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Parameters identification and dual synchronization between different chaotic and hyperchaotic systems

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Abstract

This paper investigates the adaptive dual synchronization of completely different four chaotic and hyperchaotic systems with unknown parameters. Based on the Lyapunov stability theory, an efficient adaptive synchronization controller is constructed that converges the synchronization error signals to the origin with sufficient transient speed. Suitable adaptive laws of unknown parameters are designed that converged the estimated values of the unknown parameters to the true values of the systems parameters. Two numerical examples are presented and simulation results are derived to illustrate the effectiveness of the proposed dual synchronization approach.

Keywords: Chaos, dual synchronization, adaptive control, unknown parameters, Lyapunov stability theory.

2010 MSC: 93Dxx, 93Cxx .

1. Introduction

The idea of chaos synchronization was first introduced by Pecorra and Carroll [35] in 1991. Since then, the synchronization of chaotic systems has been widely investigated in various scientific disciplines for it's theoretical as well as experimental challenges [7, 25]. Different synchronization control techniques and strategies have been developed. These include, active control [44], linear feedback control [40], nonlinear control, adaptive control, sliding mode control, projective synchronization and the lag synchronization [1–6, 15–18, 20, 27, 42, 43, 45], etc.. In practical situations, some or all parameters of a chaotic systems cannot be accessible in priori, and the effect of these uncertainties may completely break the synchronization control. The adaptive control strategy is used when some or all parameters of the chaotic systems are unknown. The significant features of the adaptive control strategy include the fast response, robustness against the uncertainties, good transient performance, and easy implementation in practical applications. Recently, an interesting synchronization phenomena discovered is the dual synchronization. In the dual synchronization, more than one pair of a drive (master) and response (slave)

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chaotic systems are synchronized using a generated signal through a linear combination of the drive systems states. An intensive research has been devoted to study the dual synchronization of chaotic systems. Tsimring and Sushchik [38] proposed the idea of dual chaos synchronization both for iterated maps and ordinary differential equations. The dual synchronization between two different pairs of chaotic oscillators using a scalar signal is studied [22], where the dual synchronization is achieved by means of specific classes of piecewise-linear maps with conditional linear coupling. In [39], the authors achieved the dual synchronization between pairs of lasers by changing the optical frequency of each laser and controlled the condition of injection locking instead of matching of internal parameters. Ning et al. [26] investigated the dual synchronization among two different chaotic systems. The authors [14] established the problem of dual synchronization using time-varying gain proportional output feedback. Gosh and Chaudhary [13] further enhanced the dual synchronization of time-delay chaotic systems. Based on the Krasovskii-Lyapunov functional approach, the authors [12] proposed a nonlinear observer-based projective-dual synchronization in modulated coupled time-delayed systems. The dual synchronization is further improved for fractional-order chaotic systems [41]. Using an adaptive and nonlinear feedback control approaches [28–34], the dual synchronization and anti-synchronization are investigated. To the best of our knowledge, the dual synchronization for pairs of non-identical unknown chaotic and hyperchaotic systems has not been investigated and has been remained an open problem. In addition, it is well-known that if the systems are completely different, then the messages of secure applications are more effective and can not be captured easily. Generally, the applications of chaos dual synchronization in secure communication make it much more important. In fact, in engineering, it is hardly the case that every component can be assumed to be identical. Being motivated by the above discussion, the main objective of this paper is to study the dual synchronization between two different unknown chaotic and hyperchaotic systems. Based on the Lyapunov stability theory, a suitable adaptive controller is constructed that accomplished the asymptotic stability of the synchronization errors at the origin. This paper also designed the update laws for the estimation of unknown parameters. Two numerical examples are presented to show the efficiency and performance of the proposed dual synchronization strategy. The rest of the paper is organized as follows. Section 2 presents the problem statement and theory of the proposed dual synchronization strategy. In Section 3, the dual synchronization among four unknown chaotic Lorenz, Lü, Chen and Genesio systems and dual synchronization among four unknown hyperchaotic Lorenz, Chen, Lü and Lorenz-Stenflo systems are investigated. This paper concludes in Section 5.

2. Problem statement

Consider the following two different systems with uncertain parameters as the drive systems:

$$\dot{x}_1 = f_1(x_1) + F_1(x_1)\alpha, \qquad \dot{y}_1 = g_1(y_1) + G_1(y_1)\beta, \tag{2.1}$$

where $x_1 \in R^n$, $y_1 \in R^n$ are the state vectors of the systems, $f_1 : R^n \to R^n$ and $g_1 : R^n \to R^n$ are two continuous vector functions, $F_1 : R^n \to R^{n\times m_1}$, $G_1 : R^n \to R^{n\times q_1}$ are two matrix functions, and $\alpha \in R^{m_1}$, $\beta \in R^{q_1}$ are the unknown parameter vectors of the two drive systems. By a linear combination of the drive systems states, a scalar signal is generated in the form of

$$\varepsilon_d = \sum_{i=1}^{n} (a_i x_1 + b_i y_1) = A^T x_1 + B^T y_1 = C^T x,$$

where $A = (a_1, a_2, \dots, a_n)^T$ and $B = (b_1, b_2, \dots, b_n)^T$ are coupling parameters to be selected later, $C = (A^T \;\; B^T)^T$, $x = (x_1^T \;\; y_1^T)^T$. This generated scalar signal is fed to the following response systems which are completely different from the drive systems.

$$\dot{x}_2 = f_2(x_2) + F_2(x_2)\delta + u_1, \qquad \dot{y}_2 = g_2(y_2) + G_2(y_2)\theta + u_2, \tag{2.2}$$

where $x_2 \in R^n$ and $y_2 \in R^n$ are the state vectors, $f_2 : R^n \to R^n$ and $g_2 : R^n \to R^n$ are two continuous vector functions, $F_2 : R^n \to R^{n\times m_2}$, $G_2 : R^n \to R^{n\times q_2}$ are two matrix functions, and $\delta \in R^{m_2}$, $\theta \in R^{q_2}$ are the unknown parameter vectors of the two response systems and $u = (u_1 \;\; u_2)^T \in R^{2n}$ is a controller. The feedback signal of the system (2.2) is given as follows:

$$\varepsilon_r = \sum_{i=1}^{n} (a_i x_2 + b_i y_2) = C^T y,$$

where $y = (x_2^T \;\; y_2^T)^T$ is a combination of the responses states.

2.1. *Adaptive dual synchronization*

System (2.1) can be written in the following form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} f_1(x_1) \\ g_1(y_1) \end{bmatrix} + \begin{bmatrix} F_1(x_1) & 0 \\ 0 & G_1(y_1) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad \dot{x} = f(x) + F(x)\Lambda. \tag{2.3}$$

where $\dot{x} = \begin{bmatrix} \dot{x}_1 & \dot{y}_1 \end{bmatrix}^T$, $f(x) = \begin{bmatrix} f_1(x_1) & g_1(y_1) \end{bmatrix}^T$, $F(x) = \begin{bmatrix} F_1(x_1) & 0 \\ 0 & G_1(y_1) \end{bmatrix}$, and $\Lambda = \begin{bmatrix} \alpha & \beta \end{bmatrix}^T$. Similarly, system (2.2) can be written in the following form:

$$\begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} f_2(x_2) \\ g_2(y_2) \end{bmatrix} + \begin{bmatrix} F_2(x_2) & 0 \\ 0 & G_2(y_2) \end{bmatrix} \begin{bmatrix} \delta \\ \theta \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad \dot{y} = g(y) + G(y)\chi + U, \tag{2.4}$$

where $\dot{y} = \begin{bmatrix} \dot{x}_2 & \dot{y}_2 \end{bmatrix}^T$, $g(y) = \begin{bmatrix} f_2(x_2) & g_2(y_2) \end{bmatrix}^T$, $G(x) = \begin{bmatrix} F_2(x_1) & 0 \\ 0 & G_2(y_2) \end{bmatrix}$, and $\chi = \begin{bmatrix} \delta & \theta \end{bmatrix}^T$ and $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$. The main goal is to dual synchronize the drive systems (2.1) and the respond systems (2.2) such that

$$\lim_{t\to\infty} \|x_2 - x_1\| = 0, \qquad \lim_{t\to\infty} \|y_2 - y_1\| = 0.$$

Remark 2.1. In this paper, we will assume that systems (2.1) and systems (2.2) satisfies $f_1(\cdot) \neq f_2(\cdot)$ and $F_1(\cdot) \neq F_2(\cdot)$, $g_1(\cdot) \neq g_2(\cdot)$ and $G_1(\cdot) \neq G_2(\cdot)$, then the structure of systems (2.1) and (2.2) are different.

