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Product and Coproduct in the Category of Fuzzy Frames

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Abstract

Frame theory is Lattice theory applied to topology. This approach to topology takes the lattices of open sets as the basic notion-it is "point free topology". There, one investigates typical properties of lattices of open sets that can be expressed without reference to points.

In this paper we generalise the concept of frame into a fuzzy frame. The category FFrm of fuzzy frame and fuzzy frame homomorphism is defined and we show that there exist products and coproducts in the category FFrm and to construct them explicitly and we conclude that the category FFrm is complete and cocomplete.

Keywords: Frame , Fuzzy frame ,Product , Coproduct,complete,cocomplete.

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1. Introduction

The first mathematician to take the notion of open set as basic to the study of continuity properties was Hausdorff in 1914. Using the lattice of open sets , Marshal stone[11] was able to give topological representation of Boolean algebras and distributive lattices and H.Wallman[9]used lattice theoretic constructs to obtain the Wallman compactification .Mekinsey and Tarski[7] studied the " algebra of topology " that is topology studied from a Lattice theoretical viewpoint. But a fundamental change in the outlook came in late fifties ; Charles Ehresmann[3]first articulated the view that a complete lattice with an appropriate distributivity property deserved to be studied in their own right rather than simply as a means to study topological spaces . He called the lattice a local lattice . Dowker and strauss[1] introduced the term frame for a local lattice and extended many results of topology to frame theory . It was with the publication of John Isbell,s [4] that the real importance of the subject emerged. Since then frame theory is studied extensively by many authors. The structure of the paper is as follows. In the second section is devoted to the basic definition and results concerning frames and fuzzy sets theory .

In section three the category **FFrm** of fuzzy frame and fuzzy frame homomorphism is defined. The main aim of section four is to show that there exist products and coproducts in the category **FFrm** and to construct them explicitly.

2. Preliminaries

Definition 2.1. A set P equipped with a partial order $\leq$ ($\leq$ is reflexive, transitive, antisymmetric) is called a partially ordered set, and usually called a poset for short.

Definition 2.2. A poset P such that for any two $a, b \in P$ there is the infimum $a \wedge b$ and supremum $a \vee b$ is called a lattice.

Definition 2.3. A complete lattice is a poset in which each subset has an infimum and a supremum. In particular, a complete lattice has the bottom$o$ and top1.

Definition 2.4. A lattice L is distributive if

$$\forall\, a, b, c \in L, \qquad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Which is equivalent to

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

A function $f : M \to N$ between two lattices is said to be increasing (decreasing) if

$$f(m_1) \leq f(m_2) \text{ Whenever } m_1 \leq m_2 \; (m_1 \geq m_2).$$

Definition 2.5. A frame is a complete lattice L satisfying the distributivity Law$(\vee A) \wedge b = \vee \{a \wedge b | a \in A\}$ for each subset $A \subseteq L$ and $b \in L$.

Definition2.6. A frame homomorphism $h: L \to M$ is a mapping preserving all suprema (including the bottom) and all finite intima (including the top1).

The category of frames and frame homomorphism will be denoted by **Frm**.

Example 2.7.

(i):Take I an the unit interval,then $I = [0,1]$ is a frame.

(ii): For every topological space $(X, \tau), \tau$ is a frame.

Definition2.8. Let X is a nonempty ordinary set, L a complete lattice. An L_fuzzy subset on X is a mapping $A: X \to L$.

Definition2.9. Let L^Xbe an L_fuzzy space. Define the partial order$\leq$ in $L^X$by:

$$\forall \ A,B \in L^X, A \leq B \Leftrightarrow \forall \ x \in X, \ A(x) \leq B(x)$$

Then $(L^X, \leq)$ is a poset.

Definition2.10. For $A, B \in L^X$ we define $A \vee B$ and $A \wedge B$ as :

$$(A \vee B)(x) = A(x) \vee B(x), \qquad (A \wedge B)(x) = A(x) \wedge B(x), \qquad \forall x \in X.$$

Thus, $\underline{o}$ and $\underline{1} \in L^X$, $\underline{o} : X \rightarrow L$ such that $\underline{o}(x) = o, \underline{1}(x) = 1$ for all $x \in X$.

Definition2.11. Let L^X, L^Y is L_fuzzy spaces, $f: X \rightarrow Y$ an ordinary mapping. Based on$f: X \rightarrow Y$, define L_fuzzy mapping $\vec{f} : L^X \rightarrow L^Y$ and its L_fuzzy reverse mapping $\overleftarrow{f}: L^Y \rightarrow L^X$,

$$\overleftrightarrow{f}(A)(y) = \vee\{A(x): x \in X, f(x) = y\}, \forall A \in L^X, y \in Y.$$

$$\overleftarrow{f}(B)(x) = B\big(f(x)\big), \forall \ A \in L^Y, x \in X.$$

3. Fuzzy Frame

Definition 3.1. Let L be a frame, then a fuzzy set $\{\mu: L \rightarrow I\}$ of L is said to be a fuzzy frame if:

(F_1) $\mu(\vee S) \geq \inf\{\mu(a) | a \in S\}$ for every arbitrary $S \subseteq L$.

