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Exponential form for Lyapunov function and stability analysis of the fractional differential equations

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Abstract

This paper deals with an exponential form for Lyapunov function, in perspective to analyze the Lyapunov characterization of the Mittag-Leffler stability and the asymptotic stability for the fractional differential equations. In addition, a new Lyapunov characterization of Mittag-Leffler stability for fractional differential equations will be introduced. The exponential form will be used to prove the Lyapunov characterization of several stability notions, used in fractional differential equations. In this paper, the Caputo fractional derivative operator will be used to do the studies.

Keywords: Caputo fractional derivative, fractional differential equations, asymptotic stability.

2010 MSC: 93D20, 26A33, 93D30.

1. Introduction

Fractional calculus has several applications in many fields in science and engineering. Recently, a lot of progress were done in fractional calculus: in [8] Aguilar et al. expose the homotopy perturbation transform method, in [1] Aguilar et al. present Cattaneo-Vernotte equation for generalized heat and particle diffusion equations, in [5] Escamilla et al. present a Gray-Scot model by using variable order fractional differential equations, in [4] Escamilla et al. present recent models such as Bateman-Feshbach-Tikochinsky (BFT) or Caldirola-Kanai (CK) oscillators using the Caputo and Fabrizio fractional derivative and the Caputo-Liouville fractional derivative, in [2] Atangana compares behaviors of the exact solutions of evolution equations in three cases: with the Caputo fractional derivative, with the Caputo and Fabrizio fractional derivative, and with the Atangana Baleanu fractional derivative. With the stability analysis, the fractional calculus has many applications in control theory. There exists special functions in control theory called the comparison functions [9], they play an important role in the stability analysis. In this paper, we are devoted to prove these functions can be used to prove some stability notions existing in fractional calculus. The comparison functions play an important role in this present paper. That makes the link between fractional calculus and control theory. Numerous works on the stability of the fractional differential equations studied in the literature have used the trajectories and the Gronwall-inequality. Here we give a useful lemma which can be used to prove many characterizations using the Lyapunov functions in the stability analysis of the fractional differential equations (FDEs). We give $\mathcal{KL}$-estimate for fractional

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differential equations which is fundamental to prove the global asymptotic stability of the FDEs. Note that the same estimate exists in the ordinary differential equations (see [9] for more information). This estimate for the FDEs is introduced in [6], we prove this estimation with supplementary assumption and use it to give the exponential form for fractional differential equations.

Fractional calculus is a generalization of ordinary differential and integration to arbitrary non integer order. Numerous fractional derivative exist in the literature : The Riemann-Liouville fractional derivative, The Caputo fractional derivative, The Atangana-Baleanu fractional derivative, etc.. The Atangana-Baleanu fractional derivative has received much interest, and it was provided in the literature to have many applications in physics and mechanics [2]. New fractional derivative operators continue to appear and make the fractional calculus a very interesting field in science and engineering. In [16, 18] Caputo proposes the definition of the fractional derivative expressed as follows

$$D^c_\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^\alpha} ds.$$

In this paper, we use the Caputo fractional derivative to establish all main results. Many contributions in the fractional calculus field use the Caputo fractional derivative. Another useful fractional derivative called Gr unwald-Letnikov derivative [6] exists in the literature and is defined as

$$\lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{(t-\alpha)/h} (-1)^j \left(\begin{array}{c} \alpha \\ j \end{array} \right) f(t-jh).$$

A new generalization for this definition can be found in [14]. In the rest of this paper, we use the Caputo idea of the fractional derivative. There exists many investigations related to the stability analysis of the FDEs in the literature: in [18] some conditions for the stability notions of the fractional differential equations using the Riemann-Liouville derivative were proposed, in [20] some conditions for the stability for a particular class of conformable differential equations were also proposed, in [19] some conditions for stability respect to small inputs were proposed to study the behavior of the solutions of the fractional differential equations with exogenous inputs, etc.