Theorem 2.2. *If the adaptive control functions is selected as*

$$U = f(x) + F(x)\hat{\Lambda} - g(y) - G(y)\hat{\chi} - ke - e_s$$

and updated by the following adaptation laws:

$$\dot{\hat{\Lambda}} = -[F(x)]^T e, \qquad \dot{\hat{\chi}} = [G(y)]^T e,$$

then the response system (2.3) can adaptive dual synchronize the drive system (2.4) globally and asymptotically, where $k > 0$ is a constant, $e_s = C^T(y - x)$ is linear coupling for dual synchronization, and $\hat{\Lambda}, \hat{\chi}$ are the estimations of the unknown parameters Λ, χ, which need to be estimated.

Proof. The error system of the adaptive control scheme between systems (2.3) and (2.4) is given as follows:

$$\dot{e} = G(y)(\chi - \hat{\chi}) - F(x)(\Lambda - \hat{\Lambda}) - ek - e_s. \tag{2.5}$$

Let us construct the following Lyapunov function candidate as

$$V = \frac{1}{2}[e^T e + \tilde{\Lambda}^T \tilde{\Lambda} + \tilde{\chi}^T \tilde{\chi}],$$

where, $\tilde{\Lambda} = \Lambda - \hat{\Lambda}, \tilde{\chi} = \chi - \hat{\chi}$. Taking the time derivative of V along the error dynamical system (2.5)which is given as follows:

$$\dot{V} = [e^T\dot{e} + \dot{\tilde{\Lambda}}^T\tilde{\Lambda} + \dot{\tilde{\chi}}^T\tilde{\chi}],$$
$$= e^T[G(y)\tilde{\chi} - F(x)\tilde{\Lambda} - ke - e_s] + e^TF(x)\tilde{\Lambda} - e^TG(x)\tilde{\chi} = -e^Tek - e^Te_s = -e^TPe.$$

Obviously, to ensure that the zero solution of the error dynamical system (2.5) is asymptotically stable, the real symmetric matrix P should be positive definite. So, based on Lyapunov stability theory [36], we know that the error dynamical system (2.5) is asymptotically stable. □

3. Dual synchronization of different chaotic systems with unknown parameters

In this section of the paper, we imply our proposed adaptive control method to achieve the asymptotical dual synchronization of non identical chaotic systems with unknown parameters. Let us consider the drive-response systems as follows:

Drive 1: Lorenz system [23] is given by

$$\dot{x_1} = \sigma(y_1 - x_1), \quad \dot{y_1} = \rho x_1 - x_1z_1 - y_1, \quad \dot{z_1} = x_1y_1 - \gamma z_1. \tag{3.1}$$

Drive 2: Lü system [24] is given by

$$\dot{x_2} = \alpha(y_2 - x_2), \quad \dot{y_2} = -x_2z_2 + \delta y_2, \quad \dot{z_2} = x_2y_2 - \beta z_2. \tag{3.2}$$

Response 1: Chen system [9]

$$\dot{x_3} = \alpha_1(y_3 - x_3) + u_1, \quad \dot{y_3} = (\delta_1 - \alpha_1)x_3 - x_3z_3 + \delta_1y_3 + u_2, \quad \dot{z_3} = x_3y_3 - \beta_1z_3 + u_3. \tag{3.3}$$

Response 2: Gensio system [11]

$$\dot{x_4} = y_4 + u_4, \quad \dot{y_4} = z_4 + u_5, \quad \dot{z_4} = -\sigma_1x_4 - \rho_1y_4 - \gamma_1z_4 + x_4^2 + u_6, \tag{3.4}$$

where σ, ρ, γ, α, δ, β and $\alpha_1, \delta_1, \beta_1, \sigma_1, \rho_1, \gamma_1$ are unknown system parameters which need to be estimated. The control input is defined as $u_i, (i = 1, 2, \ldots, 6)$. Subtracting (3.1) from (3.3) and (3.2) from (3.4), then the error system is given as follows:

$$\begin{aligned}
\dot{e_1} &= \alpha_1(y_3 - x_3) - \sigma(y_1 - x_1) + u_1,\\
\dot{e_2} &= (\delta_1 - \alpha_1)x_3 - x_3z_3 + \delta_1y_3 - \rho x_1 + x_1z_1 + y_1 + u_2,\\
\dot{e_3} &= x_3y_3 - \beta_1z_3 - x_1y_1 + \gamma z_1 + u_3,\\
\dot{e_4} &= y_4 - \alpha(y_2 - x_2) + u_4,\\
\dot{e_5} &= z_4 + x_2z_2 - \delta y_2 + u_5,\\
\dot{e_6} &= -\sigma_1x_4 - \rho_1y_4 - \gamma_1z_4 + x_4^2 - x_2y_2 + \beta z_2 + u_6,
\end{aligned} \tag{3.5}$$

where $e_1 = x_3 - x_1$, $e_2 = y_3 - y_1$, $e_3 = z_3 - z_1$, $e_4 = x_4 - x_2$, $e_5 = y_4 - y_2$, $e_6 = z_4 - z_2$. The dual synchronization of non identical chaotic systems needs the stabilization of the error system (3.5) at the origin by a suitable adaptive controller which forces the trajectories of the response systems to track the trajectories of the drive systems so that the error vector in the situation of unknown parameter i.e., $\lim_{t\to\infty} \|e_i(t)\| = 0$, (i=1,2,. . . ,6). In order to achieve that, we state the following theorem.

Theorem 3.1. *For any initial conditions, the systems (3.1)-(3.2) and (3.3)-(3.4) are globally asymptotically dual synchronized if we design the following adaptive control functions and adaptation law:*

$$\begin{aligned}
u_1 &= -\hat{\alpha}_1(y_3 - x_3) + \hat{\sigma}(y_1 - x_1) - k_1e_1 - e_s,\\
u_2 &= -(\hat{\delta}_1 - \hat{\alpha}_1)x_3 + x_3z_3 - \hat{\delta}_1y_3 + \hat{\rho}x_1 - x_1z_1 - y_1 - k_2e_2 - e_s,\\
u_3 &= -x_3y_3 + \hat{\beta}_1z_3 + x_1y_1 - \hat{\gamma}z_1 - k_3e_3 - e_s,\\
u_4 &= -y_4 + \hat{\alpha}(y_2 - x_2) - k_4e_4 - e_s,\\
u_5 &= -z_4 - x_2z_2 + \hat{\delta}y_2 - k_5e_5 - e_s,\\
u_6 &= \hat{\sigma}_1x_4 + \hat{\rho}_1y_4 + \hat{\gamma}_1z_4 - x_4^2 + x_2y_2 - \hat{\beta}z_2 - k_6e_6 - e_s,
\end{aligned} \tag{3.6}$$

and

$$\begin{array}{llll}\dot{\hat{\sigma}} = -(y_1 - x_1)e_1, & \dot{\hat{\rho}} = -x_1 e_2, & \dot{\hat{\gamma}} = z_1 e_3, & \dot{\hat{\alpha}} = -(y_2 - x_2)e_4, \\ \dot{\hat{\delta}} = y_2 e_5, & \dot{\hat{\beta}} = z_2 e_6, & \dot{\hat{\alpha}}_1 = (y_3 - x_3)e_1 - x_3 e_2, & \dot{\hat{\delta}}_1 = (x_3 + y_3)e_2, \\ \hat{\beta}_1 = -z_3 e_3, & \dot{\hat{\sigma}}_1 = -x_4 e_6, & \dot{\hat{\rho}}_1 = -y_4 e_6, & \dot{\hat{\gamma}}_1 = -z_4 e_6,\end{array} \tag{3.7}$$

where $k_i > 0, (i = 1, 2, \ldots, 6)$, $\hat{\sigma}, \hat{\rho}, \hat{\gamma}$, $\hat{\alpha}, \hat{\delta}, \hat{\beta}$ *and* $\hat{\alpha}_1, \hat{\delta}_1, \hat{\beta}_1, \hat{\sigma}_1, \hat{\rho}_1, \hat{\gamma}_1$ *are the parameter estimations of* σ, ρ, γ, α, δ, β *and* $\alpha_1, \delta_1, \beta_1, \sigma_1, \rho_1, \gamma_1$, *respectively, and*