$(F_2)$$\mu(a \wedge b) \geq \min\{\mu(a), \mu(b)\}$ $for\ all\ a, b \in L.$

$(F_3)$$\mu(e_L) = \mu(0_L) \geq \mu(a)$ for all $a \in L$where $e_L$and 0_L are respectively the unit and zero element of the frame L.

Example 3.2.consider the set R, of real numbers with usual topology τ, which is a frame. Let μ be a Fuzzy set in τ defined by,

$$\mu(u) = \begin{cases} 1 & u = R, \emptyset \\ \frac{1}{2} & u \neq R, \emptyset \end{cases} \quad \text{Where } u \in \tau$$

Then μ is a fuzzy frame of τ.

Proposition 3. 3. If μ is a fuzzy frame of L then $\mu_t = \{x \in L | \mu(x) \geq t\}$ is a subframe of L for any $t \in I$ with $t \leq \mu(e_L) = \mu(o_L)$.

Proof. For arbitrary $\{a_i\}_{i \in \Lambda} \subseteq \mu_t$ we have$\mu(\vee a_i) \geq t$ since , $\mu$is a fuzzy frame and $\mu(a_i) \geq t$ for all i . Hence $\vee a_i \in \mu_t$ .Similarly for all $a, b \in \mu_t$ we have $a \wedge b \in \mu_t$ also clearly $e_L, o_L \in \mu_t$ therefore , μ_t is a subframe of L.

Proposition3. 4. If each non-empty level subset $\mu_t, t \in I$ of a fuzzy set μ is a subframe of L ,then $\mu$is a fuzzy frame of L.

Proof. Given $\mu_t = \{x \in L | \mu(x) \geq t\}, t \in I$ is a subframe of L. μ_t being a sub frame $o_L, e_L \in \mu_t, t \in I$. In particular we have $o_L, e_L \in \mu_T$ where T the largest element of Isuch that $\mu_T \neq \emptyset$.

Hence $\mu(e_L) = \mu(o_L) = T \geq \mu(a)$ for all $a \in L$, now let S an arbitrary subset of L and let $t = \inf\{\mu(a) | a \in S\}$. Clearly we have $S \subseteq \mu_t$ hence$\vee S \in \mu_t$ and there for $\mu(\vee S) \geq \inf\{\mu(a) | a \in S\}$.

Similarly for all $a, b \in L$ we have $\mu(a \wedge b) \geq \min\{\mu(a), \mu(b)\}$ .Hence $\mu$is a fuzzy frame of L.

Theorem 3.5. μ is a fuzzy frame of L if and only if for any $\alpha \in I, \tau_\alpha = \{u \in L | \mu(u) > \alpha\}$ is a sub frame of L for any $\alpha \in I$ with $\alpha < \mu(e_L) = \mu(0_L)$.

Proof. Follows from proposition 3.3 and proposition3.4.

Definition 3.6. If there are two fuzzy frame μ_1, μ_2 on the same frame L , we say that μ_1 is stronger than μ_2 if $\mu_1(a) \geq \mu_2(a)$ for every $a \in L$.

Definition3.7. Let (L, μ) be a fuzzy frame we define the mapping $\mu^*: L \to I$ by the equality $\mu^*(a) = \mu(a^c)$ for every a in L.The number $\mu^*(a)$ will be called the degree of complement of a fuzzy set μ.

From the definition(3.1) and (3.6) one easily gets the following.

Proposition3.8. The mapping $\mu^*: L \to I$ has the following properties:

1) If $a, b \in L$,then $\mu^*(a \vee b) \geq \mu^*(a) \wedge \mu^*(b)$

3) $\mu^*(0) = \mu^*(1) = 1$

Remark. It is clear that a fuzzy frame can be equivalently defined as a pair (X, μ^*) where $\mu^*: L \to I$ satisfies the properties F_1, F_2, F_3 and is understood as the degree of closeness of fuzzy subsets. the corresponding fuzzy frame in to be defined by the equality $\mu(a) = \mu^*(a^c)$.

Definition3.9. Let (L, μ) and (M, η) be fuzzy frames and $f: L \to M$ is a frame homomorphism this mapping is called fuzzy continuous if $\mu(f^{-1})(b) \geq \eta(b), b \in M$.

