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Some approximated solutions for operator equations by using frames

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Abstract

In this paper we give some approximated solutions for an operator equation $Lu = f$ where $L: H \to H$ is a bounded and self adjoint operator on a separable Hilbert space H. We use frames in order to precondition the linear equation so that convergence of iterative methods is improved. Also we find an exact solution associated to a frame and then we seek an approximated solution in a finite dimensional subspace of H that is generated by a finite frame sequence.

Keywords: Operator equation, Separable Hilbert space, Frame, Approximated solution.

1. Introduction

Our problem is to find $u \in H$ such that

$$Lu = f, \qquad (1.1)$$

where $L: H \to H$ is a bounded and self adjoint linear operator on a separable Hilbert space H. In general it is impossible to find the exact solution of the problem (1,1), because the separable Hilbert space H is infinite dimensional. A natural approach to constructing an approximate solution is to solve a finite dimensional analog of the problem (1,1). In [2, 6, 7, 10] some approximated solutions for this equation and related problems has been developed by wavelet bases. Usually the operator under consideration is defined on a bounded domain in $\mathbb{R}^n$ or on a closed manifold. Therefore the construction of a wavelet basis with specific properties on this domain or on the manifold is needed. Motivated by some difficulties to construct an appropriate basis, we therefore suggest to use a slightly weaker concept namely, frames. In [8, 9, 11], some iterative methods for solving this system has been developed by frames.

Assume that H is a separable Hilbert space, Λ is a countable set of indices and $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset$ H is a frame for H. This means that there exist constants $0 < A_\Psi \leq B_\Psi < \infty$ such that

$$A_\Psi\|f\|_H^2 \leq \sum_{\lambda\in\Lambda}|\langle f|\psi_\lambda\rangle|^2 \leq B_\Psi\|f\|_H^2, \forall f \in H, \quad (1.2)$$

or equivalently (by the Riesz mapping)

$$A_\Psi\|f\|_{H^*}^2 \leq \|f(\Psi)\|_{\ell_{2(\Lambda)}}^2 \leq B_\Psi\|f\|_{H^*}^2, \forall f \in H^*, \quad (1.3)$$

where $f(\Psi) = \left(f(\psi_\lambda)\right)_\lambda = (\langle f|\psi_\lambda\rangle)_\lambda$.

For the frame Ψ, let $T: \ell_2(\Lambda) \to H$ be the synthesis operator

$$T((c_\lambda)_\lambda) = \sum_{\lambda\in\Lambda} c_\lambda\psi_\lambda$$

and let $T^*: H \to \ell_2(\Lambda)$ be the analysis operator

$$T^*(f) = (\langle f|\psi_\lambda\rangle)_\lambda.$$

Also let $S := TT^*: H \to H$ be the frame operator

$$S(f) = \sum_\lambda\langle f|\psi_\lambda\rangle\psi_\lambda.$$

Note that T is surjective, $T^*$is injective, T^* is the adjoint of T and because of (1.2), T is bounded. In fact we have

$$\|T\| = \|T^*\| \leq \sqrt{B_\Psi}. \quad (1.4)$$

Since $Ker(T) = \left(Ran(T^*)\right)^\perp$ then $\ell_2(\Lambda) = Ran(T^*) \oplus Ker(T)$. It was shown in [5], for the frame $(\psi_\lambda)_{\lambda\in\Lambda}$, S is a positive invertible operator satisfying $A_\Psi I_H \leq S \leq B_\Psi I_H$ and $B_\Psi^{-1} I_H \leq S^1 \leq A_\Psi^{-1} I_H$. Also, the sequence

$$\widetilde{\Psi} = \left(\tilde{\psi}_\lambda\right)_{\lambda\in\Lambda} = (S^{-1}\psi_\lambda)_{\lambda\in\Lambda}$$

is a frame (called the canonical dual frame) for H with bounds B_Ψ^{-1}, A_Ψ^{-1}. Every $f \in H$ has the expansion

$$f = \sum_\lambda\langle f|\psi_\lambda\rangle\tilde{\psi}_\lambda = \sum_\lambda\langle f|\tilde{\psi}_\lambda\rangle. \quad (1.5)$$