$$e_s = \sum_{i=1}^{3} a_i e_i + \sum_{i=1}^{3} b_i e_j, \quad j = 4, 5, 6, e = y - x, y \neq x.$$

where, e_s *is the linear coupling of the drive and response systems.*

Proof. Inserting (3.6) into (3.5) yields the following form of the error system

$$\begin{aligned}\dot{e_1} &= \tilde{\alpha}_1(y_3 - x_3) - \tilde{\sigma}(y_1 - x_1) - k_1 e_1 - e_s, \\ \dot{e_2} &= (\tilde{\delta}_1 - \tilde{\alpha}_1)x_3 + \tilde{\delta}_1 y_3 - \rho x_1 - k_2 e_2 - e_s, \\ \dot{e_3} &= -\tilde{\beta}_1 z_3 + \tilde{\gamma} z_1 - k_3 e_3 - e_s, \\ \dot{e_4} &= -\tilde{\alpha}(y_2 - x_2) - k_4 e_4 - e_s, \\ \dot{e_5} &= -\tilde{\delta} y_2 - k_5 e_5 - e_s, \\ \dot{e_6} &= -\tilde{\sigma}_1 x_4 - \tilde{\rho}_1 y_4 - \tilde{\gamma}_1 z_4 + \tilde{\beta} z_2 - k_6 e_6 - e_s,\end{aligned}$$

where $\tilde{\sigma} = \sigma - \hat{\sigma}, \tilde{\rho} = \rho - \hat{\rho}, \tilde{\gamma} = \gamma - \hat{\gamma}$, $\tilde{\alpha} = \alpha - \hat{\alpha}, \tilde{\delta} = \delta - \hat{\delta}, \tilde{\beta} = \beta - \hat{\beta}$ and $\tilde{\alpha}_1 = \alpha_1 - \hat{\alpha}_1, \tilde{\delta}_1 = \delta_1 - \hat{\delta}_1, \tilde{\beta}_1 = \beta_1 - \hat{\beta}_1, \tilde{\sigma}_1 = \sigma_1 - \hat{\sigma}_1, \tilde{\rho}_1 = \rho_1 - = \hat{\rho}_1, \tilde{\gamma}_1 = \gamma_1 - \hat{\gamma}_1$. Let us construct the following Lyapunov function candidate as

$$V = \frac{1}{2}(e^T e + \tilde{\sigma}^2 + \tilde{\rho}^2 + \tilde{\gamma}^2 + \tilde{\alpha}^2 + \tilde{\delta}^2 + \tilde{\beta}^2 + \tilde{\alpha}_1^2 + \tilde{\delta}_1^2 + \tilde{\beta}_1^2 + \tilde{\sigma}_1^2 + \tilde{\rho}_1^2 + \tilde{\gamma}_1^2).$$

Applying the above equations in the time derivative of V leads to

$$\begin{aligned}\dot{V} &= e^T \dot{e} + \tilde{\sigma}\dot{\tilde{\sigma}} + \tilde{\rho}\dot{\tilde{\rho}} + \tilde{\gamma}\dot{\tilde{\gamma}} + \tilde{\alpha}\dot{\tilde{\alpha}} + \tilde{\delta}\dot{\tilde{\delta}} + \tilde{\beta}\dot{\tilde{\beta}} + \tilde{\alpha}_1\dot{\tilde{\alpha}}_1 + \tilde{\delta}_1\dot{\tilde{\delta}}_1 + \tilde{\beta}_1\dot{\tilde{\beta}}_1 + \tilde{\sigma}_1\dot{\tilde{\sigma}}_1 + \tilde{\rho}_1\dot{\tilde{\rho}}_1 + \tilde{\gamma}_1\dot{\tilde{\gamma}}_1 \\ &= e_1\big[\tilde{\alpha}_1(y_3 - x_3) - \tilde{\sigma}(y_1 - x_1) - k_1 e_1 - e_s\big] + e_2\big[(\tilde{\delta}_1 - \tilde{\alpha}_1)x_3 + \tilde{\delta}_1 y_3 \\ &\quad - \tilde{\rho} x_1 - k_2 e_2 - e_s\big] + e_3\big[-\tilde{\beta}_1 z_3 + \tilde{\gamma} z_1 - k_3 e_3 - e_s\big] + e_4\big[-\tilde{\alpha}(y_2 - x_2) \\ &\quad - k_4 e_4 - e_s\big] + e_5\big[-\tilde{\delta} y_2 - k_5 e_5 - e_s\big] + e_6\big[-\tilde{\sigma}_1 x_4 - \tilde{\rho}_1 y_4 - \tilde{\gamma}_1 z_4 + \tilde{\beta} z_2 \\ &\quad - k_6 e_6 - e_s\big] + \tilde{\sigma}((y_1 - x_1)e_1) + \tilde{\rho}(x_1 e_2) + \tilde{\gamma}(-z_1 e_3) + \tilde{\alpha}((y_2 - x_2)e_4) \\ &\quad + \tilde{\delta}(-y_2 e_5) + \tilde{\beta}(-z_2 e_6) + \tilde{\alpha}_1(-((y_3 - x_3)e_1 - x_3 e_2)) + \tilde{\delta}_1(-(x_3 + y_3)e_2) \\ &\quad + \tilde{\beta}_1(z_3 e_3) + \tilde{\sigma}_1(x_4 e_6) + \tilde{\rho}_1(y_4 e_6) + \tilde{\gamma}_1(z_4 e_6) \\ &= -\big[(k_1 + a_1)e_1^2 + (a_1 + a_2)e_1 e_2 + (a_1 + a_3)e_1 e_3 + (a_1 + b_1)e_1 e_4 + (a_1 + b_2)e_1 e_5 \\ &\quad + (a_1 + b_3)e_1 e_6 + (k_2 + a_2)e_2^2 + (a_2 + a_3)e_2 e_3 + (a_2 + b_1)e_2 e_4 + (a_2 + b_2)e_2 e_5 \\ &\quad + (a_2 + b_3)e_2 e_6 + (k_3 + a_3)e_3^2 + (a_3 + b_1)e_3 e_4 + (a_3 + b_2)e_3 e_5 + (a_3 + b_3)e_3 e_6 \\ &\quad + (k_4 + b_1)e_4^2 + (b_1 + b_2)e_4 e_5 + (b_1 + b_3)e_4 e_6 + (k_5 + b_2)e_5^2 + (b_2 + b_3)e_5 e_6 + (k_6 + b_3)e_6^2\big] \\ &= -e^T P e,\end{aligned}$$

where, $e = [|e_1|, |e_2|, |e_3|, |e_4|, |e_5|, |e_6|]$ and P is real symmetric. From the Lyapunov theorem of stability [19], it is simple to point out that the zero equilibrium point $(e_i = 0, i = 1, \ldots, 6)$ of the error dynamical system (3.5) is globally asymptotic stable if the real symmetric matrix P is positive definite. According to Sylvester's theorem [36], P is positive definite if and only if $\Delta_i > 0, i = 1, 2, \dots, 6$, where Δ_i represents the ith order sequential sub determinant of matrix. That is, we should choose the appropriate coupled parameters. Then, we realize the adaptive dual synchronization between different chaotic systems. This completes the proof. □