Proposition3.10.Let $\tilde{f}: (L, \mu) \to (M, \eta)$ and $\tilde{g}: (M, \eta) \to (N, \gamma)$ be morphism then $\tilde{g}o\tilde{f}: (L, \mu) \to (N, \gamma)$ is a fuzzy frame homomorphism $gof: L \to M$ such that $\mu \geq \gamma ogof$.

Proof. The proof is obvious.

Since the composition is associative and the identify mapping $e: L \to L$ is fuzzy frame homomorphism with respect to any fuzzy frame on L . The following definition is justified.

Definition3.11. By **FFrm** we denote the category the objects of which are fuzzy frames and the morphisms are fuzzy frame homomorphism between them.

4. Products and coproducts of fuzzy frames

The main aim of this section is to show that there exist products and coproducts in the category **FFrm** and to construct them explicitly.

4.1The initial fuzzy frame for a mapping

Let L be a frame , (M,μ)a fuzzy frame and $f: L \to M$ is a frame homomorphism .By the initial fuzzy frame for this frame homomorphism we understand the weakest fuzzy frame τ on L such that the mapping $f:(L,\tau) \to (M,\mu)$ is fuzzy frame homomorphism.

To construct such a fuzzy frame consider the set $S = \{b = f^{-1}(a) \mid a \in M\}$ of fuzzy subframes of L. for a given $b \in S$,Let $\rho_b = \{a \mid a \in M, b = f^{-1}(a)\}$ and define $\tau(b) = \sup\{\mu(a) \mid a \in \rho_b\}$. It is obvious that $\bigcup\{\rho_b \mid b \in L\} = M$ and $\tau(f^{-1}(a)) \geq \mu(a)$ for every $a \in M$.

Let$b_1, b_2 \in S$, then$b = b_1 \wedge b_2 \in S$ and moreover,$\rho_b \subseteq \{a_1 \wedge a_2 \big| a_1 \in \rho_{b_1}, a_2 \in \rho_{b_2}\}$.

Therefore

$$\tau(b) = \sup\{\mu(a) \big| a \in \rho_b\} \geq \sup\{\sigma(a_1 \wedge a_2) \big| a_1 \in \rho_{b_1}, a_2 \in \rho_{b_2}\} \geq \sup\{\mu(a_1) \wedge \mu(a_2) \big| a_1 \in \rho_{b_1}, a_2 \in \rho_{b_2}\} = \sup\{\mu(a_1) \big| a_1 \in \rho_{b_1}\} \wedge \sup\{\mu(a_2) \big| a_2 \in \rho_{b_2}\} = \tau(b_1) \wedge \tau(b_2),$$

and hence

$$\tau(b) \geq \tau(b_1) \wedge \tau(b_2) \quad \text{for } b_1, b_2 \in S \qquad (1)$$

In a similar way we can show that for any subfamily$\{b_i \mid i \in J\}$ of S

$$\tau(\vee_i b_i) \geq \wedge_i \{\sup \mu(a_i) \big| a_i \in \rho_{b_i}\} = \wedge_i \tau(b_i) \qquad (2)$$

Moreover , it is obvious that$0 = f^{-1}(0) \in S, 1 = f^{-1}(1) \in S$ and $\tau(0) = \tau(1) = 1$

Thus $\tau: S \to I$ satisfies the axioms of definition(3.1).Now we extend τ to a mapping$\tau: L \to I$ by letting $\tau(b) = 0$ for all$b \notin S$.

It is easy to cheak that the function τ thus defined is indeed a fuzzyframe.Moreover , from the construction it is clear that τ is the weakest fuzzy frame on L making the mapping$f:(L,\tau) \to (M,\mu)$ fuzzy frame homomorphism.

4.2 Initial fuzzy frame for a family of mappings

Let now$\{(M_\alpha, \mu_\alpha) \mid \alpha \in A\}$ be a family of fuzzy frame and consider for each $\alpha \in A$ a mapping $f_\alpha: L \to M_\alpha$. Let$\tau_\alpha: L \to I$ be the initial fuzzy frame on L for f_α and let the mapping$\tau: L \to I$ be defined by the equality $\tau(b) = inf_\alpha \tau_\alpha(b)$ where $b \in L$. Since $\tau(b_1 \wedge b_2) = inf_\alpha \tau_\alpha(b_1 \wedge b_2) \geq inf_\alpha(\tau_\alpha(b_1) \wedge \tau_\alpha(b_2)) \geq inf_\alpha \tau_\alpha(b_1) \wedge inf_\alpha \tau_\alpha(b_2) = \tau(b_1) \wedge \tau(b_2)$ and $\tau(\vee_i b_i) = inf_\alpha \tau_\alpha(\vee_i b_i) \geq inf_\alpha \wedge_i \tau_\alpha(b_i) = \wedge_i inf_\alpha \tau_\alpha(b_i) = \wedge_i \tau(b_i)$.

for any collection of fuzzy subset b_i of L ,one can easily conclude that $\tau$is a fuzzy frame on L.Moreover ,it is clear from (4.1) and from the construction of τthat it is the weakest fuzzy frame on L for which mappings$f_\alpha:(L,\tau) \to (M_\alpha, \mu_\alpha)$ are fuzzy continuous. This fuzzy frame τ will be called the intial fuzzy frame for the family of mapping $\{f_\alpha: L \to M_\alpha \mid \alpha \in A\}$.