Also a complete sequence $(\psi_\lambda)_{\lambda\in\Lambda}$ in H is called a Riesz basis if there exist constants $A_\Psi, B_\Psi > 0$ such that

$$A_\Psi\|C\|_{\ell_2(\Lambda)}^2 \leq \|\sum_\lambda c_\lambda\psi_\lambda\|_H^2 \leq B_\Psi\|C\|_{\ell_{2(\Lambda)}}^2,$$

hold for all finite sequences $C = (c_\lambda)_{\lambda\in\Lambda}$. Each Riesz basis for a Hilbert space H is a frame for H. For more details see [4, 5]. For an index set $\widetilde{\Lambda} \subset \Lambda$, $(\psi_\lambda)_{\lambda\in\widetilde{\Lambda}}$ is called a frame sequence if it is a frame for its closed span.

2. Approximated inverse frames of an operator

The most straight forward approach to an iterative solution of a linear system is to rewrite the equation (1.1) as a linear fixed-point iteration. One way to do this is the Richardson iteration. The abstract method reads as follows:

write $Lu = f$ as

$$u = (I - L)u + f.$$

For given $u_0 \in H$, define for $n \geq 0$,

$$u_{n+1} = (I - L)u_n + f. \quad (2.1)$$

Since $Lu - f = 0$,

$$u_{n+1} - u = (I - L)u_n + f - u - (f - Lu) =$$
$$(I - L)u_n - u + Lu = (I - L)(u_n - u).$$

Hence

$$\|u_{n+1} - u\|_H \leq \|I - L\|_{H \to H} \|u_n - u\|_H,$$

so that (2.1) converges if

$$\|I - L\|_{H \to H} < 1.$$

It is sometimes possible to precondition (1.1) by multiplying both sides by a matrix B,

$$BLu = Bf,$$

so that convergence of iterative methods is improved. This is very effective technique for solving differential equations, integral equations, and related problems [1, 3]. We want to do this by using frames.

Definition 2.1. A frame $\Psi = (\psi_\lambda)_{\lambda \in \Lambda}$ with frame operator S, is called an approximated inverse frame of operator L on H if $\|I - SL\|_{H \to H} < 1$.

Richardson iteration, preconditioned with an approximated inverse frame of L, has the following form

$$u_{n+1} = (I - SL)u_n + Sf,$$

where S is the frame operator of the frame Ψ.

Lemma 2.2. Let $\Psi = (\psi_\lambda)_{\lambda \in \Lambda} \subset H$ be a frame for H and L be a bounded operator on H. Then the sequence $\Phi = (L\psi_\lambda)_{\lambda \in \Lambda}$ is a frame for H.

Proof. See [OCH]. ■

Proposition 2.3. If $\Psi = (\psi_\lambda)_{\lambda \in \Lambda}$ is a frame for H with frame operator S and L is a bounded self adjoint operator on H then $SL = TT_L^*$, where T is the synthesis operator of frame Ψ and T_L^* is the analysis operator of frame $\Phi = (L\psi_\lambda)_{\lambda \in \Lambda}$.

Proof. Let $f \in H$, then

$$SLf = \sum_\lambda \langle Lf | \psi_\lambda \rangle \psi_\lambda = \sum_\lambda \langle f | L\psi_\lambda \rangle \psi_\lambda = T(\langle f | L\psi_\lambda \rangle)_\lambda = TT_L^* f,$$

that means $TT_L^* = SL$. ■

Using this proposition we can say that a frame $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$ is an approximated inverse frame of an bounded self adjoint operator L if $\|I - TT_L^*\|_{H\to H} < 1$. By taking adjoint this is equivalent to $\|I - T_L T^*\|_{H\to H} < 1$.