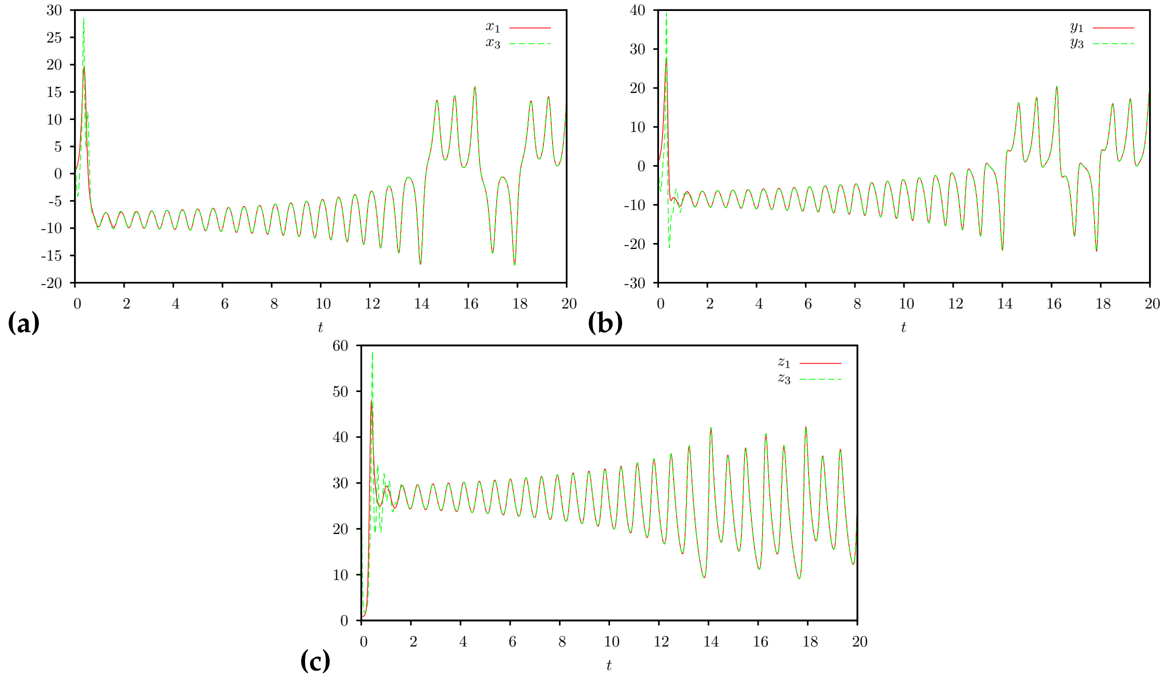

Figure 1: Evolution of the drive system (3.1) and response system (3.3) state, **(a)** signals x_1 and x_3; **(b)** signals y_1 and y_3; **(c)** signals z_1 and z_3.

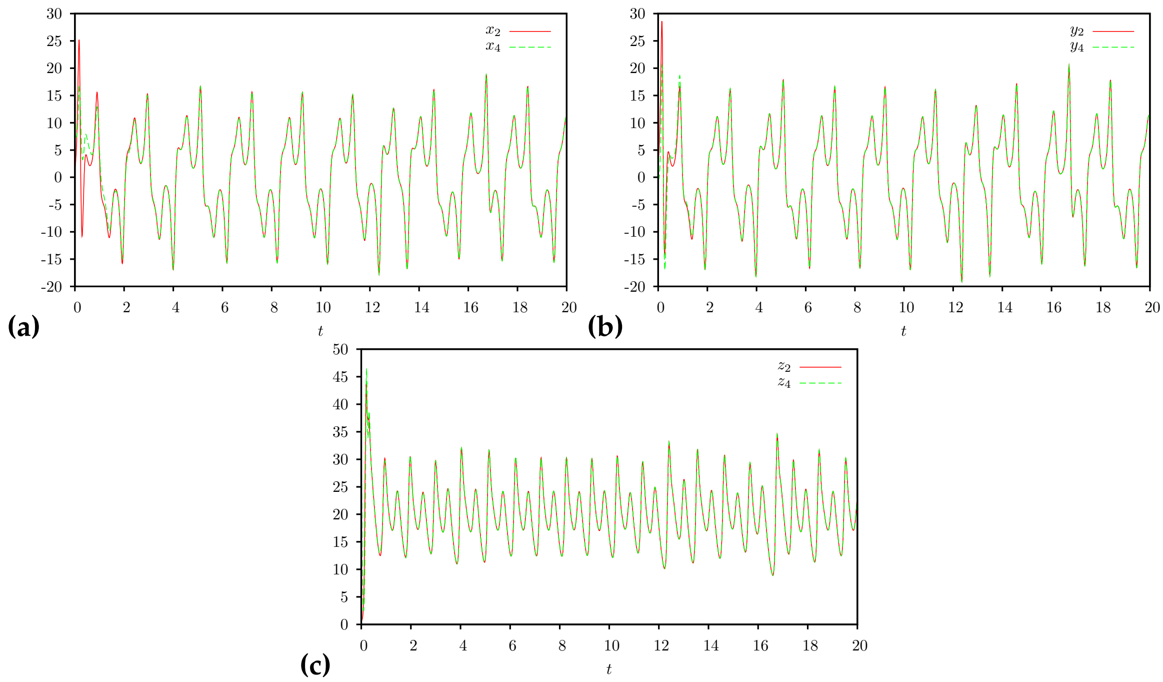

Figure 2: Evolution of the drive system (3.2) and response (3.4) system state, **(a)** signals x_2 and x_4; **(b)** signals y_2 and y_4; **(c)** signals z_2 and z_4.

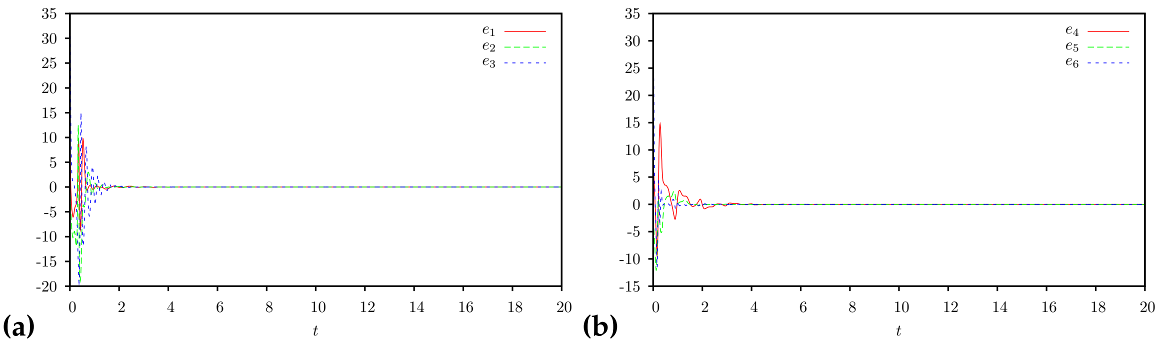

Figure 3: The dual synchronization errors **(a)** e_1, e_2, e_3 between the pair of systems (3.1) and (3.3); **(b)** e_4, e_5, e_6 between the pair of systems (3.2) and (3.4).

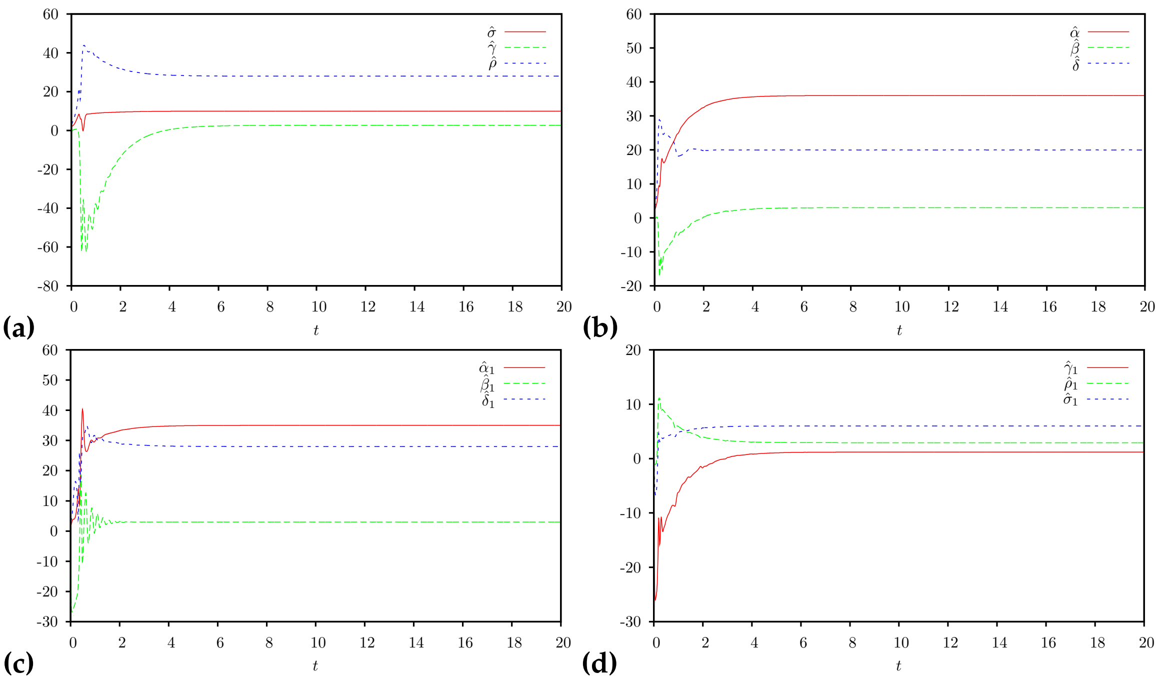

Figure 4: **(a)** Estimate values of parameters of system (3.1); **(b)** Estimate values of parameters of system (3.2); **(c)** Estimate values of parameters of system (3.3); **(d)** Estimate values of parameters of system (3.4).