In this present paper, we come with a $\mathcal{KL}$-estimate for fractional differential equations to prove the asymptotic stability of the FDEs. The stability of nonlinear systems received increased attention due to its important role in areas of science and engineering. This paper deals with useful estimate to prove Lyapunov characterization of the fractional differential equations. It treats in particular the asymptotic stability and the Mittag-Leffler stability of the fractional differential equations with the Caputo fractional derivative.

The remainder of this paper is organized as follows. In Section 2, we introduce certain necessary definitions and provided the necessary lemmas. In Section 3, we describe the class of the FDEs, and provide the main results. In Section 4, we provide an example and illustrate our main results. Our proofs, conclusions, and remarks are summarized in the Section 5.

Notations. $\mathcal{PD}$ denotes the set of all continuous functions $\chi : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ satisfying $\chi(0) = 0$ and $\chi(s) > 0$ for all $s > 0$. A class $\mathcal{K}$ function is an increasing $\mathcal{PD}$ function. The class $\mathcal{K}_\infty$ denotes the set of all unbounded $\mathcal{K}$ functions. A continuous function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to be class $\mathcal{KL}$ if $\beta(., t) \in \mathcal{K}$ for any $t \geqslant 0$ and $\beta(s, .)$ is non increasing and tends to zero as its arguments tends to infinity. Given $x \in \mathbb{R}^n$, $\|x\|$ stands for its Euclidean norm: $\|x\| := \sqrt{x_1^2 + \ldots + x_n^2}$. For a matrix A, $\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote the maximal and the minimal eigenvalue of A, respectively. If the condition $Re(\lambda_i) < 0$, $\forall$ $i = 1, 2, ..., n$, holds, then the matrix A is said Hurwitz.

2. Preliminary definitions and results

Before beginning the main results of this paper, we recall some definitions used in fractional calculus. We give the Riemann-Liouville fractional derivative and the Caputo fractional derivative. The definitions recalled here are not limited to the papers cited in this section and can be found in many other papers.

Definition 2.1 ([7, 11, 17, 18]). Given a function $f : [a, +\infty[\longrightarrow \mathbb{R}$, then the Riemann-Liouville fractional derivative of f of order α is defined as

$$D_{\alpha}^{RL}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-s)^{-\alpha}f(s)ds$$

for all $t > a$, $\alpha \in (0,1)$, and where $\Gamma(.)$ is gamma function.

Definition 2.2 ([11, 17, 18]). Given a function $f : [a, +\infty[\longrightarrow \mathbb{R}$, then the Caputo fractional derivative of f of order α is defined by

$$D_{\alpha}^{c}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{f'(s)}{(t-s)^{\alpha}}ds$$

for all $t > a$, $\alpha \in (0,1)$, and where $\Gamma(.)$ is gamma function.

Definition 2.3 ([3, 7, 17]). Given a function $f : [a, +\infty[\longrightarrow \mathbb{R}$, then the Riemann-Liouville integral of f of order α is defined by

$$I_{\alpha}^{RL}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds$$

for all $t > a$, $\alpha \in (0,1)$, and where $\Gamma(.)$ is gamma function.

In addition, if $\alpha = 1$, we recover the classical integral defined as $I_1^{RL}f(t) = \int_a^t f(s)ds$. Let recall the Mittag-Leffler function used in the structure of the solution of many fractional differential equations. The Mittag-Leffler function is defined using a series. There exists many other special functions in fractional calculus as the generalized function introduced in [12], the Mittag-Leffler function [15] and others. We have the following definition.

Definition 2.4 ([13, 15]). Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $z \in \mathbb{C}$. The Mittag-Leffler function is defined by the series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + \beta)}.$$

The exponential form is obtained if $\alpha = 1$ and $\beta = 1$. In other words the following relationship is hold $E_{1,1}(z) = \exp(z)$. The Mittag-Leffler function with one parameter follows if $\alpha = 1$, we have the following relationship $E_{\alpha}(z) = E_{\alpha,1}(z)$.

Lemma 2.5. *For all $\alpha \in (0,1)$ the function $E_{\alpha,1}(-t)$ is non increasing function respect to the argument t. In other words, the function $g(t) = E_{\alpha,1}(-t)$ is a $\mathcal{L}$ function.*

See in Figure 1, the behavior of the function g in time for different values of the order α.