From linear algebra we know that if $\|I - SL\|_{H\to H} < 1$ then SL is nonsingular and

$$(SL)^{-1} = \sum_{n=0}^{\infty}(I - SL)^n. \quad (2.2)$$

If L is a bounded self adjoint operator on H and if $\Psi = (\psi_\lambda)_{\lambda\in\Lambda}$ is an approximated inverse frame of L, then the solution u of the equation (1.1) is as follow:

$$u = (SL)^{-1}(SL)u = \sum_\lambda \langle u|L\psi_\lambda\rangle (SL)^{-1}\psi_\lambda$$
$$= \sum_\lambda \langle Lu|\psi_\lambda\rangle (SL)^{-1}\psi_\lambda = \sum_\lambda \langle f|\psi_\lambda\rangle (SL)^{-1}\psi_\lambda ,$$

also by Proposition 2.3 we have

$$u = \sum_\lambda \langle f|\psi_\lambda\rangle (TT_L^*)^{-1}\psi_\lambda.$$

Now we note that each $f \in H$ has the representation as

$$f = (T_L T^*)^{-1}(T_L T^*)f = \sum_\lambda \langle f|\psi_\lambda\rangle (T_L T^*)^{-1}(L\psi_\lambda),$$

that means $(T_L T^*)^{-1}(L\psi_\lambda)$ is a dual frame of the frame $\Psi = (\psi_\lambda)_\lambda$. Specially we have the following expansion for the exact solution u of the equation (1.1)

$$u = \sum_\lambda \langle u|L\psi_\lambda\rangle (T_L T^*)^{-1}\psi_\lambda = \sum_\lambda \langle Lu|\psi_\lambda\rangle (T_L T^*)^{-1}\psi_\lambda = \sum_\lambda \langle f|\psi_\lambda\rangle (T_L T^*)^{-1}L\psi_\lambda.$$

The following theorem tries to approximate $(T_L T^*)^{-1}L\psi_\lambda$.

Theorem 2.4. let $\Psi = (\psi_\lambda)_\lambda$ be an approximated inverse frame of a bounded self adjoint operator L. For each $N \in \mathbb{N}$ define

$$\psi_\lambda^N = \sum_{n=0}(I - TT_L^*)^n \psi_\lambda.$$

Then for each $N \in \mathbb{N}$, $\left(\psi_\lambda^N\right)_\lambda$ is a frame for H. Also if T_N denotes its synthesis operator, then

$$\|I - T_N T_L^*\|_{H\to H} \le \|I - TT_L^*\|_{H\to H}^{N+1}.$$

Proof. Let $f \in H$,

$$T_N T_L^* f = \sum_\lambda \langle f|L\psi_\lambda\rangle \psi_\lambda^N = \sum_\lambda \langle f|L\psi_\lambda\rangle \sum_{n=0}(I - TT_L^*)^n \psi_\lambda$$
$$= \sum_{n=0}(I - TT_L^*)^n \sum_\lambda \langle f|L\psi_\lambda\rangle \psi_\lambda = \sum_{n=0}(I - TT_L^*)^n TT_L^* f$$
$$= \sum_{n=0}(I - TT_L^*)^n \left(I - (I - TT_L^*)\right)f = f - (I - TT_L^*)^{N+1}f.$$

Therefore

$$f - T_N T_L^* f = (I - TT_L^*)^{N+1}f,$$

that means

$$\|I - T_N T_L^*\|_{H\to H} = \left\|(I - TT_L^*)^{N+1}\right\|_{H\to H} \le \|I - TT_L^*\|_{H\to H}^{N+1} < 1. ■$$

Now, we are going to use frames and design an algorithm in order to approximate the solution u of the problem (1.1). This algorithm works based on the frame bounds and frame operator.

Theorem 2.5. Let $\Psi = (\psi_\lambda)_\lambda$ be a frame for H with frame operator S and $L: H \to H$ be a bounded self adjoint operator on H. Suppose that A and B are the bounds of the frame $\Phi = (\phi_\lambda)_\lambda = (L\psi_\lambda)_\lambda$. Given $u_0 \in H$, define for $n \geq 1$

$$u_n = u_{n-1} + \tfrac{2}{A+B} LS(f - Lu_{n-1}).$$

Then

$$\|u - u_n\|_H \leq \left(\tfrac{B-A}{B+A}\right)\|u\|_H,$$

so that u_n converges to the exact solution u as $n \to \infty$.