3.1. Numerical simulations

In this subsection, to verify and demonstrate the effectiveness of the proposed method, we discuss the simulation results for the adaptive dual synchronization of different chaotic systems. In the numerical simulations, the fourth-order Runge-Kutta method is used to solve the systems (3.1)-(3.4) with time step size 0.001. The parameters of the drive and the response systems are set as $\sigma = 10, \rho = 28, \gamma = 8/3$, $\alpha = 36, \delta = 20, \beta = 3$, $\alpha_1 = 35, \delta_1 = 28, \beta_1 = 3, \sigma_1 = 6, \rho_1 = 2.92, \gamma_1 = 1.2$, $a_i = (1, 1, 1), b_i = (1, 1, 1), i = 1, 2, 3$ and $k_i = 4, i = 0, 1, \dots, 6$, for which condition P is positive definite. The initial conditions of the drive and response systems are taken as, $x_1(0) = 0.5, y_1(0) = 1, z_1(0) = 1$ and $x_2(0) = 1.5, y_2(0) = 2.5, z_2(0) = 0.65$, $x_3(0) = 10.5, y_3(0) = 1, z_3(0) = 37$, $x_4(0) = 10.5, y_4(0) = 1, z_4(0) = 37$, respectively. Adaptive dual synchronization of the systems (3.1)-(3.4) via adaptive control law (3.6) and (3.7) with the initial estimated parameters $\hat{\sigma}(0) = 2, \hat{\rho}(0) = 3, \hat{\gamma}(0) = -1$, $\hat{\alpha}(0) = 2, \hat{\delta}(0) = 3, \hat{\beta}(0) = -1$ and $\hat{\alpha}_1(0) = 2, \hat{\delta}_1(0) = 3, \hat{\beta}_1(0) = -1, \hat{\sigma}_1(0) = 2, \hat{\rho}_1(0) = 3, \hat{\gamma}_1(0) = -1$ are shown in Figures 1-4. Figures 1-2 show the evolution of the drive systems (3.1)-(3.2) and the response systems (3.3)-(3.4). Figure 3 shows the error signals e_1, e_2, e_3

and e_4, e_5, e_6 with initial conditions $e_1(0) = 10, e_2(0) = 0, e_3(0) = 36, e_4(0) = 9, e_5(0) = -1.5, e_6(0) = 36.35$. Figure 4 shows that the estimates $\hat{\sigma}, \hat{\rho}, \hat{\gamma}, \hat{\alpha}, \hat{\delta}, \hat{\beta}$ and $\hat{\alpha}_1, \hat{\delta}_1, \hat{\beta}_1, \hat{\sigma}_1, \hat{\rho}_1, \hat{\gamma}_1$ of the unknown parameters $\sigma, \rho, \gamma, \alpha, \delta, \beta$ and $\alpha_1, \delta_1, \beta_1, \sigma_1, \rho_1, \gamma_1$ converges to $\sigma = 10, \rho = 28, \gamma = 8/3$, $\alpha = 36, \delta = 20, \beta = 3$ and $\alpha_1 = 35, \delta_1 = 28, \beta_1 = 3, \sigma_1 = 6, \rho_1 = 2.92, \gamma_1 = 1.2$ as $t \to \infty$.

4. Dual synchronization of different hyperchaotic systems with unknown parameters

In this section, we will choose the four systems with different structure that are hyperchaotic systems, namely, hyperchaotic Lorenz system, hyperchaotic Chen system as the drives and hyperchaotic Lü, hyperchaotic Lorenz-Stenflo as the responses, we illustrate the effectiveness of the adaptive dual synchronization scheme of different hyperchaotic systems. The nonlinear differential equations that describe the different hyperchaotic systems are

Drive 1: Hyperchaotic Lorenz system [10] is given by

$$\dot{x}_1 = \alpha_1(y_1 - x_1), \quad \dot{y}_1 = \beta_1 x_1 - x_1 z_1 + y_1 - w_1, \quad \dot{z}_1 = x_1 y_1 - \gamma_1 z_1, \quad \dot{w}_1 = \theta_1 y_1 z_1. \tag{4.1}$$

Drive 2: Hyperchaotic Chen [21] is given by

$$\dot{x}_2 = \alpha_2(y_2 - x_2) + w_2, \quad \dot{y}_2 = \beta_2 x_2 - x_2 z_2 + \delta_2 y_2, \quad \dot{z}_2 = x_2 y_2 - \gamma_2 z_2, \quad \dot{w}_2 = y_2 z_2 + \theta_2 w_2. \tag{4.2}$$

Response 1: Hyperchaotic Lü [8] is given by

$$\begin{aligned}
\dot{x}_3 &= \alpha_3(y_3 - x_3) + w_3 + u_1, \\
\dot{y}_3 &= -x_3 z_3 + \delta_3 y_3 + u_2, \\
\dot{z}_3 &= x_3 y_3 - \gamma_3 z_3 + u_3, \\
\dot{w}_3 &= x_3 z_3 + \theta_3 w_3 + u_4.
\end{aligned} \tag{4.3}$$

Response 2: Hyperchaotic Lorenz-Stenflo [37] is given by

$$\begin{aligned}
\dot{x}_4 &= \alpha_4(y_4 - x_4) + \delta_4 w_4 + u_5, \\
\dot{y}_4 &= \beta_4 x_4 - x_4 z_4 - y_4 + u_6, \\
\dot{z}_4 &= x_4 y_4 - \gamma_4 z_4 + u_7, \\
\dot{w}_4 &= -x_4 - \alpha_4 w_4 + u_8,
\end{aligned} \tag{4.4}$$

where $\alpha_1, \beta_1, \gamma_1, \theta_1, \alpha_2, \beta_2, \delta_2, \gamma_2, \theta_2, \alpha_3, \delta_3, \gamma_3, \theta_3, \alpha_4, \beta_4, \gamma_4, \delta_4$ are unknown system parameters. The control input is defined as $u_i, (i = 1, 2, \ldots, 8)$. The error system can be written as follows:

$$\begin{aligned}
\dot{e}_1 &= \alpha_3(y_3 - x_3) + w_3 - \alpha_1(y_1 - x_1) + u_1, \\
\dot{e}_2 &= -x_3 z_3 + \delta_3 y_3 - \beta_1 x_1 + x_1 z_1 - y_1 + w_1 + u_2, \\
\dot{e}_3 &= x_3 y_3 - \gamma_3 z_3 - x_1 y_1 + \gamma_1 z_1 + u_3, \\
\dot{e}_4 &= x_3 z_3 + \theta_3 w_3 - \theta_1 y_1 z_1 + u_4, \\
\dot{e}_5 &= \alpha_4(y_4 - x_4) + \delta_4 w_4 - \alpha_2(y_2 - x_2) - w_2 + u_5, \\
\dot{e}_6 &= \beta_4 x_4 - x_4 z_4 - y_4 - \beta_2 x_2 + x_2 z_2 - \delta_2 y_2 + u_6, \\
\dot{e}_7 &= x_4 y_4 - \gamma_4 z_4 - x_2 y_2 + \gamma_2 z_2 + u_7, \\
\dot{e}_8 &= -x_4 - \alpha_4 w_4 - y_2 z_2 - \theta_2 w_2 + u_8,
\end{aligned} \tag{4.5}$$

where $e_1 = x_3 - x_1$, $e_2 = y_3 - y_1$, $e_3 = z_3 - z_1$, $e_4 = w_3 - w_3$, $e_5 = x_4 - x_2$, $e_6 = y_4 - y_2$, $e_7 = z_4 - z_2$, $e_8 = w_4 - w_2$. Our goal is to find proper control functions $u_i, i = 1, 2, \ldots, 8$ and parameter update rule, such that system (4.1)-(4.2) asymptotically dual synchronization systems (4.3)-(4.4), i.e., $\lim_{t\to\infty} \|e_i(t)\| = 0$, (i=1,2,. . . ,8). The following theorem shows that hyperchaotic systems (4.1)-(4.2) and (4.3)-(4.4) can be dual synchronized effectively by the following proposed adaptive controller.