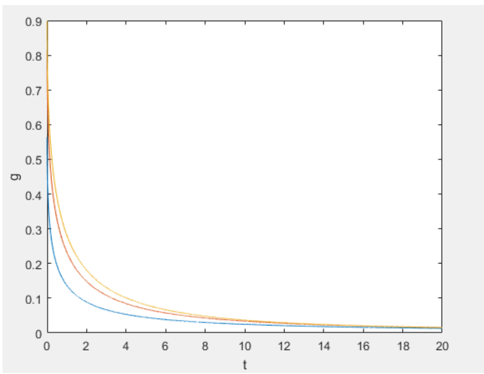

Figure 1: Behavior of g for $\alpha = 0.5$ and $\alpha = 0.75$ and $\alpha = 0.85$.

With the comparative functions in control theory, the above function is a class $\mathcal{L}$ function ($E_{\alpha,1}(-t) \in \mathcal{L}$). See definition related to the $\mathcal{L}$ function in the notation section. In general many fractional differential equations take the form $D^c_\alpha x(t) = \lambda x(t) + m(t)$. Thus, it is important to know how to get the solution of this differential equation. The structure of this solution is described in the following lemma and the proof can be found in [10].

Lemma 2.6 ([3])**.** *Let the fractional differential equation defined by* $D^c_\alpha x(t) = \lambda x(t) + m(t)$ *with initial condition defined as* $x(t_0) = \eta$*, then the unique solution is given by*

$$x(t) = \eta E_\alpha\left(\lambda(t-t_0)^\alpha\right) + \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda(t-s)^\alpha\right) m(s)ds.$$

Lemma 2.7 ([6, Comparison theorem for FDE])**.** *If* $D^c_\alpha x(t) \geqslant D^c_\alpha y(t)$ *with* $x(t_0) = y(t_0)$ *for all* $\alpha \in (0,1)$*, then* $x(t) \geqslant y(t)$.

Lemma 2.8 ($\mathcal{KL}$ estimate for FDEs)**.** *Let* $\alpha \in \mathcal{K}_\infty$ *be locally Lipschitz. Let a positive definite function* y*, which in addition is radially unbounded function, then there exists a class* $\mathcal{KL}$ *function* β *such that for all initial condition* $y(t_0) \in \mathbb{R}^+$*, the solution of the scalar differential equation* $D^c_\alpha y = -\alpha(y)$ *satisfies*

$$y(t, y_0) \leqslant \beta(\|y(t_0)\|, t - t_0).$$

The proof of this lemma was introduced on the fractional calculus by Delavari et al. in [6]. Here we will use the comparison theorem of the FDEs and add the radially unbounded condition to prove this fundamental lemma. This lemma without supplementary assumption was discussed in the literature. The Lemma 2.8 brings some precisions and can simplify many proofs using the Lyapunov characterization existing in the Literature. The first party of the proof is inspired to [11, 21].

Proof. We know that α is a class $\mathcal{K}_\infty$ function, then from the fact that $D^c_\alpha y = -\alpha(y)$ it follows that $D^c_\alpha y \leqslant 0$. Using the comparison Lemma 2.7 we have that $y(t) \leqslant y(t_0)$ for all $t \geqslant t_0$. From the assumption the function y is radially unbounded then there exists $c > 0$ such that $y(t) \geqslant c$. Furthermore, from the fact that α is an increasing function, we have

$$D^c_\alpha y = -\alpha(y) \leqslant -\alpha(c) = -\alpha(c) \times \frac{y(t_0)}{y(t_0)} = -\frac{\alpha(c)}{y(t_0)} \times y(t_0) \leqslant -\frac{\alpha(c)}{y(t_0)} \times y(t).$$