The existence of such a fuzzy frame allows us to state the following theorem:

***Theorem4.2.1* .** **FFrm** is a complete category . In particular , **FFrm** contains products and inverse limits.

4.3 Product of fuzzy frames

To construct the product in **FFrm** explicitly considers a family $\{(L_\alpha,\tau_\alpha)|\alpha \in A\}$ of fuzzy frames. The product of this family can be defined as a pair (L,τ) , where L denotes the product of all sets L_α and $\tau$is the initial fuzzy frame generated on L by the family $\{\rho_\alpha: L \to L_\alpha, \alpha \in A\}$ of all projections.

Let $(L_1,\tau_1),(L_2,\tau_2)$ be two fuzzy frame and let (L,τ) denote their product . If $b_1 \in L_1, b_2 \in L_2$ then for $b = b_1 \times b_2 \in L$ which is defined as $b = b_1 \wedge b_2$ we have

$$\tau(b) = \tau({\rho_1}^{-1}(b_1) \wedge {\rho_2}^{-1}(b_2)) \geq \tau({\rho_1}^{-1}(b_1)) \wedge \tau({\rho_2}^{-1}(b_2)) \geq \tau_1(b_1) \wedge \tau_2(b_2).$$

Hence the degree of openess of the product of two fuzzy sets in the product space is not less than the minimal dgree of openess of these sets in the corresponding fuzzy spaces.

Now let $\{(L_\alpha,\tau_\alpha)|\alpha \in A\}$ be a family of fuzzy frames and let (L,τ) denote their product . Take $b_\alpha \in L_\alpha$ for every α and let $b \in L$ denote the product of all $b_\alpha(i.e\ b\ = \wedge_\alpha \mu_\alpha(b_\alpha))$. Quite similarly as above one can show that$\tau^*(b) \geq \wedge {\tau^*}_\alpha(b_\alpha)$,and hence the degree of closednees of the product of fuzzy sets in not less than the degree of the degree of closeness of the factors.

The rest of this section is devoted to the concept of coproduct (or direct sum) of fuzzy frames and to some closely related notions.

4.4 Final fuzzy frame for a mapping

Let (L,τ) be a fuzzy frame and M is a set consider a mapping $f: L \to M$ and for every $a \in M$ let $\mu(a) = \tau\left(f^{-1}(a)\right)$.

It is easy to check that μ is a fuzzy frame on M and more over, it is the strongest fuzzy frame on M for which the mapping$f: (L,\tau) \to (M,\mu)$ is fuzzy frame homomorphism.

4.5 Final fuzzy frame for a family of mappings

Let $\{(L_\alpha,\tau_\alpha)\ |\alpha \in A\}$ be a family of fuzzy frames and for every mapping$f_\alpha: L_\alpha \to M$ where M is a set . Let μ_α denote the final topology on M for f_α. define $\mu: M \to I$ by the equality $\mu(a) = inf_\alpha \mu_\alpha(\alpha)$ for $a \in M$.

Quite similarity as in (4.2) one can show that μ is a fuzzy frame on$M$ .More over , it is easy to notice , that it is the strangest fuzzy frame on M for which all the mappings $f_\alpha: L_\alpha \to M$ are fuzzy frame continuous .

From (4.5) immediately follows such a theorem :

***Theorem4.5.1* .**The category **FFrm** is cocomplete specifically, it contains coproducts and direct limits.

4.6 Coproduct in FFrm

To construct the coproduct in **FFrm** explicity consider a family $\{L_\alpha, \tau_\alpha | \alpha \in A\}$of fuzzy frames and let $L = \bigoplus L_\alpha$ denote the direct sum of the corresponding sets. The fuzzy frame $(L, \tau)$where τ is the final topology for the family of all inclusions $i_\alpha: L_\alpha \to L$ is just the coproduct of these fuzzy frame.More over, it is easy to notice , that $\tau(b) = inf_\alpha \tau_\alpha(b_\alpha)$ and $\tau^*(b) = inf\ \tau_\alpha{}^*(b_\alpha)$ where α denotes the restriction of $b \in L$ to L_α.

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