Theorem 4.1. *Systems* (4.1)-(4.2) *and* (4.3)-(4.4) *are globally asymptotically dual synchronized if we design the following adaptive control functions and adaptation law:*

$$
\begin{aligned}
u_1 &= -\hat{\alpha}_3(y_3 - x_3) - w_3 + \hat{\alpha}_1(y_1 - x_1) - k_1 e_1 - e_s,\\
u_2 &= x_3 z_3 - \hat{\delta}_3 y_3 + \hat{\beta}_1 x_1 - x_1 z_1 + y_1 - w_1 - k_2 e_2 - e_s,\\
u_3 &= -x_3 y_3 + \hat{\gamma}_3 z_3 + x_1 y_1 - \hat{\gamma}_1 z_1 - k_3 e_3 - e_s,\\
u_4 &= -x_3 z_3 - \hat{\theta}_3 w_3 + \hat{\theta}_1 y_1 z_1 - k_4 e_4 - e_s,\\
u_5 &= -\hat{\alpha}_4(y_4 - x_4) - \hat{\delta}_4 w_4 + \hat{\alpha}_2(y_2 - x_2) + w_2 - k_5 e_5 - e_s,\\
u_6 &= -\hat{\beta}_4 x_4 + x_4 z_4 + y_4 + \hat{\beta}_2 x_2 - x_2 z_2 + \hat{\delta}_2 y_2 - k_6 e_6 - e_s,\\
u_7 &= -x_4 y_4 + \hat{\gamma}_4 z_4 + x_2 y_2 - \hat{\gamma}_2 z_2 - k_7 e_7 - e_s,\\
u_8 &= x_4 + \hat{\alpha}_4 w_4 + y_2 z_2 + \hat{\theta}_2 w_2 - k_8 e_8 - e_s,
\end{aligned}
\tag{4.6}
$$

and

$$
\begin{array}{lllll}
\dot{\hat{\alpha}}_1 = -(y_1 - x_1)e_1, & \dot{\hat{\beta}}_1 = -x_1 e_2, & \dot{\hat{\gamma}}_1 = z_1 e_3, & \dot{\hat{\theta}}_1 = -y_1 z_1 e_4, & \\
\dot{\hat{\alpha}}_2 = -(y_2 - x_2)e_5, & \dot{\hat{\beta}}_2 = -x_2 e_6, & \dot{\hat{\delta}}_2 = -y_2 e_6, & \dot{\hat{\gamma}}_2 = x_2 e_7, & \\
\dot{\hat{\theta}}_2 = -w_2 e_8 & \dot{\hat{\alpha}}_3 = (y_3 - x_3)e_1, & \dot{\hat{\delta}}_3 = y_3 e_2, & \dot{\hat{\gamma}}_3 = -z_3 e_3, & \\
\dot{\hat{\theta}}_3 = w_3 e_4, & \dot{\hat{\alpha}}_4 = (y_4 - x_4)e_5 - w_4 e_8, & \dot{\hat{\beta}}_4 = x_4 e_6, & \dot{\hat{\gamma}}_4 = -z_4 e_7, & \dot{\hat{\delta}}_4 = w_2 e_5,
\end{array}
\tag{4.7}
$$

where $k_i > 0, (i = 1, 2, \ldots, 8)$, $\hat{\alpha}_1, \hat{\beta}_1, \hat{\gamma}_1, \hat{\theta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\delta}_2, \hat{\gamma}_2, \hat{\theta}_2, \hat{\alpha}_3, \hat{\delta}_3, \hat{\gamma}_3, \hat{\theta}_3, \hat{\alpha}_4, \hat{\beta}_4, \hat{\gamma}_4, \hat{\delta}_4$ *are the parameter estimations of* $\alpha_1, \beta_1, \gamma_1, \theta_1, \alpha_2, \beta_2, \delta_2, \gamma_2, \theta_2, \alpha_3, \delta_3, \gamma_3, \theta_3, \alpha_4, \beta_4, \gamma_4, \delta_4$, *respectively, and*

$$e_s = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + b_1 e_5 + b_2 e_6 + b_3 e_6 + b_4 e_8.$$

Proof. The error system (4.5) can be written as follows

$$
\begin{aligned}
\dot{e_1} &= \tilde{\alpha}_3(y_3 - x_3) + w_3 - \tilde{\alpha}_1(y_1 - x_1) - k_1 e_1 - e_s,\\
\dot{e_2} &= \tilde{\delta}_3 y_3 - \tilde{\beta}_1 x_1 - k_2 e_2 - e_s,\\
\dot{e_3} &= -\tilde{\gamma}_3 z_3 + \tilde{\gamma}_1 z_1 - k_3 e_3 - e_s,\\
\dot{e_4} &= \tilde{\theta}_3 w_3 - \tilde{\theta}_1 y_1 z_1 - k_4 e_4 - e_s,\\
\dot{e_5} &= \tilde{\alpha}_4(y_4 - x_4) + \tilde{\delta}_4 w_4 - \tilde{\alpha}_2(y_2 - x_2) - k_5 e_5 - e_s,\\
\dot{e_6} &= \tilde{\beta}_4 x_4 - \tilde{\beta}_2 x_2 - \tilde{\delta}_2 y_2 - k_6 e_6 - e_s,\\
\dot{e_7} &= -\tilde{\gamma}_4 z_4 + \tilde{\gamma}_2 z_2 - k_7 e_7 - e_s,\\
\dot{e_8} &= -\tilde{\alpha}_4 w_4 - \tilde{\theta}_2 w_2 - k_8 e_8 - e_s,
\end{aligned}
$$

where, $\tilde{\alpha}_1 = \alpha_1 - \hat{\alpha}_1, \tilde{\beta}_1 = \beta_1 - \hat{\beta}_1, \tilde{\gamma}_1 = \gamma_1 - \hat{\gamma}_1, \tilde{\theta}_1 = \theta_1 - \hat{\theta}_1, \tilde{\alpha}_2 = \alpha_2 - \hat{\alpha}_2, \tilde{\beta}_2 = \beta_2 - \hat{\beta}_2, \tilde{\delta}_2 = \delta_2 - \hat{\delta}_2, \tilde{\gamma}_2 = \gamma_2 - \hat{\gamma}_2, \tilde{\theta}_2 = \theta_2 - \hat{\theta}_2, \tilde{\alpha}_3 = \alpha_3 - \hat{\alpha}_3, \tilde{\delta}_3 = \delta_3 - \hat{\delta}_3, \tilde{\gamma}_3 = \gamma_3 - \hat{\gamma}_3, \tilde{\theta}_3 = \theta_3 - \hat{\theta}_3, \tilde{\alpha}_4 = \alpha_4 - \hat{\alpha}_4, \tilde{\beta}_4 = \beta_4 - \hat{\beta}_4, \tilde{\gamma}_4 = \gamma_4 - \hat{\gamma}_4, \tilde{\delta}_4 = \delta_4 - \hat{\delta}_4$. The Lyapunov function candidate is formed as

$$V = \frac{1}{2}\left(e^T e + \tilde{\alpha}_1^2 + \tilde{\beta}_1^2 + \tilde{\gamma}_1^2 + \tilde{\theta}_1^2 + \tilde{\alpha}_2^2 + \tilde{\beta}_2^2 + \tilde{\delta}_2^2 + \tilde{\gamma}_2^2 + \tilde{\theta}_2^2 + \tilde{\alpha}_3^2 + \tilde{\delta}_3^2 + \tilde{\gamma}_3^2 + \tilde{\theta}_3^2 + \tilde{\alpha}_4^2 + \tilde{\beta}_4^2 + \tilde{\gamma}_4^2 + \tilde{\delta}_4^2\right).$$