Let $k = \frac{\alpha(c)}{y(t_0)}$, then we obtain that

$$D^c_\alpha y \leqslant -ky(t).$$

We know that there exists a positive continuous function m such that [19]

$$D^c_\alpha y(t) = -ky(t) - m(t).$$

Using the Lemma 2.6, we have the following equality

$$y(t) = y(t_0)E_\alpha\left(-k(t-t_0)^\alpha\right) - \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-k(t-s)^\alpha\right) m(s)ds.$$

With the fact that $\int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-k(t-s)^\alpha\right) m(s)ds \geqslant 0$, we obtain the following inequality

$$y(t) \leqslant y(t_0)E_\alpha\left(-k(t-t_0)^\alpha\right).$$

By the Lemma 2.5 the function $E_\alpha\left(-k(t-t_0)^\alpha\right) \in \mathcal{L}$ and multiplying it by the variable s, we get $sE_\alpha\left(-k(t-t_0)^\alpha\right)$ which is a class $\mathcal{KL}$ function. Let $\beta(\|y(t_0)\|, t-t_0) = sE_\alpha\left(-k(t-t_0)^\alpha\right)$, we obtain that

$$y(t) \leqslant \beta(\|y(t_0)\|, t - t_0).$$

□

This lemma will serve to find an exponential form for Lyapunov function which is important to obtain the Mittag-Leffler stability and the asymptotic stability. We have the following lemma.

Lemma 2.9 (Exponential form for Lyapunov function for FDEs)**.** *If there exists a continuously differentiable function $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$, and class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ satisfying the following conditions*

1. $\alpha_1 (\|x\|) \leqslant V(t,x) \leqslant \alpha_2 (\|x\|)$;
2. $V(t,x)$ *has Caputo fractional derivative of order* α *for all* $t > t_0 \geqslant 0$;
3. $D^c_\alpha V(t,x) \leqslant -\alpha_3 (\|x\|)$,

then there exits a positive constant k *such that*

$$D^c_\alpha V(t,x) \leqslant -kV(t,x).$$

This lemma will have many consequences as we will see in the next section, related to the stability analysis of the FDEs. It's important to note that by the first assumption, α_1 is an increasing function, thus $\alpha_1 (\|x(t)\|) \geqslant \alpha_1 (\|x(t_0)\|)$. The first assumption prove V is radially unbounded then it holds in particular that $V(t,x) \geqslant \alpha_1 (\|x(t_0)\|)$. For the rest of the proof we repeat the proof of Lemma 2.8 as is described in the following proof.

Proof. It follows from assumption 3 that $D^c_\alpha V(t,x) \leqslant -\alpha_3 (\|x\|) \leqslant 0$. Using comparison Lemma 2.7 we have that $V(t) \leqslant V(t_0)$ for all $t \geqslant t_0$. Another fact, we have that $V(t,x) \geqslant \alpha_1 (\|x(t_0)\|)$. This remark is fundamental for the rest of the proof. Due to the fact the function α is an increasing function, using assumptions 1 and 3 we have the following

$$\begin{aligned} D^c_\alpha V(t,x) \leqslant -\alpha_3 (\|x\|) &\leqslant -\alpha_3 \circ \alpha_2^{-1}(V(t,x)) \\ &\leqslant -\alpha_3 \circ \alpha_2^{-1} \circ \alpha_1(\|x(t_0)\|) \\ &= -\alpha_3 \circ \alpha_2^{-1} \circ \alpha_1(\|x(t_0)\|) \times \frac{V(t_0)}{V(t_0)} \\ &= -\frac{\alpha_3 \circ \alpha_2^{-1} \circ \alpha_1(\|x(t_0)\|)}{V(t_0)} \times V(t_0) \\ &\leqslant -\frac{\alpha_3 \circ \alpha_2^{-1} \circ \alpha_1(\|x(t_0)\|)}{V(t_0)} \times V(t,x). \end{aligned}$$

Let $k = \frac{\alpha_3 \circ \alpha_2^{-1} \circ \alpha_1(\|x(t_0)\|)}{V(t_0)}$, then we obtain that

$$D^c_\alpha V(t,x) \leqslant -kV(t,x).$$

□

3. Stability analysis of fractional differential equations

The fractional differential equations considered in this section are represented by the following equation

$$D^c_\alpha x = f(t,x), \tag{3.1}$$

where $x \in \mathbb{R}^n$ is a state variable, and $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ a continuous locally Lipschitz function satisfying in particularly $f(t,0) = 0$. Given the initial condition $x_0 \in \mathbb{R}^n$, the solution of (3.1) starting at $x(t_0)$ at time $t = t_0$ is denoted by $x(.) = x(., x(t_0))$.