Proof. By definition of u_n

$$\begin{aligned} u - u_n &= u - u_{n-1} - \tfrac{2}{A+B} LSL(u - u_{n-1}) \\ &= \left(I - \tfrac{2}{A+B}LSL\right)(u - u_{n-1}) = \left(I - \tfrac{2}{A+B}LSL\right)^2 (u - u_{n-2}) = \cdots \\ &= \left(I - \tfrac{2}{A+B}LSL\right)^n (u - u_0), \end{aligned}$$

thus

$$\|u - u_n\|_H \leq \left\|I - \tfrac{2}{A+B}LSL\right\|_{H\to H}^n \|u\|_H \,. \quad (2.3)$$

But for every $v \in H$ we have

$$\begin{aligned} \left\langle \left(I - \tfrac{2}{A+B}LSL\right)v \middle| v\right\rangle &= \|v\|_H^2 - \tfrac{2}{A+B}\langle LSLv|v\rangle \\ &= \|v\|_H^2 - \tfrac{2}{A+B}\textstyle\sum_\lambda |\langle v|\phi_\lambda\rangle|^2 \\ &\leq \|v\|_H^2 - \tfrac{2A}{A+B}\|v\|_H^2 = \left(\tfrac{B-A}{B+A}\right)\|v\|_H^2. \end{aligned}$$

The last inequality obtains by the frame property of the frame $\Phi = (L\psi_\lambda)_{\lambda\in\Lambda}$. Similarly we have

$$-\left(\tfrac{B-A}{B+A}\right)\|v\|_H^2 \leq \left\langle \left(I - \tfrac{2}{A+B}LSL\right)v \middle| v\right\rangle.$$

So we conclude that

$$\left\|I - \tfrac{2}{A+B}LSL\right\|_{H\to H} \leq \tfrac{B-A}{B+A}. \quad (2.4)$$

Combining this inequality with (2.3) give

$$\|u - u_n\|_H \leq \left(\tfrac{B-A}{B+A}\right)^n \|u\|_H,$$

and so u_n converges to u as $n \to \infty$. ■

3. Using frames in Galerkin methods

In this section we will find a sequence of solutions u_i of finite dimensional analogs of the problem (1.1), related to the some finite frame sequences, such that converges to the exact solution u of the problem (1.1). By (1.5), u has the expansion

$$u = \textstyle\sum_{\lambda\in\Lambda}\langle u|\phi_\lambda\rangle (S')^{-1}\phi_\lambda,$$

where $(\psi_\lambda)_\lambda$ is a frame for H and S′ is the frame operator of the frame $(\phi_\lambda)_\lambda = (L\psi_\lambda)_\lambda$. Since L is self adjoint then

$$u = \textstyle\sum_{\lambda\in\Lambda}\langle u|L\psi_\lambda\rangle (S')^{-1}\phi_\lambda = \sum_{\lambda\in\Lambda}\langle Lu|\psi_\lambda\rangle (S')^{-1} = \sum_{\lambda\in\Lambda}\langle f|\psi_\lambda\rangle (S')^{-1}\phi_\lambda. \quad (3.1)$$

Proposition 3.1. Let $\Psi = (\psi_\lambda)_\lambda \subset H$ be a frame for H and $\widetilde{\Lambda} \subset \Lambda$ be a finite subset of Λ such that not all elements of $\Psi_{\widetilde{\Lambda}} = (\psi_\lambda)_{\lambda\in\widetilde{\Lambda}}$ are zero. Then $\Psi_{\widetilde{\Lambda}}$ is a frame sequence.

Proof. See [5]. ■

Let $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots \nearrow \Lambda$ be finite subsets of Λ where not all elements of Λ_1 are zero. By Proposition 3.1 each Ψ_{Λ_i} and consequently Φ_{Λ_i} is a frame sequence. We assume that Φ_{Λ_i} are frame sequences with common bounds A and B. If L is invertible, it follows that Φ_{Λ_i} are frame sequences with common bounds $\frac{A}{\|L^{-1}\|^2}$ and $B\|L\|^2$. Let $V_i = \operatorname{span}\Phi_{\Lambda_i} = \operatorname{span}(L\psi_\lambda)_{\lambda\in\Lambda_i}$ and $S'_i: V_i \to V_i$ be its frame operator, then for every $f \in V_i$ we have

$$S'_i f = \sum_{\lambda\in\Lambda_i}\langle f|L\psi_\lambda\rangle L\psi_\lambda = L(\sum_{\lambda\in\Lambda_i}\langle f|L\psi_\lambda\rangle\psi_\lambda) = L(\sum_{\lambda\in\Lambda_i}\langle Lf|\psi_\lambda\rangle\psi_\lambda) = LS_iLf,$$

