505	2.4596
The proposed scheme	1.06447	1.20620	1.75511	2.4595

Example 2. Consider the following differential equation of fractional order

$$\frac{\partial^\alpha f}{\partial x^\alpha}+f(x)=x+\frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{1-\alpha}, \alpha\in(0,1], f(0)=0, 0\le x\le 1,$$
(24)

The exact solution for $\alpha=1$ is $f(x)=x$.

The graph of the error term, for different values of approximation of series solution is presented in Figure 1 by using formal Monte Carlo method.

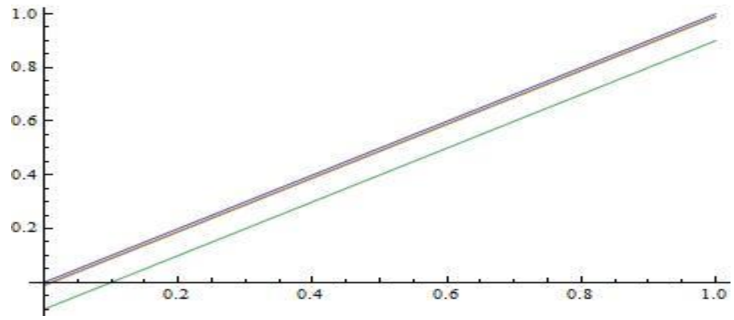

Figure 1: The graph of the error term, for different values of approximation of series solution

5. Conclusion

In this paper, the Monte Carlo algorithm has been successfully applied to finding the solution of fractional order equations. The convergence and stability of the method was examined in examples.
The results show that the proposed scheme is a powerful Mathematical tool for solving equations of fractional order. In other words, the proposed approach is also a promising method to solve other nonlinear equations. Finally, we pointed out that the corresponding analytical and numerical solutions are obtained using Mathematica.

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