Note that the solution of the fractional differential equation defined by (3.1) exists because the function f is continuous and locally Lipschitz. The theory of the existence is not the subject of this paper, thus we suppose it. Let recall some stability notions used in fractional calculus. We begin with the definition of the stability of the fractional differential equations.

Definition 3.1 ([21]). The trivial solution of the fractional differential equation (3.1) is said to be stable if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon)$ such that for any initial condition $\|x(t_0\| < \delta$, the solution $x(t)$ of the system (3.1) satisfies the inequality $\|x(t)\| < \epsilon$ for all $t > t_0$.

The trivial solution of the system $D^c_\alpha x = f(t, x)$ is said to be asymptotically stable if it is stable and furthermore $\lim_{t\to+\infty} x(t) = 0$.

Definition 3.2 ([21]). We denote by $C_\infty((0, +\infty), \mathbb{R}^n)$ the set of function $x \in C_\infty((0, +\infty), \mathbb{R}^n)$ such that $D^c_\alpha x(t)$ exists and is continuous on $(0, +\infty)$.

For the main results of this paper, we recall the definitions of the Mittag-Leffler stability and the asymptotic stability.

Definition 3.3 ([11]). The origin of the fractional differential equation defined by (3.1) is said to be Mittag-Leffler stable, if for any initial condition $\|x(t_0)\|$ its solution satisfies

$$\|x(t, x_0)\| \leqslant [m(\|x(t_0)\|)E_\alpha(\lambda(t - t_0)^\alpha)]^{\frac{1}{b}},$$

where $b > 0$, and m is locally Lipschitz on a domain contained in $\mathbb{R}^n$ with a Lipschitz constant K and satisfies $m(0) = 0$.

Definition 3.4 ([9]). The origin of the fractional differential equation defined by (3.1) is said to be globally uniformly asymptotic stable if there exists a class $\mathcal{KL}$ function β such that for any initial condition $\|x(t_0)\|$ its solution satisfies

$$\|x(t, x_0)\| \leqslant \beta(\|x(t_0)\|, t - t_0).$$

To see the applications of the $\mathcal{KL}$-estimate for the FDEs, we make the following theorem to prove the global uniform asymptotic stability.

Theorem 3.5. *Let $x = 0$ be an equilibrium point for the fractional differential equations (3.1) and there exists a positive continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$, and class $\mathcal{K}_\infty$ functions χ_2, χ_3, χ_4, satisfying the following conditions*

1. $\chi_2(\|x\|) \leqslant V(t, x) \leqslant \chi_3(\|x\|)$;
2. $V(t, x)$ *has Caputo fractional derivative of order* α *for all* $t > t_0 \geqslant 0$;
3. $D^c_\alpha V(t, x) \leqslant -\chi_4(\|x\|)$,

then the origin of the fractional differential equation (3.1) is globally uniformly asymptotically stable.

Above theorem is originated to Yan Li et al. in [11] and has already proved. In the application of the above theorem, one may calculate $\chi_2^{-1} \circ (2\mu)$, where $\mu \in \mathcal{KL}$, the result can also depend to the Mittag-Leffler function. And then we make the following theorem to prove the Mittag-Leffler stability.

Theorem 3.6. *Let $x = 0$ be an equilibrium point for the fractional differential equation (3.1) and there exists a positive continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$, and class $\mathcal{K}_\infty$ functions χ_2, χ_3, χ_4 satisfying the following conditions*

1. $\chi_2(\|x\|) \leqslant V(t, x) \leqslant \chi_3(\|x\|)$;
2. $V(t, x)$ *has Caputo fractional derivative of order* α *for all* $t > t_0 \geqslant 0$;
3. $D^c_\alpha V(t, x) \leqslant -\chi_4(\|x\|)$.