therefore $S'_i = L_iS_iL_i$, where L_i is the restriction of L on V_i, and S_i is the frame operator of the frame sequence Ψ_{Λ_i}. Specially we obtain $(S'_i)^{-1} = L_i^{-1}S_i^{-1}L_i^{-1}$ and so

$$\sup_{i\in\mathbb{N}} \left\|\left(S_i^{'}\right)^{-1}\right\| \leq \|L^{-1}\|^2/A\,, \forall\, i \in \mathbb{N}. \quad (3.2)$$

Now we are ready to present the main result that is the following theorem.

Theorem 3.2. Let $\Psi_\Lambda, \Phi_\Lambda$ and Λ_i be as above. Assume that

$$u_i = \sum_{\lambda\in\Lambda_i}\langle f|\psi_\lambda\rangle\left(S_i^{'}\right)^{-1}\phi_\lambda,$$

then $u_i \to u$ as $i \to \infty$.

Proof. Let $(c_\lambda)_{\lambda\in\Lambda} \in \ell_2(\Lambda)$ and $\sum_{\lambda\in\Lambda} c_\lambda\phi_\lambda = 0$. Using (3.2) we have

$$\left\|\left(S_i^{'}\right)^{-1}\sum_{\lambda\in\Lambda_i} c_\lambda\phi_\lambda\right\|_{V_i} \leq \left\|\left(S_i^{'}\right)^{-1}\right\| \left\|\sum_{\lambda\in\Lambda_i} c_\lambda\phi_\lambda\right\|_{V_i} \leq (\|L^{-1}\|^2/A_1)\left\|\sum_{\lambda\in\Lambda_i} c_{\lambda\phi_\lambda}\right\|_{V_i},$$

hence

$$\left(S_i^{'}\right)^{-1}\sum_{\lambda\in\Lambda_i} c_\lambda\phi_\lambda \to 0, \text{as } i \to \infty. \quad (3.3)$$

Since $\left(\langle f|\psi_\lambda\rangle\right)_{\lambda\in\Lambda} \in \ell_2(\Lambda)$ and $\ell_2(\Lambda) = \operatorname{Ran}(T^*) \oplus \operatorname{Ker}(T)$ then

$$\left(\langle f|\psi_\lambda\rangle\right)_{\lambda\in\Lambda} = (\langle f'|\phi_\lambda\rangle)_{\lambda\in\Lambda} + (c_\lambda)_{\lambda\in\Lambda}$$

for some $f^{'} \in H$ and $(c_\lambda)_{\lambda\in\Lambda} \in \operatorname{Ker}(T)$. Thus

$$\sum_{\lambda\in\Lambda_i}\langle f|\psi_\lambda\rangle\left(S_i^{'}\right)^{-1}\phi_\lambda = \sum_{\lambda\in\Lambda_i}\langle f'|\phi_\lambda\rangle\left(S_i^{'}\right)^{-1}\phi_\lambda + \sum_{\lambda\in\Lambda_i} c_\lambda\left(S_i^{'}\right)^{-1}\phi_\lambda,$$

therefore

$$\sum_{\lambda\in\Lambda_i}\langle f|\psi_\lambda\rangle\left(S_i^{'}\right)^{-1}\phi_\lambda = \sum_{\lambda\in\Lambda_i}\langle f'|\phi_\lambda\rangle\left(S_i^{'}\right)^{-1}\phi_\lambda + \left(S_i^{'}\right)^{-1}\sum_{\lambda\in\Lambda_i} c_\lambda\phi_\lambda. \quad (3.4)$$