Applying the above equations in the time derivative of V leads to the time derivative of V leads to

$$
\begin{aligned}
\dot{V} &= e^T \dot{e} + \tilde{\alpha}_1 \dot{\tilde{\alpha}}_1 + \tilde{\beta}_1 \dot{\tilde{\beta}}_1 + \tilde{\gamma}_1 \dot{\tilde{\gamma}}_1 + \tilde{\theta}_1 \dot{\tilde{\theta}}_1 + \tilde{\alpha}_2 \dot{\tilde{\alpha}}_2 + \tilde{\beta}_2 \dot{\tilde{\beta}}_2 + \tilde{\delta}_2 \dot{\tilde{\delta}}_2 + \tilde{\gamma}_2 \dot{\tilde{\gamma}}_2 + \tilde{\theta}_2 \dot{\tilde{\theta}}_2\\
&\quad + \tilde{\alpha}_3 \dot{\tilde{\alpha}}_3 + \tilde{\delta}_3 \dot{\tilde{\delta}}_3 + \tilde{\gamma}_3 \dot{\tilde{\gamma}}_3 + \tilde{\theta}_3 \dot{\tilde{\theta}}_3 + \tilde{\delta}_4 \dot{\tilde{\delta}}_4 + \tilde{\beta}_4 \dot{\tilde{\beta}}_4 + \tilde{\gamma}_4 \dot{\tilde{\gamma}}_4 + \tilde{\delta}_4 \dot{\tilde{\delta}}_4\\
&= e_1\Big[\tilde{\alpha}_3(y_3 - x_3) - \tilde{\alpha}_1(y_1 - x_1) - k_1 e_1 - e_s\Big] + e_2\Big[\tilde{\delta}_3 y_3 - \tilde{\beta}_1 x_1 - k_2 e_2\\
&\quad - e_s\Big] + e_3\Big[-\tilde{\gamma}_3 z_3 + \tilde{\gamma}_1 z_1 - k_3 e_3 - e_s\Big] + e_4\Big[\tilde{\theta}_3 w_3 - \tilde{\theta}_1 y_1 z_1 - k_4 e_4 - e_s\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ e_5\Big[\tilde{\alpha}_4(y_4-x_4)+\tilde{\delta}_4 w_4-\tilde{\alpha}_2(y_2-x_2)-k_5e_5-e_s\Big]+e_6\Big[\tilde{\beta}_4x_4-\tilde{\beta}_2x_2\\
&-\tilde{\delta}_2y_2-k_6e_6-e_s\Big]+e_7\Big[-\tilde{\gamma}_4z_4+\tilde{\gamma}_2z_2-k_7e_7-e_s\Big]+e_8\Big[-\tilde{\alpha}_4w_4-\tilde{\theta}_2w_2\\
&-k_8e_8-e_s\Big]+\tilde{\alpha}_1((y_1-x_1)e_1)+\tilde{\beta}_1(x_1e_2)+\tilde{\gamma}_1(-z_1e_3)+\tilde{\theta}_1(y_1z_1e_4)\\
&+\tilde{\alpha}_2((y_2-x_2)e_5)+\tilde{\beta}_2(x_2e_6)+\tilde{\delta}_2(y_2e_6)+\tilde{\gamma}_2(-z_2e_7)+\tilde{\theta}_2(w_2e_8)\\
&+\tilde{\alpha}_3(-(y_3-x_3)e_1)+\tilde{\delta}_3(-y_3e_2)+\tilde{\gamma}_3(z_3e_3)+\tilde{\theta}_3(-w_3e_4) \qquad (4.8)\\
&+\tilde{\alpha}_4(-(y_4-x_4)e_5-w_4e_8)+\tilde{\beta}_4(-x_4e_6)+\tilde{\gamma}_4(z_4e_7)+\tilde{\delta}_4(-w_2e_5)\\
=&-\Big[(k_1+a_1)e_1^2+(a_1+a_2)e_1e_2+(a_1+a_3)e_1e_3+(a_1+a_4)e_1e_4+(a_1+b_1)e_1e_5\\
&+(a_1+b_2)e_1e_6+(a_1+b_3)e_1e_7+(a_1+b_4)e_1e_8+(k_2+a_2)e_2^2+(a_2+a_3)e_2e_3\\
&+(a_2+a_4)e_2e_4+(a_2+b_1)e_2e_5+(a_2+b_2)e_2e_6+(a_2+b_3)e_2e_7+(a_2+b_4)e_2e_8\\
&+(k_3+a_3)e_3^2+(a_3+a_4)e_3e_4-(a_3+b_1)e_3e_5+(a_3+b_2)e_3e_6+(a_3+b_3)e_3e_7\\
&+(a_3+b_3)e_3e_8+(k_4+a_4)e_4^2+(a_4+b_1)e_4e_5+(a_4+b_2)e_4e_6+(a_4+b_3)e_4e_7\\
&+(a_4+b_4)e_4e_8+(k_5+b_1)e_5^2+(b_1+b_2)e_5e_6+(b_1+b_3)e_5e_7+(b_1+b_4)e_5e_8\\
&+(k_6+b_2)e_6^2+(b_2+b_3)e_6e_7+(b_2+b_4)e_6e_8+(k_7+b_3)e_7^2+(b_3+b_4)e_7e_8\\
&+(k_8+b_4)e_8^2\Big]\\
=&-e^TPe,
\end{aligned}
$$

where, $e_i > 0, (i = 1, 2, \ldots 8)$ and P is is real symmetric. From the Lyapunov theorem of stability [19], it is simple to point out that the zero equilibrium point $(e_i = 0, i = 1, \ldots, 8)$ of the error dynamical system (4.5) is globally asymptotic stable if the real symmetric matrix P is positive definite. According to Sylvester's theorem [36], P is positive definite if and only if $\Delta_i > 0, i = 1, 2, \ldots, 8$, where Δ_i represents the ith order sequential sub determinant of matrix. That is, we should choose the appropriate coupled parameters. Then, we realize the adaptive dual synchronization between different hyperchaotic systems. This completes the proof. □

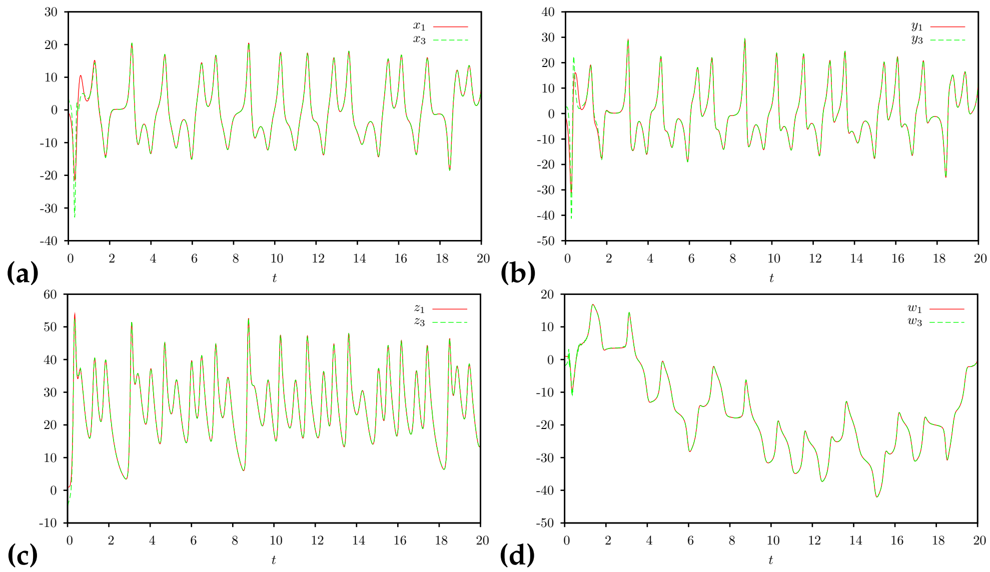

Figure 5: Evolution of the drive system (4.1) and response system (4.3) state, **(a)** signals x_1 and x_3; **(b)** signals y_1 and y_3; **(c)** signals z_1 and z_3; **(d)** signals w_1 and w_3.

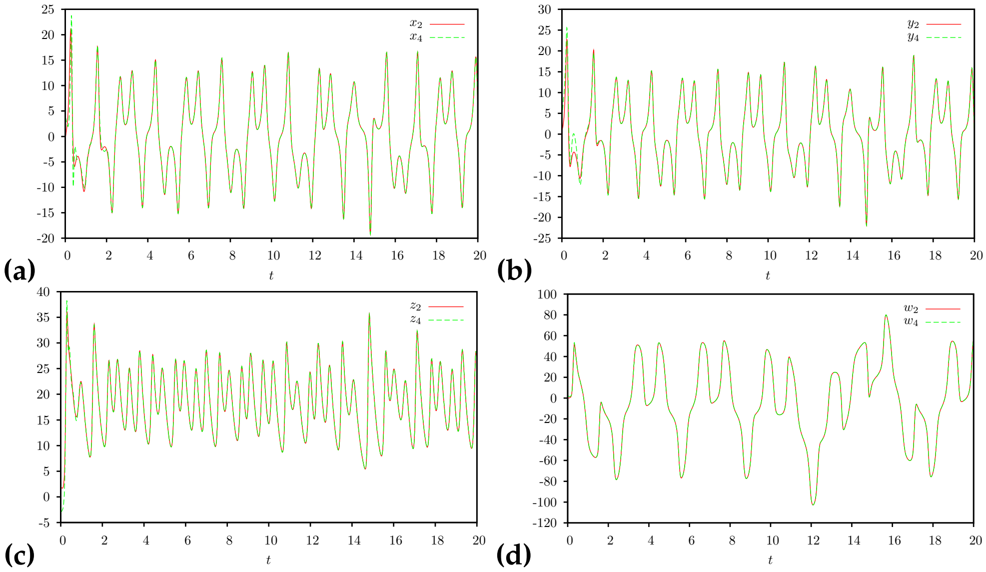

Figure 6: Evolution of the drive system (4.2) and response system (4.4) state, **(a)** signals x_2 and x_4; **(b)** signals y_2 and y_4; **(c)** signals z_2 and z_4; **(d)** signals w_2 and w_4.