If in addition

$$\chi_2^{-1}(2\mu(\chi_3(s), t - t_0)) \leqslant rsE_\alpha(-k(t - t_0)^\alpha),$$

where r is a positive constant, then the origin of the fractional differential equation (3.1) is Mittag-Leffler stable.

In many problems after using a given Lyapunov function, the α-derivative along the trajectories of the fractional differential equation under consideration, the result can also depend on the used Lyapunov function, for that we give the following theorem to prove the Mittag-Leffler stability.

Theorem 3.7. *Let $x = 0$ be an equilibrium point for the fractional differential equation (3.1) and there exists a positive continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$, and satisfying the following conditions*

1. $\|x(t)\|^a \leqslant V(t, x)$;
2. $V(t, x)$ *has Caputo fractional derivative of order* α *for all* $t > t_0 \geqslant 0$;
3. $D^c_\alpha V(t, x) \leqslant -kV(t, x)$,

where k is non negative constant. Then the origin of the fractional differential equation (3.1) is Mittag-Leffler stable.

Let see now a particular Lyapunov characterization of the Mittag-Leffler stability given in the following theorem.

Theorem 3.8. *Let $x = 0$ be an equilibrium point for the fractional differential equation (3.1) and there exists a positive continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$, and a class $\mathcal{K}_\infty$ function χ satisfying the following conditions*

1. $a\chi(\|x\|) \leqslant V(t, x) \leqslant b\chi(\|x\|)$;
2. $V(t, x)$ *has Caputo fractional derivative of order* α *for all* $t > t_0 \geqslant 0$;
3. $D^c_\alpha V(t, x) \leqslant -k\chi(\|x\|)$,

where a, b, k are non negative constants. Then the origin of the fractional nonlinear system (3.1) is Mittag-Leffler stable.

4. Illustrative example

In this section, we give an example to illustrate the Lyapunov characterization of the Mittag-Leffler stability and the asymptotic stability given in the Section 3. Let the following linear fractional differential equation defined as

$$D^c_\alpha x = Ax + Bx, \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state variable, A is an Hurwitz matrix in $\mathbb{R}^{n\times n}$, and B is an matrix in $\mathbb{R}^{n\times n}$. Let a Lyapunov candidate function defined by $V(t, x) = x^T Px$ where $A^T P + PA = -Q$ and $P = I_n$. The α-derivative of V along the trajectories of (4.1) yields that

$$\begin{aligned} D^c_\alpha V(t, x) \leqslant 2x^T PD^c_\alpha x &= [Ax + Bx]^T Px + x^T P [Ax + Bx] \\ &= x^T A^T Px + (Bx)^T Px + x^T PAx + x^T P(Bx) \\ &= x^T \left(A^T P + PA\right) x + (Bx)^T Px + x^T P(Bx) \\ &\leqslant -\lambda_{min}(Q) \|x\|^2 + 2\lambda_{max}(P) \|B\| \|x\|^2 \\ &= -[\lambda_{min}(Q) - 2\lambda_{max}(P) \|B\|] \|x\|^2 . \end{aligned}$$

Let that $k = \lambda_{min}(Q) - 2\lambda_{max}(P) \|B\| < 0$, then all assumptions of the Theorem 3.7 are satisfied with $\lambda_{min}(P) \|x\|^2 \leqslant V(t, x)$. That prove the trivial solution of the fractional differential equation (4.1) is Mittag-Leffler stable, furthermore (4.1) is asymptotic stable due to the linearity of the fractional differential equation. We can see that after calculating the state of the fractional differential equation (4.1), we obtain the following inequality

$$\|x(t)\| \leqslant [V(t_0, x_0)E_\alpha (-k(t - t_0)^\alpha)]^{1/2} .$$

5. Proofs of the theorems

5.1. *Proof of Theorem 3.5*

Let $x = 0$ be an equilibrium point for the fractional differential equation (3.1) and there exists a positive continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$, and class $\mathcal{K}_\infty$ functions χ_2, χ_3, χ_4 satisfying the following conditions

1. $\chi_2(\|x\|) \leqslant V(t, x) \leqslant \chi_3(\|x\|)$;

2. $V(t,x)$ has Caputo fractional derivative of order α for all $t > t_0 \geqslant 0$;
3. $D^c_\alpha V(t,x) \leqslant -\chi_4(\|x\|)$.