Also because of $(c_\lambda)_{\lambda\in\Lambda} \in \operatorname{Ker}(T)$, it follows

$$u = \sum_{\lambda\in\Lambda}\langle f|\psi_\lambda\rangle(S')^{-1}\phi_\lambda = \sum_{\lambda\in\Lambda}\langle f'|\phi_\lambda\rangle\left(S^{'}\right)^{-1}\phi_\lambda + \left(S^{'}\right)^{-1}\sum_{\lambda\in\Lambda} c_\lambda\phi_\lambda = f^{'},$$

combining this with (3.3) and (3.4) induce the result as $i \to \infty$. ∎

Now consider the operator equation (1.1), where L is a bounded, self adjoint and H-elliptic operator from H to H^*, that is $(Lv)(v) \geq C\|v\|^2$ for a constant C and every $v \in H$. In this case, by Lax-Milgram Lemma, the equation has a unique solution. Our problem is equivalent to find u such that

$$(Lu)v = f(v), \forall\, v \in H. \quad (3.5)$$

Now let Ψ, Φ, Λ, and Λ_i be as the previous section. Projecting the problem onto$H_i = \operatorname{span}\Psi_{\Lambda_i}$, we can again apply the Lax-Milgram Lemma and conclude that the problem

$$(Lu_i)(v) = f(v), \forall\, v \in H_i, \quad (3.6)$$

has a unique solution. If there exists a constant M such that $\left(S_i^{'}\right)^{-1} \leq M, \forall\, i \in \mathbb{N}$, similar to the previous section we can show that

$$u = \sum_{\lambda\in\Lambda}\langle f|\psi_\lambda\rangle\left(S^{'}\right)^{-1}\phi_\lambda,$$

and

$$u_i = \sum_{\lambda\in\Lambda_i}\langle f|\psi_\lambda\rangle\left(S_i^{'}\right)^{-1}\phi_\lambda.$$

The following theorem holds.

Theorem 3.3. Let $L: H \to H^*$ be a bounded, self adjoint and H-elliptic. If there exists a constant M such that $\left(S_i^{'}\right)^{-1} \leq M, \forall\, i \in \mathbb{N}$, then $u_i \to u$ as $i \to \infty$ and there exists a constant c such that

$$\|u - u_i\|_H \leq c \inf_{v\in H_i}\|u - v\|_H.$$

Proof. Subtracting (3.6) from (3.5) with $v \in H_i$, we obtain

$$L(u - u_i)v = 0, \forall\, v \in H_i. \quad (3.7)$$

By the ellipticity of L,

$$C\|u - u_i\|_H^2 \leq \left(L(u - u_i)\right)(u - u_i) = \left(L(u - u_i)\right)(u - v) \leq \|L\|\|u - u_i\|_H\|u - v\|_H,$$

Therefore

$$C\|u - u_i\|_H \leq \|L\|\|u - v\|_H, \forall\, v \in H_i,$$

that means

$$\|u - u_i\|_H \leq (\|L\|/C) \inf_{v\in H_i}\|u - v\|_H,$$

as we desired. ■

We can express the problem (3.6) in the form of a linear system of the finite dimensional space H_i. If we write $u_i = \sum_{j=1}^{i} \xi_j \phi_j$ then (3.6) is equivalent to a linear system

$$A_i\xi = b,$$

where $\xi = \left(\xi_j\right) \in \mathbb{R}^i$ is the unknown vector, $A_i = \left(\left(L\phi_j\right)\left(\phi_j\right)\right)_{i\times i}$ is the stiffness matrix and $b = \left(f\phi_j\right) \in \mathbb{R}^i$. Since L is bounded and elliptic then we can define a bounded and elliptic bilinear form $a: H \times H \to R$ by $a(u, v) = (Lu)v$. That is there exist constants M and C such that

$$|a(u, v)| \leq M\|u\|_H\|v\|_H, a(v, v) \geq C\|v\|_H^2.$$

Now we consider the equivalent problem to find

$$u \in H, a(u, v) = f(v), \forall\, v \in H.$$

In order to seeking the approximated solution $u_i \in H_i$, we project this problem onto H_i,

$$u_i \in H_i, a(u_i, v) = f(v), \forall\, v \in H_i.$$

This solution procedure is called the Galerkin method. Since $H_i \subset H_{i+1}$ and $\overline{\bigcup_{i\geq 1} H_i} = H$, then there exist a sequence $(v_i)_{i\geq 1}, v_i \in H_i$, such that $\|u - v_i\|_H \to 0$ as $i \to \infty$. Applying the previous theorem,

$$\|u - u_i\|_H \leq c\|u - v_i\|_H.$$

Therefore we conclude $\|u - u_i\|_H \to 0$ as $i \to \infty$, that means the Galerkin method converges.

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