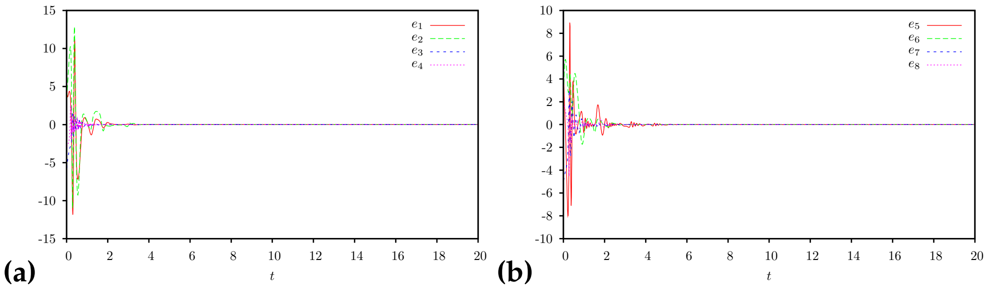

Figure 7: The dual synchronization errors **(a)** e_1, e_2, e_3, e_4 between the pair of systems (4.1) and (4.3); **(b)** e_5, e_6, e_7, e_8 between the pair of systems (4.2) and (4.4).

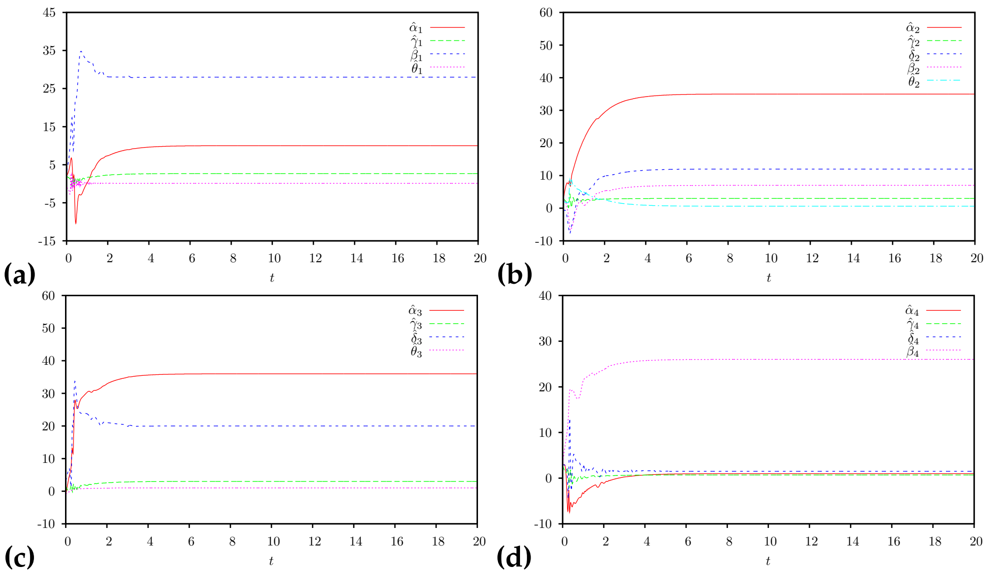

Figure 8: **(a)** Estimate values of parameters of system (4.1); **(b)** Estimate values of parameters of system (4.2); **(c)** Estimate values of parameters of system (4.3); **(d)** Estimate values of parameters of system (4.4).

4.1. Numerical simulations

In the numerical simulations, we demonstrate the effectiveness of the proposed method and discuss the simulation results for the adaptive dual synchronization of different chaotic systems, the fourth-order Runge-Kutta method is used to solve the systems (4.1)-(4.4) with time step size 0.001. The parameters of the drive and the response systems are set as $\alpha_1 = 10, \beta_1 = 28, \gamma_1 = 8/3, \theta_1 = 0.1, \alpha_2 = 35, \beta_2 = 7, \delta_2 = 12, \gamma_2 = 3, \theta_2 = 0.6, \alpha_3 = 35, \delta_3 = 20, \gamma_3 = 3 , \theta_3 = 1, \alpha_4 = 1, \beta_4 = 26, \gamma_4 = 0.7, \delta_4 = 1.5$ and $a_i = (1,1,1,1), b_i = (1,1,1,1), i = 1,2,4$ and $k_i = 4, i = 0,1,\ldots,8$ for which condition P is positive definite. The initial conditions of the drive and response systems are taken as $x_1(0) = -1, y_1(0) = -1, z_1(0) = 1, w_1(0) = 1,$ $x_2(0) = -1, y_2(0) = 1, z_2(0) = 2, w_2(0) = 0,$ $x_3(0) = 5, y_3(0) = 2, z_3(0) = -5, w_3(0) = -2$ and $x_4(0) = -4, y_4(0) = 3, z_4(0) = -3, w_4(0) = 2$, respectively. Adaptive dual synchronization of the systems (4.1)-(4.4) via adaptive control law (4.6) and (4.7) with the initial estimated parameters $\hat{\alpha}_1(0) = 2, \hat{\beta}_1(0) - 1 =, \hat{\gamma}_1(0) = 3, \hat{\theta}_1(0) = 2, \hat{\alpha}_2(0) = 3, \hat{\beta}_2(0) = 2, \hat{\delta}_2(0) = -1, \hat{\gamma}_2(0) = 3, \hat{\theta}_2(0) = 2, \hat{\alpha}_3(0) = -1, \hat{\delta}_3(0) = 3, \hat{\gamma}_3(0) = 2, \hat{\theta}_3(0) = -1, \hat{\alpha}_4(0) = 3, \hat{\beta}_4(0) = 3, \hat{\gamma}_4(0) = 3, \hat{\delta}_4(0) = 3$ are shown in Figures 5-8. Figures 5-6 show the evolution of the drive systems (4.1)-(4.2) and the response systems (4.3)-(4.4). Figure 7 shows the error signals e_1, e_2, e_3, e_4 and e_5, e_6, e_7, e_8, between the pair of drive systems and the pair of response systems with initial conditions $e_1(0) = 6, e_2(0) = 3, e_3(0) = -6, e_4(0) = -3, e_5(0) = -3, e_6(0) = 2, e_7(0) = -5, e_8(0) = 2$. Figure 8 shows that the estimates $\hat{\alpha}_1, \hat{\beta}_1, \hat{\gamma}_1, \hat{\theta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{\delta}_2, \hat{\gamma}_2, \hat{\theta}_2, \hat{\alpha}_3,$ $\hat{\delta}_3, \hat{\gamma}_3, \hat{\theta}_3, \hat{\alpha}_4, \hat{\beta}_4, \hat{\gamma}_4, \hat{\delta}_4$ of the unknown parameters converges to $\alpha_1 = 10, \beta_1 = 28, \gamma_1 = 8/3, \theta_1 = 0.1, \alpha_2 = 35, \beta_2 = 7, \delta_2 = 12, \gamma_2 = 3, \theta_2 = 0.6, \alpha_3 =, \delta_3 = 20, \gamma_3 = 3 , \theta_3 = 1, \alpha_4 = 1, \beta_4 = 26, \gamma_4 = 0.7, \delta_4 = 1.5$ as $t \to \infty$.

5. Conclusions

In this paper, based on the Lyapunov theorem of stability, an adaptive control strategy is proposed to achieve the dual synchronized among four non-identical unknown chaotic and four non-identical unknown hyperchaotic systems are investigated.The graphical results fully certified the theoretical findings. The error states converged to the origin with sufficient transient speed. The estimation of unknown parameters approached to the true values of the unknown systems parameters. Numerical experimental results further validated the robustness and effectiveness of the proposed methodology.

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