Commutating the assumptions (1) and (3) we have that

$$D^c_\alpha V(t,x) \leqslant -\chi_4\left(\chi_3^{-1}(V(t,x)\right).$$

Using $\mathcal{KL}$-estimate for fractional differential equations in the Lemma 2.8 there exists a class $\mathcal{KL}$ function μ such that

$$V(t,x) \leqslant \mu\left(\chi_3(\|x_0\|), t-t_0\right).$$

Using again the first assumption we obtain that

$$\chi_2(\|x\|) \leqslant \mu\left(\chi_3(\|x_0\|), t-t_0\right).$$

Recalling that $\chi_2^{-1}(a+b) \leqslant \chi_2^{-1}(2a)+\chi_2^{-1}(2b)$ as $\chi_2 \in \mathcal{K}_\infty$ and $a, b \in \mathbb{R}$, then we have

$$\|x(t)\| \leqslant \chi_2^{-1}(2\mu\left(\chi_3(\|x_0\|), t-t_0\right)).$$

Let the function $\beta(\|x_0\|, t-t_0)) = \chi_2^{-1}(2\mu\left(\chi_3(\|x_0\|), t-t_0\right)) \in \mathcal{KL}$. We obtain that

$$\|x(t)\| \leqslant \beta(\|x_0\|, t-t_0)).$$

We conclude that the origin of the fractional differential equation (3.1) is globally uniformly asymptotically stable.

5.2. *Proof of Theorem* 3.6

Let $x = 0$ be an equilibrium point for the fractional differential equation (3.1) and there exists a positive continuouse function $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$, and class $\mathcal{K}_\infty$ functions χ_2, χ_3, χ_4 satisfying the following conditions

1. $\chi_2(\|x\|) \leqslant V(t,x) \leqslant \chi_3(\|x\|)$;
2. $V(t,x)$ has Caputo fractional derivative of order α for all $t > t_0 \geqslant 0$;
3. $D^c_\alpha V(t,x) \leqslant -\chi_4(\|x\|)$.

Using the assumptions (1) and (3) we get that

$$D^c_\alpha V(t,x) \leqslant -\chi_4\left(\chi_3^{-1}(V(t,x)\right).$$

Using the exponential form for Lyapunov function for FDEs in Lemma 2.9 there exists a positive constant k such that

$$D^c_\alpha V(t,x) \leqslant -kV(t,x).$$

Doing same reasoning as in the proof of Lemma 2.8 we obtain that

$$V(t,x) \leqslant V(t_0,x_0)E_\alpha\left(-k(t-t_0)^\alpha\right).$$

Using again the first assumption we have that

$$\chi_2(\|x\|) \leqslant V(t_0,x_0)E_\alpha\left(-k(t-t_0)^\alpha\right).$$

Recalling that $\chi_2^{-1}(a+b) \leqslant \chi_2^{-1}(2a)+\chi_2^{-1}(2b)$ as $\chi_2 \in \mathcal{K}_\infty$ and $a, b \in \mathbb{R}$, then we have

$$\|x(t)\| \leqslant \chi_2^{-1}(2V(t_0,x_0)E_\alpha\left(-k(t-t_0)^\alpha\right)).$$

If in addition

$$\chi_2^{-1}(2\mu\left(s, t-t_0\right)) \leqslant csE_\alpha\left(-k(t-t_0)^\alpha\right),$$

where $\mu(s, t-t_0) = sE_\alpha\left(-k(t-t_0)^\alpha\right)$, we have the following inequality

$$\|x(t)\| \leqslant cV(t_0,x_0)E_\alpha\left(-k(t-t_0)^\alpha\right).$$

Then the origin of the fractional differential equations (3.1) is Mittag-Leffler stable.

5.3. *Proof of Theorem* 3.7

Let $x = 0$ be an equilibrium point for the fractional differential equation (3.1) and there exists a positive continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$, satisfying the following conditions

1. $\|x(t)\|^a \leqslant V(t,x)$;
2. $V(t,x)$ has Caputo fractional derivative of order α for all $t > t_0 \geqslant 0$;
3. $D^c_\alpha V(t,x) \leqslant -kV(t,x)$,

where k is non negative constant. By the condition in assumption (3) we have that

$$D^c_\alpha V(t,x) \leqslant -kV(t,x).$$

Doing the same reasoning as in the proof of the Lemma 2.8 we obtain that

$$V(t,x) \leqslant V(t_0,x_0)E_\alpha\left(-k(t-t_0)^\alpha\right).$$

Using the first assumption we have that

$$\|x(t)\| \leqslant [V(t_0,x_0)E_\alpha\left(-k(t-t_0)^\alpha\right)]^{1/a}.$$

Then the origin of the fractional differential equations (3.1) is Mittag-Leffler stable.

5.4. *Proof of Theorem* 3.8

Let $x = 0$ be an equilibrium point for the fractional nonlinear system (3.1) and there exists a positive continuous function $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$, and class $\mathcal{K}_\infty$ functions χ satisfying the following conditions

1. $a\chi(\|x\|) \leqslant V(t,x) \leqslant b\chi(\|x\|)$;
2. $V(t,x)$ has Caputo fractional derivative of order α for all $t > t_0 \geqslant 0$;
3. $D^c_\alpha V(t,x) \leqslant -k\chi(\|x\|)$,

where a, b, k are non negative constants. Using the conditions in the assumptions (1) and (3) we have that

$$D^c_\alpha V(t,x) \leqslant -k\chi\left(\chi^{-1}(\frac{1}{b}V(t,x(t))\right).$$

We observe that we obtain directly the following inequality

$$D^c_\alpha V(t,x) \leqslant -\frac{k}{b}V(t,x(t)).$$

Doing same reasoning as in the proof of Lemma 2.8 we obtain that

$$V(t,x) \leqslant V(t_0,x_0)E_\alpha\left(-\frac{k}{b}(t-t_0)^\alpha\right).$$

Using again the first assumption we get that

$$\chi(\|x\|) \leqslant V(t_0,x_0)E_\alpha\left(-\frac{k}{b}(t-t_0)^\alpha\right).$$

Let that $\beta(\|x_0\|, t-t_0)) = V(t_0,x_0)E_\alpha\left(-\frac{k}{b}(t-t_0)^\alpha\right)$ which is a class $\mathcal{KL}$ function. Recall the fact that $\beta(s, t-t_0)) = \chi(E_\alpha\left(-\frac{k}{b}(t-t_0)^\alpha\right)\chi(s))$. Note that a transformation of the function β done here with the function χ is particular and restrictive. That is one of the decomposition which we can obtain with comparison function. This decomposition is motived by result which we want to obtain. Then we have that

$$\|x(t)\| \leqslant \chi^{-1}(\chi(E_\alpha\left(-\frac{k}{b}(t-t_0)^\alpha\right))\chi(V(t_0,x_0))).$$

Explicitly by above inequality, we obtain that

$$\|x(t)\| \leqslant \chi(V(t_0,x_0))E_\alpha\left(-\frac{k}{b}(t-t_0)^\alpha\right).$$

Then the origin of the fractional differential equation (3.1) is Mittag-Leffler stable.

6. Conclusion

We have discussed in this paper the Mittag-Leffler stability and globally uniformly asymptotic stability of the fractional differential equations using Caputo fractional derivative. It contributes to give particularly the existence of $\mathcal{KL}$-estimate and give the exponential form for Lyapunov function for FDEs. Theses tools have many applications in Lyapunov characterization of the stability of the fractional differential equations. This paper offer also basic properties in comparison functions.

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